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Elżbieta OLSZOWSKA

FIXED POINTS, BASINS OF ATTRACTION AND JULIA SETS ASSOCIATED WITH STUDYING THE GLOBAL CONVERGENCE OF ITERATION METHOD APPLIED TO SOLVING NONLINEAR EQUATIONS

Summary. Global convergence of iteration methods of solving nonlinear equations is studied in this paper. Newton, König, Schröder and Döring methods applied to looking for zeros of polynomials are considered. This involves the iteration of rational functions over the complex Riemann sphere, which is described by the classical theory of Julia and Fatou. The investigations of existence of attractive and indifferent extraneous fixed points or cycles of the iteration functions are presented. It is shown that such fixed points always exist for Döring iteration method used for solving polynomial of degree greater than 2. The existence of extraneous fixed points is also demonstrated for Schröder iteration functions. König iteration methods have only repulsive fixed points. Theoretical considerations are helped by microcomputer plots of Julia sets and basins of attraction of fixed point and cycles. Also existence of attractive and indifferent extraneous fixed points or cycles of Newton and Schröder method applied to finding zeros of cubic polynomials is examined experimentally. It is made by detecting orbits of the free critical points of these functions.

INTRODUCTION

More than hundred years ago lord Cayley [3], who was studying the Newton's method of solving nonlinear equations posed the following question: What is the set $W(z^*)$ of all initial values $z_o \in C$ for which the iteration sequence $z_{n+1} = z_n - g(z_n)/g'(z_n)$ converges to a given root z^* of $g(z)$. Two French mathematicians - Julia and Fatou [9], [11] - took up this problem. To answer Cayley's question they developed their fascinating theory of the iteration of rational functions in the complex plane. Curry [4] applied this theory to investigate a global convergence of Newton's method. Vrscay and Gilbert [19] used the Julia-Fatou theory to study a global convergence of Schröder and König iteration functions. They found that the König method of order three had never attractive or indifferent fixed points besides the roots of the solving polynomial. In this paper we'll show that their statement may be extended for Konig functions of any order. Apart from iteration methods considered in previous papers we shall examine convergence of iteration method due to Döring. We shall prove that this iteration function applied to solving polynomial of degree greater than two has always indifferent fixed points. In Curry's and Vrscay's papers a lot of observations are based on computer experiments. We have repeated some of this experiments and made some new ones concerning with Döring method and basins of attraction of iteration methods applied to solving cubic polynomials.

1. RATIONAL ITERATION FUNCTIONS AND JULIA-FATOU THEORY

Let $\bar{C} = C \cup \{\infty\}$ denote the Riemann sphere with suitably defined spherical metric. We now consider a rational function $R(z) = P(z)/Q(z)$, where $P: \bar{C} \longrightarrow \bar{C}$ and $Q: \bar{C} \longrightarrow \bar{C}$ are polynomials with complex coefficients and no common factors, and $d=\deg(R)\equiv\max\{\deg P, \deg Q\}\geq 2$. Since we are interested in the iteration of functions, for the convenience we introduce the notation:

$$R^{o}(z) = z, \qquad R^{n}(z) = R(R^{n-1}(z)). \tag{1}$$

Given a point $z_o \in \bar{C}$, the sequence $\{z_n\}_o^\infty$ defined by

$$z_{n+1} = R^{n}(z_o) \tag{2}$$

is called the forward orbit of z_o and is denoted $0^+(z_o)$.

If $R^n(z_o)=z_o$ for some n, then z_o is a periodic point and $0^+(z_o)$ is a periodic orbit (often called a cycle). If n is the first natural number such that $R^n(z_o)=z_o$, then n is called the period of the orbit (or cycle). If $n=1$, z_o is simply called a fixed point of R. The cycle of period n is attractive, indiferent, or repulsive, depending on whether the multiplier $|R^n(z_i)'|$ is less than, equal to or greater than one, respectively. Attractive cycle $\{z_1,\ldots,z_{n-1}\}$ such that the multiplier $|R^n(z_i)'|$ equels to 0 is also called superattractive cycle.

Let p be an attractive fixed point of $R(z)$. The basin of attraction of p is defined as the set of all initial values $z_o \in \bar{C}$ for which the sequence $\{z_n\}$ converges to p:

$$W(p) = \{z_o \in \bar{C}:\ R^n(z_o) \longrightarrow \infty,\ \ n \longrightarrow \infty\}. \tag{3}$$

Let $\gamma = \{p_1, p_2, \dots, p_{n-1}\}$ be an attractive cycle. The basin of attraction of this cycle is given by:

$$W(\gamma) = \bigcup_{i=1}^{n-1} W(p_i), \quad \text{where} \quad W(p_i)=\{z_o\in\bar{C}:\ R^{nk}(z_o)\longrightarrow p_i,\ k\longrightarrow\infty\}. \tag{4}$$

Let U be an open subset of $\bar{C}$, let $\mathcal{F} = \{f_i \mid i\in I\}$ be a family of rational functions defined on U with values in $\bar{C}$. We say that $\mathcal{F}$ is the normal family if every sequence $\{f_n\}$ contains a subsequence $\{f_{nj}\}$ which converges uniformly on compact subsets of U. The set of points in $\bar{C}$ for which the family of maps is not normal is called the Julia set. A more working description is that $J(R)$ is the closure of all repulsive cycles of R.

Some important properties of Julia sets are listed below.

(1) $J\neq\emptyset$

(2) The repulsive cycles are dense in J.

(3) if $z^*\in J$, then $J=cl\{z\in\bar{C} : \exists_{n\geq 1}\ R^n(z^*)=z^*\}$.

(4) J is invariant with respect to R, i.e., $R(J) = J = R^{-1}(J)$.

(5) The Julia sets with respect to R and with respect to R^k, $k\in N$, are the same.

(6) If J has interior points, then $J=\bar{C}$.

(7) If D is any domain such that $D \cap J = J^* \neq \emptyset$, then there exists an integer n such that $J=R^n(J^*)$.

(8) If $z\in J$ and U is any neighborhood of z, then $\{R^n(U)\}$ covers $\bar{C}$ except for at most two points.

(9) If p is an attractive fixed point of R, then $\partial W(p)=J$.

