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ON THE IMAGES OF STABILITY AND NON-EFFECTIVITY GROUPS UNDER THE EQUIVARIANT MAPPINGS

Summary. In the paper we consider the images and coimages of the stability and non-effectivity groups of the subsets of fibres of the abstract objects under the equivariant mappings. Some facts concerning these problems were proved by E. Zaporowski in the paper [5].

We state (theorem 3) that for every nonempty subset A of the fibre X of the object (X, G, F) the image $\varphi(G_A)$ of the stability group G_A of the set A is a subgroup of the stability group $H_{\alpha(A)}$ of the set $\alpha(A)$.

The non-effectivity groups are found to posses the same property (theorem 4).

In the theorem 5 we prove that for every nonempty subset B of the fibre of the codomain of the equivariant mapping (α, φ) , $\varphi(G) \cap H_B$ is a subgroup of the group $\varphi(G_{\alpha^{-1}(B)})$.

And, moreover, if α is a surjection then the groups $\varphi(G) \cap H_B$ and $\varphi(G_{\alpha^{-1}(B)})$ are identically equal.

The respective property is proved for the non-effectivity groups (theorem 6).

We also present some examples showing the necessity of the conditions given in the theorems 3, 4 and 5.

INTRODUCTION

In the paper [5] E. Zaporowski proved that the equivariant mappings maps the stability group of the element $x \in X$ into the stability group of the image of the element x . We give more properties of the stability and non-effectivity groups of the subsets of the fibre of an abstract object.

1. Preliminaries

An abstract object (object) is a triple (X, G, F) where X is a nonempty set, G is a group and $F: X \times G \rightarrow X$ is a mapping satisfying the translation and identity conditions (cf. [2], p.12).

An equivariant mapping from the object (X, G, F) into the object (Y, H, f) is a pair (α, φ) , where $\alpha: X \rightarrow Y$ is a mapping and $\varphi: G \rightarrow H$ is an homomorphism which satisfies the condition:

$$\alpha(F(x, g)) = f(\alpha(x), \varphi(g)), \quad x \in X, \quad g \in G \quad (1)$$

A stability group of nonempty set $A \subset X$ we denote by

$$G_A := \{g \in G; F(A, g) = A\}.$$

A non-effectivity group of nonempty set $A \subset X$ we denote by

$$\tilde{G}_A := \{g \in G; F(x, g) = x, \quad \text{for } x \in A\}.$$

In the case $A = \emptyset$ we assume that $G_\emptyset = \tilde{G}_\emptyset = G$ (cf. [3], p.23, 25).

2. Let us consider an equivariant mapping

$$(\alpha, \varphi) : (X, G, F) \rightarrow (Y, H, f) \quad (2)$$

In [5] E. Zaporowski proved

Theorem 1. For every equivariant mapping (2) and every $x \in X$ the group $\varphi(G_x)$ is a subgroup of the stability group $H_{\alpha(x)}$. If α is an injection and φ - isomorphism then the groups G_x and $H_{\alpha(x)}$ are isomorphic. ■

Let us consider a set

$$G'_A := \{g \in G; F(x, g) \in A, \quad \text{for } x \in A\}. \quad (3)$$

The set G'_A is not a group in general, but it contains (as a set the stability group G_A .

Lemma. A subgroup H of the group G is a subgroup of the stability group G_A of a nonempty subset A of the fibre X of the object (X, G, F) iff $F(x, g) \in A$, for $x \in A$, $g \in H$.

$$(X, G, F) \text{ iff } F(x, g) \in A, \text{ for } x \in A, g \in H. \quad (4)$$

Proof. Necessity of the condition (4) is obvious. Let us assume that the group H satisfies the condition (4). A mapping $F_g : A \longrightarrow A$ defined by $F_g(x) := F(x, g)$, for $x \in A$, is a bijection for $g \in H$.

Indeed, the translation condition and $g^{-1} \in H$ implies $F_g F_{g^{-1}} = F_e = id_A$ and $F_{g^{-1}} F_g = F_e = id_A$. ■

From that lemma it follows that:

Theorem 2. For every nonempty subset A of the fibre X of the object (X, G, F) the set $G'_A := \{g \in G; g \in G_A^* \text{ and } g^{-1} \in G_A'\}$ with the action induced by the action of G is a group identically equal to the stability group G_A . ■

Theorem 3. If (2) is an equivariant mapping then for every nonempty $A \subset X$, $\varphi(G_A)$ is a subgroup of $H_{\alpha(A)}$. If α is an injection then $\varphi(G_A) = H_{\alpha(A)} \cap (G)$. ■

Proof. $\varphi(G_A)$ is a group of course. From the lemma it follows that it is enough to prove that $f(y, h) \in \alpha(A)$ for $h \in \varphi(G_A)$ and $y \in \alpha(A)$. For every $y \in \alpha(A)$. For every $y \in \alpha(A)$ and $h \in \varphi(G_A)$ there exists $x \in A$ and $g \in G$ such that $\alpha(x) = y$ and $\varphi(g) = h$. From (1) it follows that $f(y, h) = f(\alpha(x), \varphi(g)) = \alpha(F(x, g)) \in \alpha(A)$.

