ZESZYTY NAUKOWE POLITECHNIKI ŚLĄSKIEJ 1993

Seria: MATEMATYKA-FIZYKA z. 68 Nr kol. 1147

I. ŁOJCZYK-KRÓLIKIEWICZ

DIFFIRENTIAL-FUNCTIONAL INEQUALITIES OF PARABOLIC AND ELLIPTIC TYPE IN BOUNDED DOMAIN

Summary. In the first part of the paper the inequalities of parabolic type are considered, with the linear boundary conditions. It is assumed the Lipschitz condition in its nonlinear form and the additional inequality (5). This inequality is a generalization of well known condition for linear equation, in which it is assumed that the coefficient at the unknown function is bounded from above. This assumption allows us to omit the condition of quasi-monotonicity of the function. The theorem concerning the inequalities results in the uniqueness theorem on the solution to the system (11) with the boundary condition given in Def. 5. In the second part, analogous theorems for the elliptic systems of the form (16) are considered.

All the theorems of this paper one can considered in an unbounded domain without introducing a stifling divisor. It suffices to assume that at all infinite points of the domain we have

$\limsup_{|x|\to\infty} (u^i-v^i)(x) \le 0$ for $i=1,\dots,m$. Obviously, using the methods of proofs given in the papers [5] and [7], we can prove all the above given theorems for the solutions irregular on the boundary with solutions nonlinear boundary conditions.

1. INTRODUCTION

The purpose of this note is to prove the Theorems 4 and 5, concerning differential-functional inequalities, given without proofs in the paper [3]. Moreover an analogous problem for the elliptic system is considered.

The boundary problems for the systems

$$u^i_t \le f^i(t,x,u,u^i_x,u^i_{xx},u(t,\cdot)) \tag{1}$$

$$v^i_t \ge f^i(t,x,v,v^i_x,v^i_{xx},v(t,\cdot)) \tag{2}$$

for $i \in \{1,2,\dots,m\} = I$, in an arbitrary set D, were investigated in papers [2], [4], [5]. In the first paper we have applied the method of

M. Krzyżański [1], based on the so-called quasi-extremum. This method requires assumption of a strong Lipschitz condition, and also a strong assumption on the parabolicity of f (see [2]).

In papers [4], [5] the stifling divisors were introduced, what in the case a of bounded set D is superfluous.

We want to stress that the Lipschitz condition (4) introduced here has a nonlinear form as in [5]. Besides, it is possible to investigate the irregular solutions in unbounded domains in the same way as in which [5] and [7] (but without assuming the existence of stifling divisors) under the same weak assumptions, but demanding that the solutions satisfy certain inequality at infinity. We follow the idea of [8] (see Remark 6).

The elliptic systems are examined in [6], [7], but under the assumption of the existence of the stifling divisor.

2. NOTATIONS AND DEFINITIONS FOR PARABOLIC SYSTEMS

Let $E \subset \mathbb{R}^{1+n}_{(t,x)}$ be an open, set such that the projection of E onto the t-axis is the interval $(0,T)$, $T \le \infty$.

Notation 1. We call the parabolic interior E of the set $\bar{E}$ all the set of points $(\bar{t},\bar{x},) \in \bar{E}$, which have, for $\rho>0$, the lower half-neighbourhood

$$\left\{(t,x): \sum_{i=1}^{n} (x_i-\bar{x}_i)^2+(t-\bar{t})^2 < \rho^2, t<\bar{t}\right\},$$

belonging to E. This parabolic interior we denote shortly by D. Obviously $E \subset D \subset \bar{E}$.

Next we denote: $S_o = \bar{E} \cap \{(t,x): t=0\}$ and $\sigma = [\partial E \cap \{(t,x): 0<t\le T\}] \setminus D$. We assume that S_o is bounded non -empty set.

Notation 2. The set $\Sigma=S_o\cup\sigma$ we call the parabolic boundary of the set D. It is evident that $D\cup\Sigma=\bar{D}$.

For every τ, $0 < \tau \le T$, we denote by S_τ the projection of the set $D\cap\{(t,x): t=\tau\}$ onto the space $\mathbb{R}^n$. S_τ is an open, bounded, non-empty set for every τ.

Notation 3. We denote by $Z(S_t)$ the class of all functions $z(t,\cdot): S_t \longrightarrow \mathbb{R}^m$, where $z(t,\cdot)(x)=z(t,x)$ for every $t>0$.