To illustrate the definitions we have just introduced we make few observations about the Julia sets of the family of quadratic polynomials $P_c(z)=z^2+c$. The polynomial $R_o(z)=z^2$ has two superattractive fixed points $p_1=0$ and $p_2=\infty$. All other periodic points are repulsive and belong to the unit circle $l=\{z: |z|=1\}$ is the boundary of the two basins of attraction $W(0)=\{z: |z|<1\}$ and $W(\infty)=\{z: |z|>1\}$. The Julia set of the polynomial $R_{-2}(z)=z^2-2$ is the interval $[-2;2]$. The basin of attraction of the unique attractive fixed point is the complement of the Julia set $W(\infty)=\bar{C}\setminus[-2;2]$. For all other values of c the Julia sets $J(P_c(z))$ are nonstandard sets, namely fractals [2]. Examples of such sets are in fig. 1. The topology and the shape of Julia sets of $P_c(z)$ greatly depend on the value of c. This set can be a loop-free curves, a curves with multiple points, a tree that does not surround a domain or a totally disconnected dust. The set in center of fig. 1 consists of those

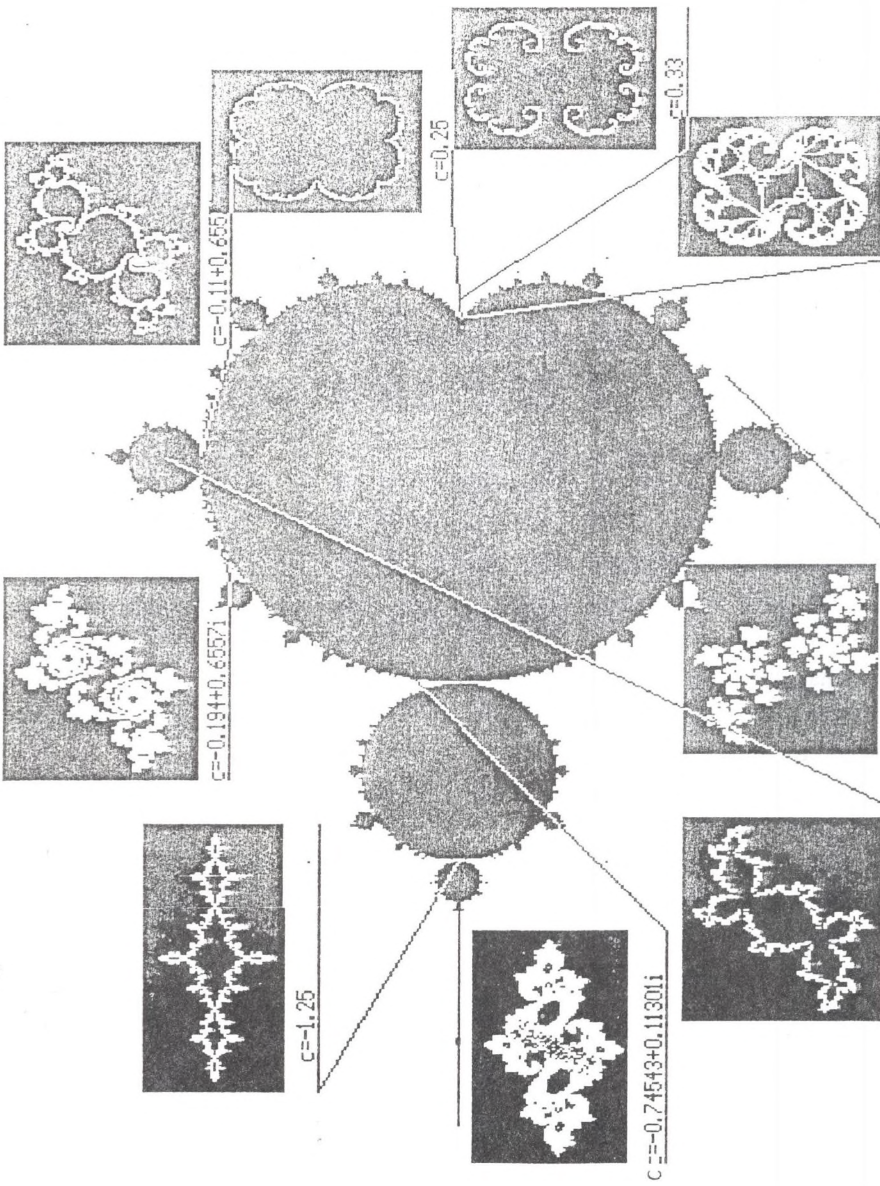


Fig. 1. Mandlbrot set and Julia sets of the polynomial $P_c(z)$

values of c for which $J(P_c)$ is connected. This set is called Mandelbrot set and will be denoted by M. The central part of this M-set which has the shape of cardioid consists of those values of c for which $P_c(z)$ has an attractive or indifferent fixed point besides ∞. The circle straight to the left from the cardioid contains all c's for which $P_c(z)$ has attractive or indifferent cycle of period 2, and the nearest circle that follow to the left corresponds to attractive or indifferent cycles of periods 4, 8 etc. Other circles that touch the cardioid correspond to cycles of order $n>2$. Fig. 1 shows dependence between position of value of c in M-set and shape of Julia set $J(P_c)$.

2. ITERATION FUNCTIONS OF ORDER TWO, THREE AND FOUR

Let $f:C\to C$ be analytic on a compact subset T of the complex plane C, having fixed point $p\in T$, i.e. $f(p)=p$. Let e_n denote the error associated with *n*th iterate

$$e_n = z_n - p \tag{5}$$

If there exist a number m and $C(C\neq 0)$ such that $\lim\limits_{n\to\infty} \dfrac{|e_{n+1}|}{|e_n|^m} = C$, then f is said to be an iteration function of order m. By Taylor's formula we have

$$e_{n+1} = F(e_n + p) =$$

$$= \frac{1}{m!} f^m (p)e_n^{(m)} + 0\,[(e_n)^{(m+1)}], \tag{6}$$

where m is the smallest integer for which $f^m(p)\neq 0$. So the condition

$\lim\limits_{n\to\infty} \dfrac{|e_{n+1}|}{|e_n|^m} = C$ $(C\neq 0)$ implies

$$f'(p) = f''(p) = \dots f^{(m-1)}(p) = 0.$$

If $g(z_o) = 0$ and $g'(z_o)\neq 0$, then the Newton iteration function

$$N(z) = z - \frac{g(z)}{g'(z)}, \tag{7}$$

constructed to solve the equation $g(z)=0$, is an iteration function of order two

The Schröder iteration function

$$S_m(z) = z + \sum_{n=1}^{m-1} c_n[-g(z)]^n, \tag{8}$$

where the coefficients $c_n(z)$ are given by

$$c_n(z) = \frac{1}{n!}\left[\frac{1}{g'(z)}\frac{d}{dz}\right]^{n-1}\frac{1}{g'(z)}, \tag{9}$$

is a generalization of Newton's one.