To prove the second part of the theorem it is enough to show that $H_{\alpha(A)} \cap \varphi(G) \subset \varphi(G_A)$. For every $h \in H_{\alpha(A)} \cap \varphi(G)$ there exists $g \in G$ such that $\varphi(g) = h$. For every $x \in A$ from (1) we obtain

$$\alpha(F(x, g)) = f(\alpha(x), \varphi(g)) = f(\alpha(x), h) \in \alpha(A)$$

and

$$\alpha(F(x, g^{-1})) = f(\alpha(x), \varphi(g^{-1})) = f(\alpha(x), h^{-1}) \in \alpha(A).$$

Therefore we obtain that $F(x, g) \in A$ and $F(x, g^{-1}) \in A$, because α is an injection. So $g, g^{-1} \in G_A'$. From theorem 2 it follows that $g \in G_A$. ■
For the non-effectivity groups we can analogically prove

Theorem 4. If (2) is an equivariant mapping then for every nonempty $A \subset X$, $\varphi(\tilde{G}_A)$ is a subgroup of $\tilde{H}_{\alpha(A)}$. If α is an injection then $\varphi(G_A) = \tilde{H}_{\alpha(A)} \cap \varphi(G)$. ■

Corollary 1. If a pair (2) is such that α is an injection, and φ is an isomorphism, then for every nonempty $A \subset X$ the groups G_A and $H_{\alpha(A)}$ are isomorphic and the groups \tilde{G}_A and $\tilde{H}_{\alpha(A)}$ are isomorphic. ■

3. For the stability and non-effectivity groups of the subsets of codomains of an equivariant mapping we have

Theorem 5. If (2) an equivariant mapping then for every nonempty $B \subset Y$, $\varphi(G) \cap H_B$ is a subgroup of the group $\varphi(G_{\alpha^{-1}(B)})$. If α is a surjection then

the groups $\varphi(G) \cap H_B$ and $\varphi(G_{\alpha^{-1}(B)})$ are identically equal.

Proof. In the case $\alpha^{-1}(B) = \emptyset$ theorem is true. Let us assume $\alpha^{-1}(B) \neq \emptyset$. For every $h \in \varphi(G) \cap H_B$ there exists $g \in G$ such that $\varphi(g) = h$. For every $x \in \alpha^{-1}(B)$ from (1) it follows that $\alpha(F(x, g)) = f(\alpha(x), h) \in B$ and $\alpha(F(x, g^{-1})) = f(\alpha(x), h^{-1}) \in B$. Therefore $g, g^{-1} \in G_{\alpha^{-1}(B)}$, and from theorem 2 we obtain $g \in G_{\alpha^{-1}(B)}$ so $h \in \varphi(G_{\alpha^{-1}(B)})$. To prove the second partf the

theorem it is enough to show that if α is a surjection then $\varphi(G_{\alpha^{-1}(B)}) \subset \varphi(G) \cap H_B$. For every $h \in \varphi(G_{\alpha^{-1}(B)})$ and $y \in B$ there exists $g \in G_{\alpha^{-1}(B)}$ and $x \in \alpha^{-1}(B)$ such that $\varphi(g) = h$, $\alpha(x) = y$, $F(x, g) \in \alpha^{-1}(B)$ and $F(x, g^{-1}) \in \alpha^{-1}(B)$. From (1) we obtain $f(y, h) = f(\alpha(x), \varphi(g)) = \alpha(F(x, g)) \in B$ and $f(y, h^{-1}) = f(\alpha(x), \varphi(g^{-1})) = \alpha(F(x, g^{-1})) \in B$, so from theorem 2 we obtain that $h \in H_B$. ■

Theorem 6. If (2) is such that α is a surjection then for every nonempty $B \subset Y$ the group $\varphi(G_{\alpha^{-1}(B)})$ is a subgroup of the group \tilde{H}_B .