For every set $E\subset\mathbb{R}^{n+1}$ we denote

$$E_{kT_o} = E \cap \{((k-1)T_o, kT_o]\cdot \mathbb{R}^n\} \quad \text{for } k\in\mathbb{N}. \tag{3}$$

Notation 4. Let $f=(f^1,\dots,f^m)$ be a function defined on a set of arguments (t,x,s,q,r,z), where $(t,x)\in D$, $s\in\mathbb{R}^m$, $q\in\mathbb{R}^n$, $r\in\mathbb{R}^{n^2}$, $z\in Z(S_t)$.

Definition 1. Let $\sigma^i\subseteq\sigma$, for $i\in I$, be a set on which two functions $h^i:\sigma^i\to\mathbb{R}_+$ and $g^i:\sigma^i\to\mathbb{R}_+$ are defined. For certain $i\in I$, σ^i can be empty. From every point $(t,x)\in\sigma^i$ there emerges a half-line at the given direction $\nu^i(t,x)$, such that its open interval, beginning at the point (t,x) is contained in D. We require that ν^i is orthogonal to the t-axis.

Remark 1. In all theorems, where the existence of stifling divisor is assumed, it suffices to require that the angle between ν^i and the positive direction of t-axis is not smaller than $\pi/2$ for every $(t,x)\in\sigma^i$ (see [4] and [5]).

Definition 2. By $C_\sigma(D)$ we denote the class of functions $u:\bar{D}\to\mathbb{R}^m$, continuous on $\bar{D}$, which for every $i\in I$ have the derivatives $\frac{du^i}{d\nu^i}$ on σ^i, as well as the derivatives u^i_t, $u^i_x=(u^i_{x_j})$, $u_{xx}=(u^i_{x_jx_k})$, $j,k=1,\dots,n$, continuous on D.

Remark 2. In the theorems concerning parabolic systems, the solutions can be irregular on the boundary in the sense given in the Definition 2 of the paper [5]. It is sufficient to introduce a simple modification of the proofs, according to the idea of the proofs given in [5].

Definition 3. We say that the function $w\in C_\sigma(D)$ satisfies the boundary inequalities if, for $i\in I$, we have

1^o $w^i(t,x)\le 0$ on $\Sigma\setminus\sigma^i$

2^o $F^i(w^i)(t,x) = h^i(t,x)w^i(t,x)-g^i(t,x)\frac{d}{d\nu^i}w^i(t,x)\le 0$ on σ^i

for $i\in I$.

Remark 3. The boundary inequalities can be formulated in the nonlinear form according to the Assumption E in [4] or, in case of irregular solutions, according to Definition 6 in [5].

Definition 4. Let $u\in C_\sigma(D)$. We say that the function f^i is parabolic with respect to u in a certain $D_1\subset D$, if for every pair of symmetric matrices $r, \bar{r}$ such that

$$r \le \bar{r} \Leftrightarrow \sum_{i,j=1}^{n}(r_{ij}-\bar{r}_{ij})\lambda_i\lambda_j \le 0,$$

we have

$f^i(t,x,u,u_x^i,r,u(t,\cdot)) - f^i(t,x,u,u_x^i,\bar{r},u(t,\cdot)) \le 0$

for every $(t,x) \in D_1$ (see [8]).

3. PARABOLIC SYSTEMS

We will prove two symmetric theorems concerning the system of parabolic inequalities and a uniqueness theorem as a conclusion from the first of them.

Assumption A_1. Let $u,v\in C_\sigma(D)$ and let

$$N^i = \{(t,x)\in D:\ u^i(t,x) > v^i(t,x)\} \text{ for } i\in I.$$

We assume that

$$u_t^i(t,x) \le f^i(t,x,u(t,x),u_x^i(t,x),u_{xx}^i(t,x),u(t,\cdot)),$$

$$v_t^i(t,x) \ge f^i(t,x,v(t,x),v_x^i(t,x),v_{xx}^i(t,x),v(t,\cdot))$$

for every $(t,x)\in N^i$.

This 2m-system we will write shortly in the form of (1) and (2).