If z_o is a simple root of $g(z)$, then

$$S'(z_o) = S''(z_o) = \dots = S^{m-1}(z_o) = 0.$$

The Schröder iteration functions of order two $S_2(z_o)$ correspond to Newton's method. The higher order Schröder functions are presented below:

$$S_3(z) = S_2(z) - \frac{g''(z)}{2[g'(z)]^3}[g(z)]^2, \tag{10}$$

$$S_4(z) = S_3(z) - \frac{1/2[g''(z)]^2 - 1/6g'(z)g'''(z)}{[g'(z)]^3}. \tag{11}$$

The König iteration functions of order m [19] corresponding to g are given by

$$K_m(z) = z + (m-1)\frac{[1/g(z)]^{(m-2)}}{[1/g(z)]^{(m-1)}}. \tag{12}$$

The case $m=2$ again corresponds to Newton's method. The König functions of order 3 or 4 are presented below:

$$K_3(z) = z + \frac{2g(z)g'(z)}{g(z)g''(z)-2(g'(z))^2}, \tag{13}$$

$$K_4(z) = z + \frac{3g(z)\,[g(z)g''(z)-2(g'(z))^2}{6(g'(z))^3 - 6g(z)g'\,(z)g''(z) + (g(z))^2g'''(z)}. \tag{14}$$

The function $K_3(z)$ is also Döring iteration function of order 3. Döring iteration function of order 4 is given by

$$D_4(z) = z - \frac{f(f''f-2(f')^2)^2}{4(f')^5+2f^2f'(f'')^2 - 6f(f')^3f''+2/3f^2(f')^2f'''}. \quad (15)$$

3. FIXED POINTS OF ITERATION FUNCTIONS APPLIED TO SOLVING NONLINEAR EQUATIONS

For all methods presented in the previous section roots of g are super-atractive fixed points of iteration functions. The point ∞ is the repulsive fixed point for all those functions. Newton's method has no other fixed points, because in this case the fixed point condition $N(p)=p$ implies that $g(p)=0$. For higher order Schröder iteration functions $S_m(z)$ $(m\geq 3)$ the condition $S_m(p)=p$ implies that either (1) $g(p)=0$ or (2) $T_m(p)=0$, where

$$T_m(z) = \sum_{n=1}^{m-1} c_n(z)[-g(z)]^{(n-1)}. \quad (16)$$

Fixed points of iteration functions, which are not roots of g, will be called extraneous fixed points. Such points have also Döring and König iteration functions of higher order (order greater than 2).

If an extraneous fixed point is attractive or indifferent, then it may trap an iteration sequence, giving erroneous result for a root of g. It often takes place in the case of Döring iteration function, because this function usually has indifferent fixed points.

THEOREM 1. Let point z_o be an extraneous fixed point of D. Suppose that $f'(z_o)\neq 0$. Then the point z_o is an indifferent fixed point.

Proof. We introduce the following notations:

$$m(z) = 4(f'(z))^5 + 2f^2(z)(f''(z))^2 - 6f(z)(f'(z))^3f''(z) + \frac{2}{3} f^2(z)(f'(z))^2f'''(z), \quad (17)$$

$$g(z) = f(z)f''(z) - 2(f'(z))^2. \quad (18)$$

Then we can write (15) in the form

$$D(z) = z - \frac{f(z)\ (g(z))^2}{m(z)}. \tag{19}$$

Derivative of D is given by

$$D'(z)=1 - \frac{g(z)\ [f'(z)g(z) - 2f(z)g'(z)]m(z) - f(z)g^2(z)m'(z)}{m^2(z)}. \tag{20}$$

Expression (17) may be written as

$$m(z) = 2g^2(z)f'(z) + \frac{2}{3}\ f^2(z)(f'(z))^2f'''(z). \tag{21}$$

Since z_o is an extraneous fixed point, it follows that $g(z_o)=0$. By assumptions that $f'(z_o)\neq 0$ and $f'''(z_o)\neq 0$, we have $m(z)\neq 0$. Then $D'(z_o)=1$ follows from (20).

COROLLARY. Let f be a polynomial of degree $n\geq 3$. Then the function D constructed from the polynomial f has at least 3 indifferent fixed points.

Proof. If the assumptions of Theorem 2 are satisfied for all roots of the polynomial g then the function D has $2n-2$ extraneous indifferent fixed points. Now suppose that any root of g is also a root of f' then it would have to be a root of $f''(z)$, too. So there would be at most $2n-5$ points z_i such that the assumptions of Theorem 2 would not be fulfilled. This result is in a sharp contrast with König iteration functions.

THEOREM 2. For $m\geq 3$ all fixed points of $K_m(z)$ which are not roots of $g(z)$ are repulsive.

Proof. The extraneous fixed point condition implies

$$[1/g(z)]^{(m-2)} = 0. \tag{22}$$

The derivative of $K_m(z)$ is given by

$$K_m'(z) = 1 + (m-1)\ \frac{[(1/g(z))^{(m-1)}]^2 - (1/g(z))^{(m-2)}(1/g(z))^m}{[(1/g(z))^{(m-1)}]^2} \tag{23}$$

If $[1/g(z_o)]^{(m-1)} \neq 0$ then $K_m'(z_o) = m$.

In order to show that z_o is a repulsive fixed point when $[1/g(z_o)]^{(m-1)}= 0$ we introduce polynomials $W_m(z)$ defined by

$$[1/g(z)]^{(m)} = \frac{W_m(z)}{[g(z)]^{(m+1)}} \qquad m=1,2,3,... \tag{24}$$

Differentiation of equation (24) gives the following recurrence relation for the $W_m(z)$:

$$W_{(m+1)} = W'_m(z)g(z) - (m+1)W_m(z)g'_m(z) \tag{25}$$

The Konig functions constructed from the polynomial $g(z)$ may now be written as:

$$K_m(z) = z + (m+1)g(z) \frac{W_m(z)}{W'_m(z)g(z) - (m+1)W_m(z)g'_m(z)} \tag{26}$$