Proof. For every $h \in \varphi(G_{\alpha^{-1}(B)})$ and $y \in B$ there exists $g \in G_{\alpha^{-1}(B)}$ and $x \in \alpha^{-1}(B)$ such that $\varphi(g) = h$ and $\alpha(x) = y$. Because for that x and g we have $F(x, g) = x$, then from (1) we obtain $f(y, h) = f(\alpha(x), \varphi(g)) = \alpha(F(x, g)) = \alpha(x) = y$, so $h \in \tilde{H}_B$. ■

Let us remark that if α is not a surjection, then theorems 5 and 6 are not true. Indeed, let us consider the scalar $(\{0\}, R_*, F)$ and the object (R, R_*, f) where R denotes the set of real numbers, R_* - the multiplicative group of the real numbers and $f(x, g) := gx$, for $x \in R$ and $g \in R_*$. Considering the mapping $(\alpha, id_{R_*}) : (\{0\}, R_*, F) \rightarrow (R, R_*, f)$ where $\alpha(0) := 0$, with the respect to the notation of the theorems 5 and 6 we obtain $\varphi(R_*) \cap H_{\{0, 1\}} = \{1\} \neq R_* = \varphi(R_* \setminus \{0\}) = \varphi(R_*^{-1}(\{0, 1\}))$ and $\tilde{H}_R = \{1\} \neq R_* = \varphi(\tilde{R}_*^{-1}(R))$. Let us now consider the mapping $(\beta, id_{R_*}) : (R, R_*, f) \rightarrow (\{0\}, R_*, F)$ where $\beta(x) := 0$, for $x \in R$.

With the respect to the notation of the theorems 3, 4 and 6 we have:
 $\varphi(R_* \setminus \{1\}) = \varphi(\tilde{R}_* \setminus \{1\}) = \{1\} \neq R_* = \tilde{R}_* \beta(1) = R_* \beta(1)$ and $\varphi(R_*) \cap \tilde{R}_* \setminus \{0\} = R_* \neq \{1\} = \tilde{\varphi}(R_*^{-1} \setminus \{0\})$.

It proves that in the theorems 3 and 4 α should be an injection and that the theorem 6 does not state of the equality of the groups $\varphi(\tilde{G}_*^{-1})$ and $\varphi(G) \cap \tilde{H}_B$.

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ОБРАЗЫ ГРУПП УСТОЙЧИВОСТИ И НЕЭФФЕКТИВНОСТИ ПРИ ЭКВИВАРИАНТНЫХ ОТОБРАЖЕНИЯХ

Резюме. В работе рассматриваются образы и прообразы групп устойчивости и незэффективности подмножеств абстрактных объектов при эквивариантных отображениях.

Некоторые факты, касающиеся этих проблем, были доказаны Е. Запоровским.

Покажем (теорема 3), что для любого непустого подмножества A слоя объекта (X, G, F) образ $\phi(G_A)$ группы устойчивости G множества A есть подгруппа группы устойчивости H $\alpha(A)$ множества $\alpha(A)$.

Аналогичный факт будет доказан для групп независимости (теорема 4).

В теореме 5 докажем, что для любого непустого подмножества B слоя ко-области эквивариантного отображения (α, ϕ) группа $\phi(G) \cap H_B$ есть подгруппа группы $\phi(G \alpha^{-1}(B))$. Более того, если α есть сюръективное отображение, то группы $\phi(G) \cap H_B$ и $\phi(G \alpha^{-1}(B))$ совпадают.

Аналогичный факт для групп независимости будет доказан в теореме 6.

Будут также представлены примеры показывающие существенность условий, приведенных в теоремах 3, 4 и 5.

OBRASY GRUP STABILNOŚCI I NIEEFEKTYWNOŚCI

PRZY ODWZOROWANIACH EKWIARIANTNYCH

S t r e s z c z e n i e

W pracy rozważamy obrazy i przeciwoobrazy grup stabilności i nieefektywności podzbiorów włókien obiektów abstrakcyjnych przy odwzorowaniach ekwiariantnych.

Pewne fakty dotyczące tych problemów zostały udowodnione przez E. Zaporskiego w pracy [5].

Stwierdzimy (twierdzenie 3), że dla każdego niepustego podzbioru A włókna X obiektu (X, G, F) obraz $\phi(G_A)$ grupy stabilności G_A zbioru A jest podgrupa grupy stabilności $H_{\alpha(A)}$ zbioru $\alpha(A)$.

Analogiczny fakt udowodnimy dla grupy nieefektywności (twierdzenie 4).

W twierdzeniu 5 udowodnimy, że dla każdego niepustego podzbioru B włókna kodziedziny odwzorowania ekwiariantnego (α, φ) grupa $\varphi(G) \cap H_B$ jest podgrupą grupy $\varphi(G \alpha^{-1}(B))$. Co więcej, jeśli α jest surjekcją, to grupy

$\varphi(G) \cap H_B$ i $\varphi(G \alpha^{-1}(B))$ są identyczne.

Analogiczny fakt dla grup nieefektywności udowodnimy w twierdzeniu 6.

Podamy także kilka przykładów pokazujących istotność założeń w podanych w twierdzeniach 3, 4 i 5.