Assumption B_1. There exists $M:\{t,x,s,q,s(t,\cdot)\}\to\mathbb{R}^m$, where $s(t,\cdot)\in C(S_t)$ for every $t>0$, such that for $i\in I$ and every pair of arguments of f^i we have

$$\operatorname{sgn}(s^i - \bar{s}^i)[f^i(t,x,s,q,r,s,(t,\cdot)) - f^i(t,x,\bar{s},\bar{q},r,\bar{s}(t,\cdot))] \le \le M^i(t,x,s-\bar{s},q-\bar{q},s(t,\cdot)-\bar{s}(t,\cdot)) \tag{4}$$

on the set N^i,

for arbitrarily fixed $r\in\mathbb{R}^{n^2}$. Next we assume that for arbitrary $z:D\to\mathbb{R}^m$, bounded from above, at every point of the set N^i in which $\max_p z^p(t,x) = z^i(t,x)>0$ we have

$$M^i(t,x,z(t,x),0,z(t,\cdot)) \le K \sup_{S_t} z^i(t,x) \tag{5}$$

for a certain $K\in\mathbb{R}$.

Remark 4. If f^i do not depend on the last argument and if M^i are linear, that means

$$M^i(t,x,z(t,x),0) \equiv \sum_{j=1}^{n} c_j^i(t,x)z^j(t,x),$$

then from (5) it follows that $c_i^i(t,x) \le K$, since we can put $z^j(t,x) = 0$ for $i \ne j$. In the theorems concerning the uniqueness of the solution to the linear equation there appears the well known assumption $c(t,x) \le c_o$, a natural generalization of which is our (5).

Theorem 1. Let $u,v \in C_\sigma(D)$ be functions for which Assumption A_1 holds, and let f^i be parabolic with respect to u in the set N^i, for $i \in I$. We assume that all the conditions of Assumption B_1 hold for f. If u-v bounded from above in $\bar{D}$ satisfies the boundary inequalities according to the Definition 3, then $u \le v$ in D.

Proof. If $K>0$ (see (5)), then we put $T_o \in (0, \frac{1}{2K})$, if $K \le 0$, then $T_o > 0$ can be fixed arbitrarily. We suppose that in $D_o = D_{T_o}$ (see (3)) there are points belonging to N^i. Hence $\max_i [\sup_{D_o} (u^i - v^i)(t,x)] = H > 0$. There exists $j \in I$, such We create the auxiliary function

$$w(t,x) = u^j(t,x) - v^j(t,x) - \frac{\lambda}{T_o} t,$$

where λ is arbitarily fixed so that

$$\frac{H}{2} < \lambda < H. \tag{6}$$

For $t \in [0, T_o]$ we have

$$H \ge \sup_{D_o} w(t,x) = \omega \ge \sup_{D_o} [u^j(t,x) - v^j(t,x) - \lambda] = H - \lambda > 0.$$

The function w does not reach the least upper bound ω on $(\Sigma \setminus \sigma^j)_{T_o}$ (see Definition 3 p. 1^o). At all points of σ^j we have

$$F^j(w)(t,x) = F^j(u^j - v^j)(t,x) - h^j(t,x) \frac{\lambda t}{T_o} < 0, \tag{7}$$

in virtue of Definition 3 p. 2^o. Now we suppose that w attains ω at $(t_1, x_1) \in (\sigma^j)_{T_o}$, but then from (7) it follows that

$$g^j(t_1,x_1) \frac{dw}{d\nu^j} > h^j(t_1,x_1)\, \omega > 0$$

whence $\frac{dw}{d\nu^j}(t_1,x_1) > 0$, which contradicts our supposition.

Therefore there exists $(\bar{t},\bar{x})$ belonging to the parabolic interior of D_o such that w reaches maximum at this point, and so we have

$$w_t(\bar{t},\bar{x}) \geq 0, \tag{8}$$

$$w_{x_j}(\bar{t},\bar{x}) = 0 \quad \text{for} \quad j=1,\dots,n \tag{9}$$

and

$$\sum_{i,j=1}^{n} w_{x_i x_j}(\bar{t},\bar{x})\lambda_i\lambda_j \leq 0 \quad \text{for every} \quad \lambda \in \mathbb{R}^n. \tag{10}$$