Since $W_m(z_o) = 0$ and $W_{m+1}(z_o) = 0$, it follows that $W'_m(z_o) = 0$. Then there exists an integer number $n>1$ and a polynomial $V_m(z)$ such that

$$W_m(z) = (z-z_o)^n V_m(z) \tag{27}$$

Substituting the derivative $W_m(z)$ given by

$$W'_m(z) = n(z-z_o)^{n-1}[V_m(z)+(z-z_o)V'_m(z)] \tag{28}$$

into (12) we obtain

$$K_m(z) = z + \tag{29}$$

$$+ (m-1)g(z) \frac{(z-z_o)^n V_m(z)}{(z-z_o)^{n-1}\{[nV_m(z)+(z-z_o)V'_m(z)]g(z)-(m-1)(z-z_o)V_m(z)g'_m(z)}$$

An easy computation shows that z_o is a fixed point of König function $K_m(z)$. To show that this point is repulsive differentiate equation (26) to give

$$K'_m(z) = 1 + (m-1)g'(z)\,\frac{W(z)}{W'(z)g(z)-(m-1)W(z)g'(z)} +$$

$$+ (m-1)g(z)\,\frac{W'(z)[W'(z)g(z)-(m-1)W(z)g'(z)]}{[W'(z)g(z)-(m-1)W(z)g'(z)]^2} - \tag{30}$$

$$-(m-1)g(z)\frac{[W''(z)g(z)+W'(z)g'(z)-(m-1)W'(z)g'(z)-(m-1)W(z)g''(z)]W(z)}{[W'(z)g(z)-(m-1)W(z)g'(z)]^2}$$

Using (30), (27) and (28) it is a routine matter to shaw that

$$K'_m(z_o) = 1 + \frac{m-1}{n} \tag{31}$$

Since $\left|1 + \frac{m-1}{n}\right|>1$ for all $m,n>1$ we conclude that $|K'_m(z_o)|>1$.

4. BASINS OF ATTRACTION OF ROOTS OF THE POLYNOMIAL $g(z)=z^n-1$

The Newton's method applied to polynomial $g(z)=z^2-1$ was analyzed by Cayley [3], who found that in this case

$$A(+1) = \mathcal{L} = \{z\colon \mathrm{Re}(z)>0\}$$

$$A(-1) = \mathcal{L} = \{z\colon \mathrm{Re}(z)<0\}$$

and that

$\partial A(+1) = \partial A(-1) = \varphi$, where φ denote the imaginary axis

Let $\phi(z)$ be a Mobius transformation

$$\phi(z) = \frac{z+1}{z-1} \quad \text{with} \quad \phi^{-1}(w) = \frac{w-1}{w+1} \tag{32}$$

then $\phi N\phi^{-1}(w)=w^2$, $\phi(\mathcal{L}) = \{w\colon |w|<1\}$, $\phi(\mathcal{R}) = \{w\colon |w|>1\}$ and $\phi(\varphi)= \mathcal{C}$, the unit circle. In the same way, using Möbius transformation, Vrscay and Gilbert [19] found that the imaginary axis is the Julia set for all König functions $K_m(z)$ associated with the polynomial $g(z)=z^2-1$.

Basins of attraction of roots of $g(z)=z^2-1$ associated with Schröder and Döring methods were found experimentally. Also in the case of Döring method imaginary axis appeared to be the Julia set of iteration function. The Schröder functions have fixed points that don't lie on the imaginary axis. For example $S_3(z)$ has two repulsive fixed point at $z = \pm \frac{1}{\sqrt{5}}$ and $S_4(z)$ has four repulsive fixed points at $z = \pm \frac{(2\pm\sqrt{7})^{1/2}}{\sqrt{11}}$. Since repulsive fixed points must lie on the Julia set $J(S_m)$, the imaginary axis mustn't be Julia set of $S_m(z)$. The basin of attraction A(1) and A(-1) associated with Schroder functions are seen on fig. 1 and 2.

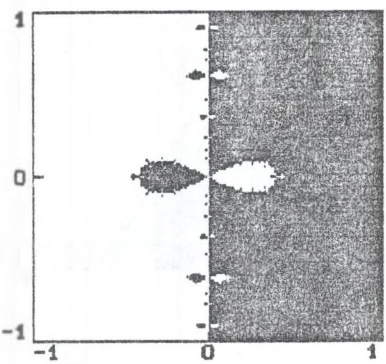


Fig. 2a. Basins of attraction of roots the polynomial $g(z)=z^2-1$ for iteration function $S_3(z)$

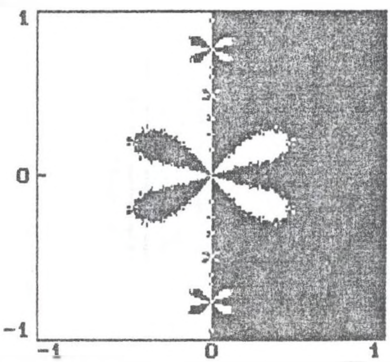


Fig. 2b. Basins of attraction of roots of polynomial $g(z)= z^2- 1$ for the iteration function $S_4(z)$

Let us now consider the polynomial $P_3(z) = z^3-1$. In this case there are at least 3 basins of attraction $A(z_i)$, where $z_o=1$, $z_{1,2} = -1/2\pm\sqrt{3}/2$
In accordance with property (9) of Julia sets we have

$$J = \partial A(z_o) = \partial A(z_1) = \partial A(z_2)$$

From this property, it follows that Julia set for all iteration functions associated with polynomial of degree greater than two must have very complicated and strange shapes. Fig. 2,3 allow to compare basins of attraction for Newton, Schröder, König nad Döring functions applied to $g(z)= = z^3-1$ and $g(z) = z^4-1$, accordingly.

If the Doring method is applied to polynomial $f(z) = z^n-1$ then equation:

$$ff''-2(f')^2=0$$

has $n+1$ solutions: $z_k = \sqrt[n]{(n-1)/(n+1)}\ \exp(2k\Pi i/n)$, where $k = 0,1,\dots,n-1$ and $z_n=0$. Point z_o is always repulsive fixed point of $D(z)$. If $n>2$, then $f'(z_k)\neq 0$ and $f'''(z_k)\neq 0$ for $k=0,\dots,n-1$. So the assumptions of the theorem 1 are hold for all z_k $(k=0,\dots,n-1)$ and the function $D(z)$ associated with $g(z) = z^n-1$ has exactly n extraneous indifferent fixed points. On fig. 2 and 3 basins of attraction of all those points are white.