Because $(\bar{t},\bar{x}) \in N^j$, we have

$$w_t(\bar{t},\bar{x}) + \frac{\lambda}{T_o} = u_t^j(\bar{t},\bar{x}) - v_t^j(\bar{t},\bar{x}) \leq f^j(\bar{t},\bar{x},u,u_x^j,u_{xx}^j,u(\bar{t},\cdot)) -$$

$$- f^j(\bar{t},\bar{x},u,u_x^j,v_{xx}^j,u(\bar{t},\cdot)) + f^j(\bar{t},\bar{x},u,u_x^j,v_{xx}^j,u(\bar{t},\cdot)) -$$

$$- f^j(\bar{t},\bar{x},v,v_x^j,v_{xx}^j,v(\bar{t},\cdot)).$$

The first difference on the right hand of the last inequality is non-positive in virtue of the parabolicity of f^j with respect to the u, provided that (10) holds. To the second difference we apply successively: first inequality (4) from Assumption B_1, and then condition (5) ence $w_t(\bar{t},\bar{x}) \leq KH - \frac{\lambda}{T_o}$.
We have assumed (6), therefore $w_t(t,x) \leq H(K - \frac{1}{2T_o}) < 0$ for T_o fixed at the beginning, which contradicts (8).
Our supposition that N^i is a non-empty set for certain i, has brought us to the contradiction, so $u \leq v$ in D_o.

Repeating the above reasoning for $t \in (kT_o,(k+1)T_o]$, we obtain $u \leq v$ in the whole D.

As a conclusion from Theorem 1 we obtain the uniqueness theorem for the system

$$u_t^i = f^i(t,x,u,u_x^i,u_{xx}^i,u(t,\cdot)) \quad \text{for} \quad i \in I \tag{11}$$

with the following boundary value conditions:

Assumption C. Let $u \in C_\sigma(D)$ satisfy the conditions:

1^o $u^i(t,x) = \varphi_1^i(t,x)$ on $\Sigma \setminus \sigma^i$;

2^o $F^i(u^i)(t,x) = \varphi_2^i(t,x)$ on σ^i

for given $\varphi_1^i : (\Sigma \setminus \sigma^i) \to \mathbb{R}$ and $\phi_2^i : \sigma^i \to \mathbb{R}$.

Theorem 2. Let $u, v \in C_\sigma(D)$ be two bounded solutions to the system (11) in D, with the same boundary value conditions, given in Assumption C. We assume that for every $i \in I$, f^i are parabolic with respect to the solutions, and that f^i satisfy Assumption B_1. Then $u \equiv v$ in D.

We omit standard proof (see [2] the proof of Theorem 2).

Now we will formulate symmetric theorem to Theorem 1.

Assumption A_2. Let $u, v \in C_\sigma(D)$ and let

$$N^j = \{(t,x) \in D : u^j(t,x) < v^j(t,x)\}, \quad \text{for } j \in I.$$

We assume that

$$u_t^j \geq f^j(t,x,u,u_x^j,u_{xx}^j,u(t,\cdot)),$$

$$v_t^j \leq f^j(t,x,v,v_x^j,v_{xx}^j,v(t,\cdot)), \quad \text{for every } (t,x) \in N^j.$$

Assumption B_2. We keep the first part of Assumption B_1, in particular, inequality (4). Now we assume, that for every $z : D \to \mathbb{R}^m$ bounded from below in D, at every point of the set N^j in which $\min_p z^p(t,x) = z^j(t,x) < 0$ we have

$$M^j(t,x,z(t,x),0,z(t,x)) \leq -K \inf_{S_t} z^j(t,x) \tag{12}$$

for a certain fixed $K \in \mathbb{R}$.

Theorem 3. Let $u, v \in C_\sigma(D)$ be the functions for which the Assumption A_2 holds and let f^j be parabolic with respect to the u in the set N^j, $j \in I$. We assume that B_2 holds. If $v-u$, bounded from above satisfies boundary inequalities according to Definition 3, then $v \leq u$ in D.

We omit the proof, which is quite symmetrical to the proof of Theorem 1.

Remark 4. In [3], inequalities (5) and (12) were introduced in a little stronger form, namely

$$M^i(t,x,z(t,x),0,z(t,\cdot)) \le K \max_p \sup_{S_t} [(-1)^{k+1} z^p(t,x)]$$

at all points of N^i at which $\max_p [(-1)^{k+1} z^p(t,x)] > 0$, for $k=1,2$.
It is not necessary to investigate the maximum of all functions $z^p(t,x)$ (or $-z^p(t,x)$), it is sufficient to take into consideration only these points of N^i, at which this maximum reaches exactly z^i, which is assumed now in B_1 and B_2. This form of Assumptions B_k, $k=1,2$, is very convenient for application in the theorems of [3], which we are going to show now.

In Assumption E of Theorem 1 in [3], we have assumed inequality (8), which we now repeat below for $k=1$.

If $(t,x)\in D$ and $i\in I$, then

$$f^i(t,x,\phi(t)\overset{1}{w}(x),\phi(t)\overset{1i}{w}_x(x),0,\phi(t)\overset{1}{w}(\cdot)) - f^i(t,x,0,0,0) \le$$
$$\le -\phi(t)\, L_1^i(t,x,\overset{1}{w}(x),\overset{1i}{w}_x(x),\overset{1}{w}(\cdot)), \tag{13}$$

where $1 < \overset{1}{w} < K_o$ and $\phi(t) \ge 0$ is bounded from above. Moreover the following inequality holds

$$L_1^i(t,x,\overset{1}{w}(x),\overset{1i}{w}_x(x),\overset{1}{w}(\cdot)) - \lambda \overset{1i}{w}(x) > 0, \tag{14}$$

where $\lambda\in\mathbb{R}_+$ is an arbitrary constant.

From (13) and (14) it follows that

$$f^i(t,x,\phi(t)\overset{1}{w}(x),\phi(t)\overset{1i}{w}_x(x),0,\phi(t)\overset{1}{w}(\cdot)) - f^i(t,x,0,0,0,0) <$$
$$< -\phi(t)\lambda\overset{1i}{w}(x) \le 0. \tag{15}$$

We see that (15) is a particular case of 5 with K=0 is (15), which we have assumed additionally in [3].

4. NOTATIONS AND DEFINITIONS FOR ELLIPTIC SYSTEMS

Notation 5. Let $C(G)$ be the class of functions $z{:}G\to\mathbb{R}^m$, continuous in G, where $G\subset\mathbb{R}^n$ is an open, bounded set.

Notation 6. Let $f=(\overset{1}{f},\dots,\overset{m}{f})$ be function defined on a set of arguments (x,s,q,r,z) where $x\in G$, $s\in\mathbb{R}^m$, $q\in\mathbb{R}^n$ $r\in\mathbb{R}^{n^2}$, $z\in C(G)$.

Definition 5. Let s^i, for $i\in I$, be a subset of the boundary ∂G, on which two functions $g^i: s^i \to \mathbb{R}_+$ and $h^i: s^i \to \mathbb{R}_+$ are defined. For certain index i, s^i can be empty. From every $x\in s^i$, there emerges a half-line of the given direction $l^i(x)$ such that its open interval beginning at the point x is contained in G.

Notation 7. Let us denote $S^i = \partial G \setminus s^i$ for $i\in I$.

Definiton 7. By $C_s(G)$ we denote the class of functions $u:\bar{G}\to\mathbb{R}^m$, continuous in $\bar{G}$, which satisfy the following conditions: every u, for every $i\in I$, has continous derivatives $u^i_x\in\mathbb{R}^n$, $u^i_{xx}\in\mathbb{R}^{n^2}$ (cf. definition 2) in the domain G, and at every point x of s^i there exists the directional derivative $\frac{d}{dl^i(x)} u^i(x)$.

Definition 8. We say that the function $w\in C_s(G)$ satisfies boundary inequalities if

1^o $w^i(x) \le 0$ on S^i,

2^o $F^i_1(w^i)(w) = h^i(x)w^i(x) - g^i(x)\frac{dw^i}{dl^i}(x) \le 0$ on s^i.

Definition 9. Let $u\in C_s(G)$. We say that f^i is elliptic in $G_o\subset G$ with respect to the u, if $r \le \bar{r} \Rightarrow f^i(x,u,u^i_x,r,u(\cdot)) \le f^i(x,u,u^i_x,\bar{r},u(\cdot))$ for every $x\in G_o$ (cf. Definition 4).

5. ELLIPTIC SYSTEMS

Assumption C_1. Let $u,v\in C_s(G)$. Denote by $N^i = \{(x)\in G: u^i(x) > v^i(x)\}$. We assume that

$$f^i(x,u,u^i_x,u^i_{xx},u(\cdot)) \ge f^i(x,v,v^i_x,v^i_{xx},u(\cdot)) \tag{16}$$

on the set N^i for $i\in I$.