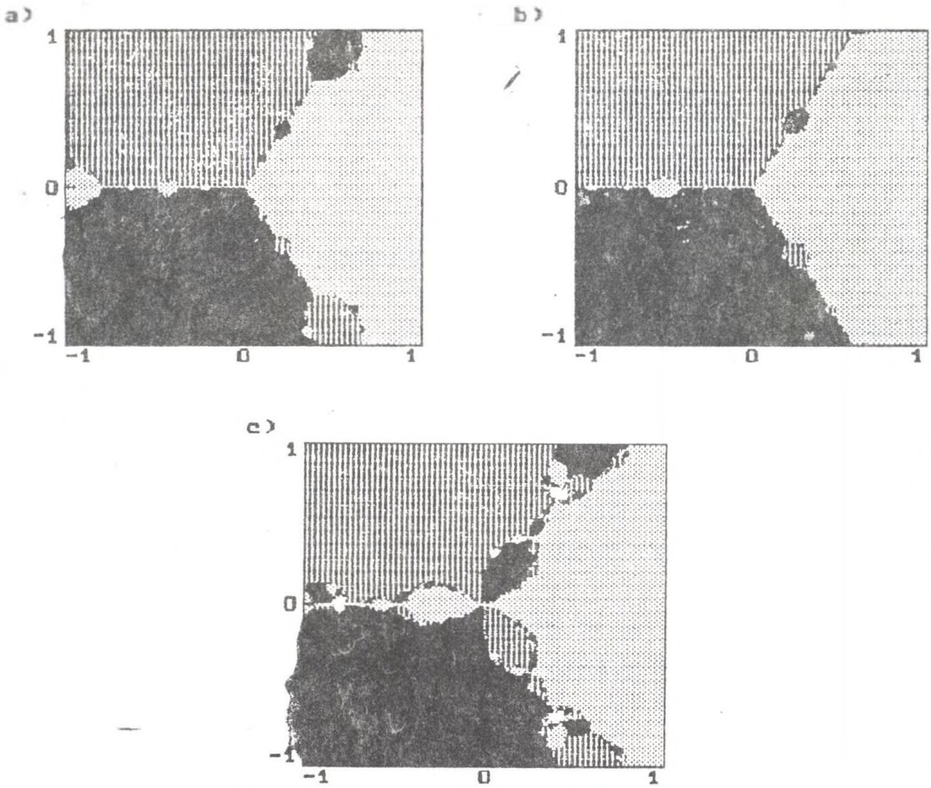


Fig. 3. Basins of attraction of roots of plynomial $g(z) = z^3 -1$ for the K_3 method (a), for the K_4 method (b) and for the Döring method (c)

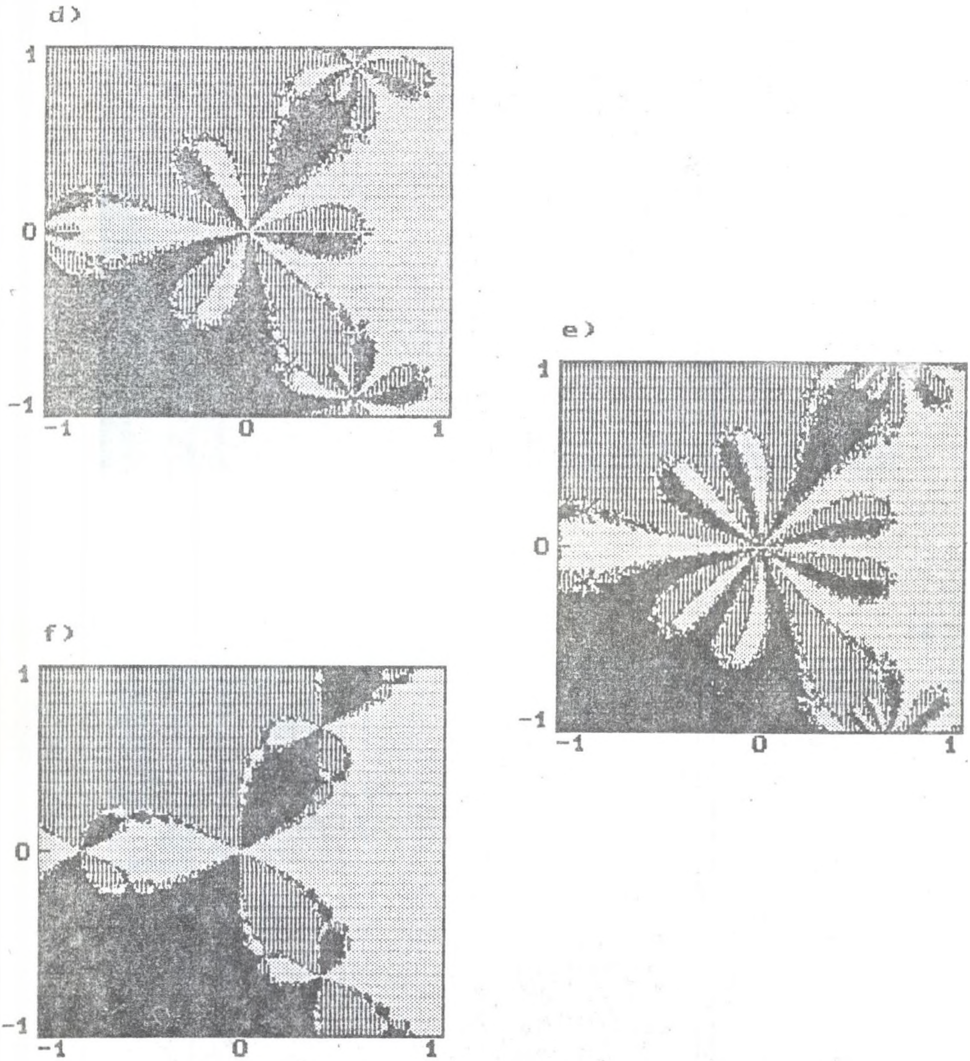


Fig. 3. Basins of attraction of roots of plynomial $g(z) = z^3 - 1$ for the S_3 method (d), for the S_4 method (e) and for Newton's method (f)