Assumption D_1. There exists $M:\{(x,s,q,s(\cdot)\} \to \mathbb{R}^m$, where $s\in C(G)$, such that for every pair of arguments of f^i we have

$$\begin{aligned} &\operatorname{sgn}(s^i-\bar{s}^i)[f^i(x,s,q,r,s(\cdot))-f^i(x,\bar{s},\bar{q},r,\bar{s}(\cdot))] \le \\ &\le M^i(x,s-\bar{s},q-\bar{q},s(\cdot)-\bar{s}(\cdot)) \end{aligned} \tag{17}$$

on the set N^i for arbitrarily fixed $r\in\mathbb{R}^{n^2}$.

Next we assume that for every $z: G \to \mathbb{R}^m$, bounded from above, at every point of the set N^i for which $\max_p z^p(x) = z^i(x) > 0$ we have

$$M^i(x, z(x), 0, z(\cdot)) \le K_1 \sup_G z^i(x) \tag{18}$$

for a certain $K_1 > 0$.

Theorem 4. Let $u, v \in C_s(G)$ be functions for which Assumption C_1 is held. Let f^i be elliptic with respect to u in N^i and let it satisfy Assuption D_1. If u-v satisfies the boundary inequalities according to Definition 8, then $u \le v$ in G.

Proof. Suppose that $\max_i \sup_G (u^i - v^i)(x) = H_1 > 0$. There exists such $j \in I$ that $\sup_G (u^j - v^j)(x) = H_1$. The function $w(x) = u^j(x) - v^j(x)$ cannot reach this H_1 onthe boundary S^j since we have assumed part 1^o of Definition 8. At every point of s^j, in virtue of part 2^o of the same definition, we have

$$F_1^j (u^j - v^j)(x) \le 0. \tag{19}$$

If $w(\bar{x}) = H_1$ at a certain $\bar{x} \in s^j$, then from (19) it results that at this point $\frac{dw}{dl^j}(\bar{x}) > 0$, which contradicts the definition of l.u.b. So the point $\bar{x}$ at which w attains its maximum is an interior point of G belonging to N^j. Then

$$w_{x_k}(\bar{x}) = 0, \quad k=1,\dots,n \tag{20}$$

and

$$\sum_{j,k=1}^{n} w_{x_j x_k}(\bar{x}) \lambda_j \lambda_k \le 0 \quad \text{for every } \lambda \in \mathbb{R}^n. \tag{21}$$

We investigate now the difference

$$P = f^j(\bar{x}, u(\bar{x}), u_x^j(\bar{x}), u_{xx}^j(\bar{x}), u(\cdot)) - f^j(\bar{x}, v(\bar{x}). v_x(\bar{x}), \bar{v}_{xx}^j(x), \bar{v}(\cdot)).$$

Applying successively the ellipticity of f^j, (21), (17), (18) and (20), we see that $P \le M^j(\bar{x}, u(\bar{x}) - v(\bar{x}), 0, u(\cdot) - v(\cdot)) \le H_1 K_1 < 0$ which contradicts the assumed inequality (16) and finishes the proof.

Now we can formulate the uniqueness theorem.

Consider the system

$$f^i(x,u,u_x^i,u_{xx}^i,u(\cdot)) = 0 \quad \text{for} \quad i\in I \tag{22}$$

in the set G with the boundary value conditions: $u^i(x) = \psi_1^i(x)$ on S^i, $F^i(u^i)(x) = = \psi_2^i(x)$ on s^i for given ψ_1^i, ψ_2^i $(i \in I)$.

Theorem 5. Let $u,v\in C_s(G)$ be two solutions to the system (22) in G with the same boundary value conditions given above. We assume that for every $i\in I$, f^i are elliptic with respect to the both u and v and that f^i satisfy the Assumption D. Then $u\equiv v_1$ in G.

The symmetric theorem to Theorem 1 is obvious.

Remark 5. For elliptic systems we can also consider solutions irregular on the boundry in the sense given in [7] together with the nonlinear form of boundary conditions, similarly to [6] and [7].

Remark 6. All the theorems of the present paper, both for parabolic and elliptic systems, can be proved in unbounded domains without introducing stifling divisors,but under the assumption that at all infinite points of the domain $\limsup_{|x|\to\infty} (u^i-v^i)(x) \le 0$ for $i\in I$.