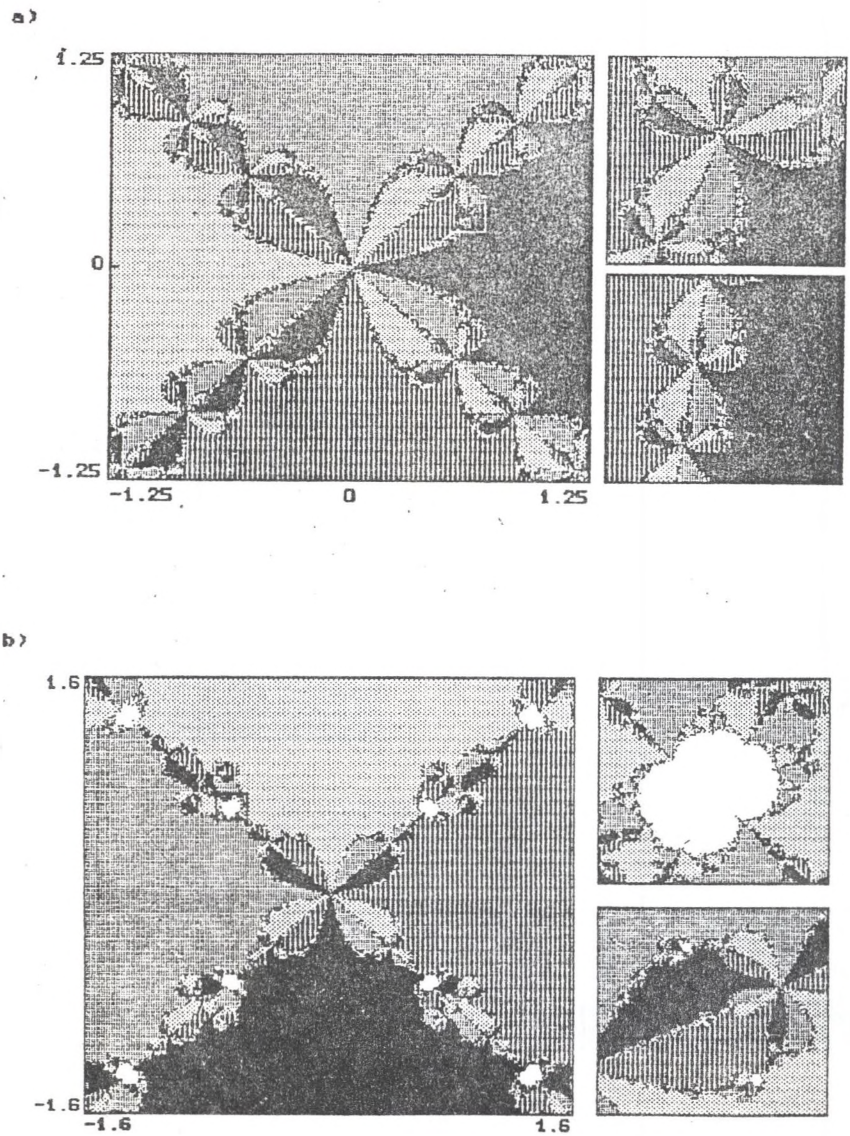
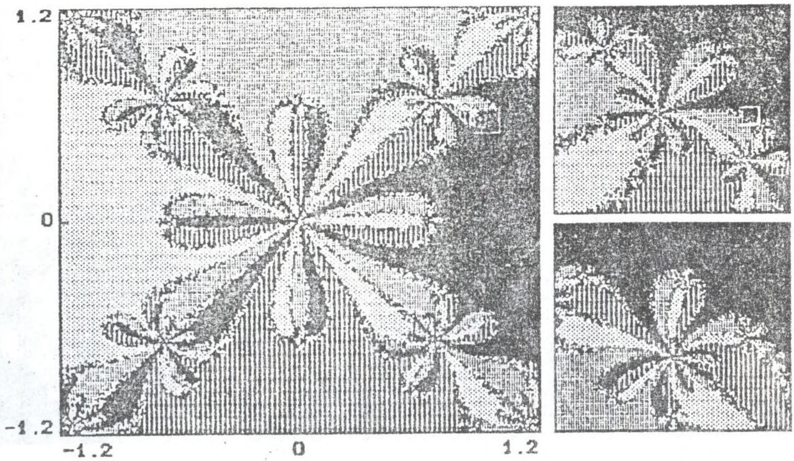


Fig. 4. Basins of attraction of roots of polynomial $g(z)=z^4-1$ for Newton method (a) and Döring method (b)

c).



d)

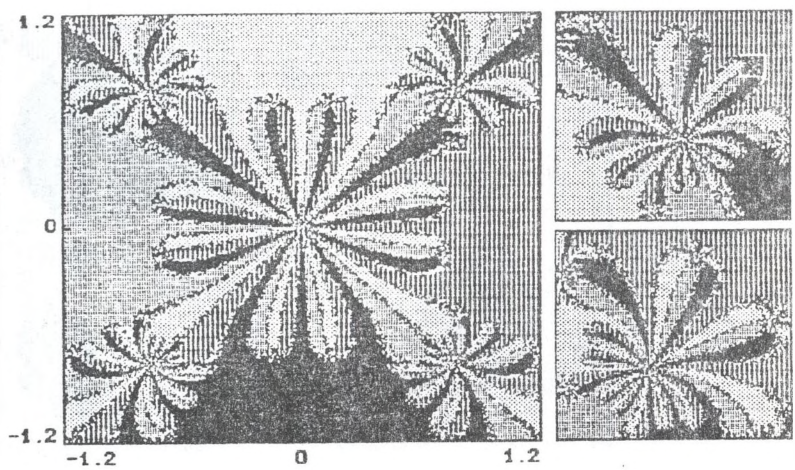
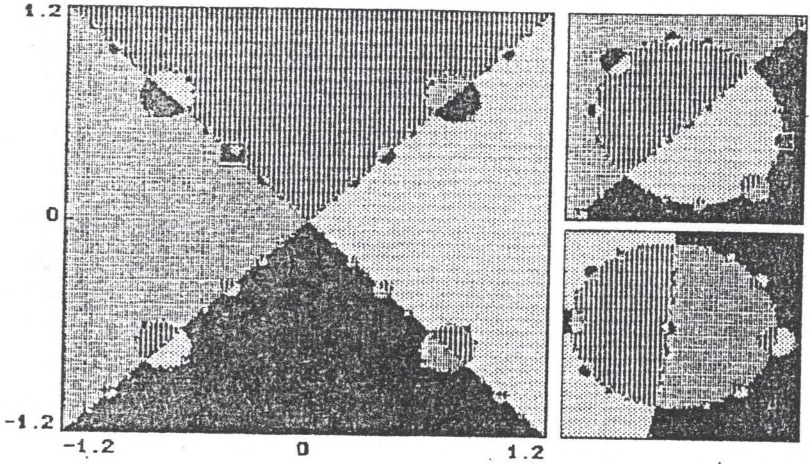


Fig. 4. Basins of attraction of roots of polynomial $g(z)=z^4-1$ for the S_3 method (c) and the S_4 method (d)

e)



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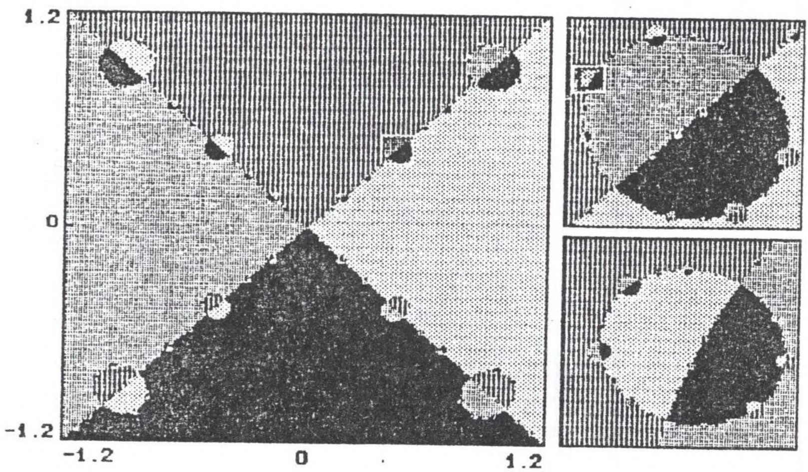


Fig. 4. Basins of attraction of roots of polynomial $g(z)=z^4-1$ for the K_3 method (e) and the K_4 method (f)

5. ATTRACTIVE EXTRANEOUS FIXED POINTS AND CYCLES OF CUBIC POLYNOMIALS

Here we'll study a one-parameter family of cubic polynomials

$$g_A(z) = z^3 + (A-1)z - A$$

In our consideration critical points of iteration functions will play an important role:

Critical points $z=c_i$ of iteration function $R(z)$ are those points for which the equation $R(z)=v$ admits multiple roots.

There are simply those points where the derivative $R'(c_i)$ vanishes.

Using the following theorem due to Fatou:

If $R(z)$, a rational function, has an attractie periodic cycle, then the orbit of at least one critical point will converge to it.

We'll determine the set of the parameters A ($A \in \mathbb{C}$), such that iteration function applied to solving the $g_A(z)$ has attractive or indifferent extraneous point or cycle.

Newton's iteration function $Ng_A(z)$ has four cirtical points. Three of them are roots of $g_A(z)$, so the fourth cirtical point located at $z=0$ is the only one available to converge to an atractive periodic cycle, if such a cycle exists. Black areas on fig. 5a represent A-values for which attractive cycles are observed. The black set in fig.5b, which is a magnification of the region $[0.22;0.39]\times[1.58;1.69]$ of fig. 5a looks remarkably like Mandelbrot set. As in the case of quadratic maps, these sets represents zones of stable cycles which undergo period-doubling bifurcations. For example, the major cordioid in fig. 5b represents A-values for which there exists attractive two cycles (No extraneous fixed points can occur for Newton's method). The adjacent circular region corresponds to attractive 4-cycles, etc. Fig. 1 and fig. 6 allow to compare basins of attraction of attractive points or cycles the quadratic polynomials $P_c(z)$ and basins of attraction of attractive points or cycles of Newton's iteration function, when positions of the parameters A in the black set on fig. 6 correspond to positions of the parameters c in Mandelbrot set.

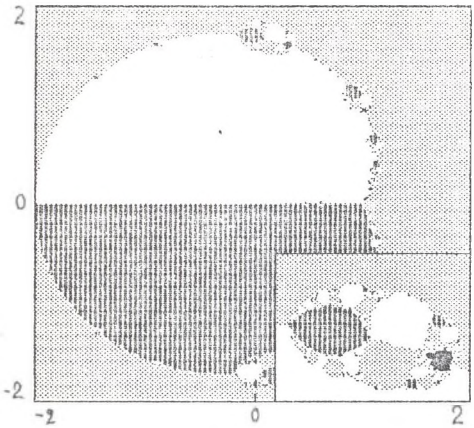


Fig. 5a. Convergence of the critical point z=0 of the iteration function $Ng_A(z)$

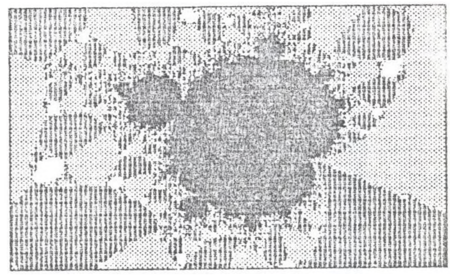


Fig. 5b. A magnification of the Mandelbrot-like set in fig 5a

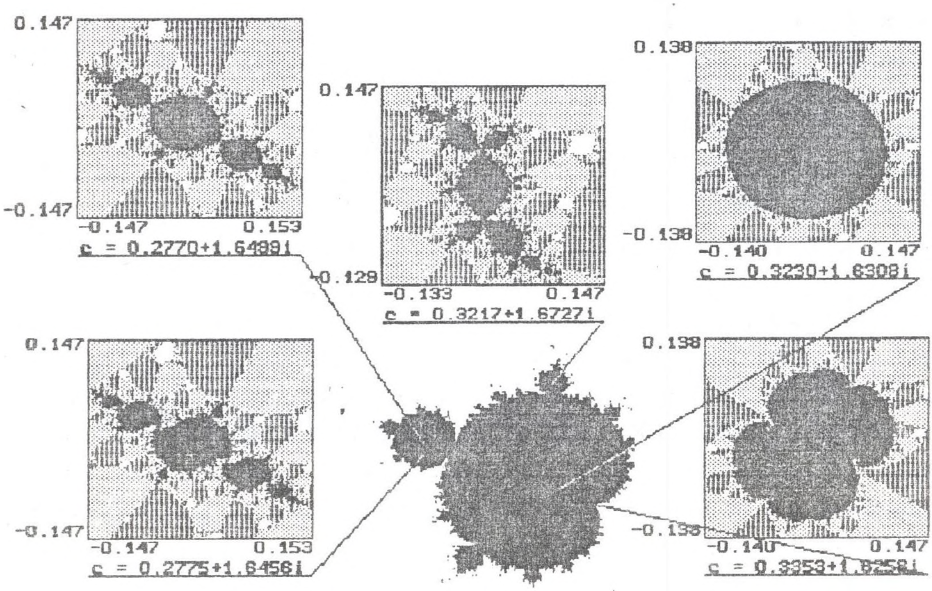
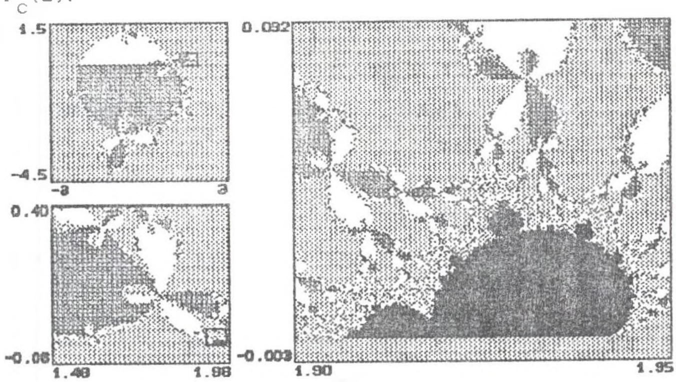