6. EXAMPLE

Now we are going to give an example. As one can see immediately, the equation $as_{xx} + s = 0$, $a>0$, has two solutions in $[0,\pi\sqrt{a}]$ which have the value zero for $x_1=0$ and $x_2=\pi\sqrt{a}$, namely $s \equiv 0$ and $s = \sin(x/\sqrt{a})$. Notice that for $s\ge\bar{s}$ the inequality (17) takes now the form $f(x,s,q,r,s(\cdot)) - f(x,s,q,r,s(\cdot)) = s-\bar{s}$ that means $M(x,s-\bar{s},q-\bar{q},s(\cdot)-\bar{s}(\cdot)) = s-\bar{s}$. But setting $z(x) > 0$ we have $M(x,z(x),0,z(\cdot)) = z(x)$ and therefore $K_1=1$ and (18) does not hold.

Considering another equation $as_{xx} + s - b\int_0^{\pi\sqrt{a}} s(x)dx = 0$, where $a>0$, $b>0$ we see that $s(x)\equiv 0$ is the solution with the same boundary value conditions. We check Assumption D_1: for $s,\bar{s}\in C(G) = C([0,\pi\sqrt{a}])$ such that $\bar{s}\ge s$, we have

$$f(x,s,q,r,s(\cdot)) - f(x,\bar{s},\bar{q},r,\bar{s}(\cdot)) = \bar{s}(x) - b\int_0^{\pi\sqrt{a}} (s(x)-\bar{s}(x))dx \le$$

$$\le \sup_{[o,\pi\sqrt{a}]} (s(x)-\bar{s}(x))(1-b\pi\sqrt{a}) < 0$$ for a and b fixed above. Therefore the uniqueness of the above solution follows from Theorem 5.

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NIERÓWNOŚCI RÓŻNICZKOWO-FUNKCJONALNE TYPU PARABOLICZNEGO I ELIPTYCZNEGO W OBSZARZE OGRANICZONYM

S t r e s z c z e n i e

W pierwszej części pracy rozważane są układy nierówności typu parabolicznego przy liniowym warunku brzegowym. Zakłada się nieliniową postać warunku Lipschitza i dodatkową nierówność (5). Nierówność ta stanowi uogólnienie znanego warunku dla równań liniowych, w którym żąda się ograniczoności od góry współczynnika przy funkcji niewiadomej. Założenie powyższe pozwala pominąć założenie o quasi-monotoniczności funkcji f. Z twierdzeń o nierównościach wynika twierdzenie o jednoznaczności rozwiązania układu (11) przy odpowiednich warunkach brzegowych (Assumption C). W drugiej części pracy, analogiczne twierdzenia są rozważane dla układów eliptycznych postaci (16).

Wszystkie twierdzenia tej pracy można rozważać w obszarze nieograniczonym bez wprowadzania dzielnika tłumiącego, zakładając tylko, że we wszystkich punktach niewłaściwych obszaru jest spełniony warunek $\lim_{|x|\to\infty} \sup (u^i - v^i)(x) \le 0$ dla $i=1,\dots,m$.

W sposób oczywisty, stosując metody dowodów podane w pracach [5] i [7], można otrzymać wszystkie podane w pracy twierdzenia dla rozwiązań nieregularnych na brzegu przy nieliniowych warunkach brzegowych.

ДИФФЕРЕНЦИАЛЬНО-ФУНКЦИОНАЛЬНЫЕ НЕРАВЕНСТВА ПАРАБОЛИЧЕСКОГО И ЭЛЛИПТИЧЕСКОГО ТИПОВ В ОГРАНИЧЕННОЙ ОБЛАСТИ

Резюме. В первой части этой работы рассматриваются системы неравенств параболического типа с линейном граничном условием. Предполагается нелинейная форма условия Липшица и добовочное неравенство (5). Это неравенство является обобщением знакомого условия для линейных уравнений, в котором предполагается ограничение сверху коэффициента при неизвестной функции. Вышеуказанное предположение позволяет снять условие о квазимонотонности функции f. Из теорем о неравенствах вытекает теорема о единственности для системы (11), при соответственных граничных условиях (определение 5). Во второй части работы, рассуждаются аналогичные теоремы для эллиптических систем вида (16).

Все теоремы этой работы можно рассматривать также в неограниченной области без ввода заглушающего делителя. Достаточно только предположить, что во всех несобственных точках области выполняется условие

$$\lim_{|x|\to\infty} \sup (u^i - v^i)(x)=0 \quad \text{для} \quad i = 1,\dots m$$

Очевидно, используя методы доказательств из работ [5] и [7], можно доказать все вышеизложенные теоремы для нерегулярных решений на границе при нелинейных граничных условиях.