Fig. 6. The Mandelbrot-like set for Newton's method and basins of attraction of extraneous cycles

Schröder iteration function of order theree $S_3g_A(z)$ has two critical points besides roots of the polynomial $g_A(z)$. Black area on fig. 7 is a set of those A-values for which an orbit of a critical point of the $S_3g_A(z)$ $z = \sqrt{(A-1)/15}$ converges to an extraneous fixed point or cycle. Also in the case of this function the basins of attraction of the extraneous fixed points or the cycles are alike to corresponding basins of attraction of the polynomials $P_c(z)$.



Rys. 7. Convergence of the cirtical point $z = \sqrt{(A-1)/15}$ of the iteration function $S_3(z)$

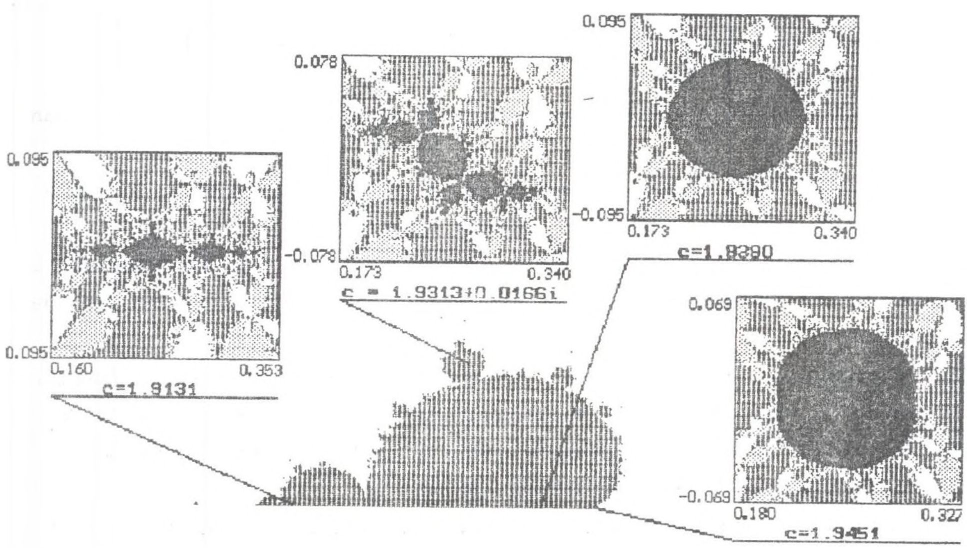


Fig. 8. The Mandelbrot-like set for the S_3 method and basins of attraction of extraneous fixed points or cycles

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НЕПОДВИЖНЫЕ ТОЧКИ, ОБЛАСТИ СХОДИМОСТИ, МНОЖЕСТВА ЖЮЛИЯ И ГЛОБАЛЬНАЯ СХОДИМОСТЬ ИТЕРАЦИОННЫХ МЕТОДОВ РЕШЕНИЯ НЕЛИНЕЙНЫХ УРАВНЕНИЙ

Резюме. В работе представлен анализ глобальной сходимости методов решения многочленов. Мы проследили сходимость методов Ньютона, Шредера, Кенига и Деринга. Если применить один из этих методов для решения многочлена получается измеримая функция. Проблемам итераций измеримых функций посвещена классическая теория Жюлия-Фату. Множества Жюлия исследованных методов и области притяжения мы получили с помощью компьютера. Одной из наиболее важных проблем связанных с глобальной сходимостью итерационных методов является существование устойчивых циклов и неподвижных точек. Мы доказали две теоремы о неподвижных точках: Все неподвижны точки функций Кенига, кроме корней многочлена, являются неустойчивыми; функция Деринга для решения многочленов порядка выше чем два бсегда обладает критическо-устойчивой неподвижной точку. Экспериментально обнаружено, что итерационные функции Шредера имеют одинаково устойчивые, критическо-устойчивые и неустойчивые неподвижные точки.

PUNKTY STAŁE, BASENY ZBIEŻNOŚCI ORAZ ZBIORY JULII A ZBIEŻNOŚĆ GLOBALNA METOD ITERACYJNYCH ROZWIĄZYWANIA RÓWNAŃ NIELINIOWYCH

S t r e s z c z e n i e

W pracy rozważane są metody Newtona, Königa, Schrödera i Döringa zastosowane do znajdowania zer wielomianów. Z metodami tymi związane jest zagadnienie iteracji funkcji wymiernych w przestrzeni Riemanna, któremu poświęcona jest teoria Julii i Fatou. W pracy jest badane istnienie przyciągających i obojętnych dodatkowych punktów stałych lub cykli. Zostało pokazane, że punkty takie zawsze posiada funkcja iteracyjna Doringa zastosowana do rozwiązywania wielomianów stopnia wyższego od dwóch. Istnienie takich punktów stwierdzono również dla funkcji iteracyjnych Schrodera. Z kolei dodatkowe punkty stałe metod iteracyjnych Königa są zawsze odpychające. Rozważania teoretyczne są uzupełnione rysunkami zbiorów Julii i basenów zbieżności uzyskanymi za pomocą komputera. Istnienie cykli lub dodatkowych punktów stałych przyciągających bądź obojętnych dla funkcji iteracyjnych zastosowanych do wyznaczania pierwiastków wielomianów stopnia trzeciego jest również badane eksperymentalnie poprzez śledzenie orbit punktów krytycznych funkcji iteracyjnej, nie będących jednocześnie pierwiastkami rozwiązywanego wielomianu.