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BASIC PROPERTIES OF THE MORPHISMS OF THE CATEGORY OF ABSTRACT OBJECTS

Summary. The categorial aspects of the theory of abstract objects were explored by M. Kucharzewski in the paper [1].

In the present paper we consider the properties of the morphisms of the category of abstract objects and its subcategories.

Some facts related to these problems were given by E. Zaporowski in the paper [5].

In the present paper we state the from of monomorphisms, epimorphisms in the subcategories of the category OA (corollaries 1 and 2).

We also show some construction of the abstract object which is generated by an equivariant mapping. We state when this object is identically equal to the codomain of the morphism which generates it (theorem 3).

Wer prove that for every morphism (α,ϕ) of the category OA the coimage $\overset{-1}{\alpha}(B)$ of any invariant subset B of the fibre Y of the codomain (Y,H,f) of (α,ϕ) (theorem 4).

In the theorem 5 we prove the property of monomorphisms of OA connected with concomitants.

INTRODUCTION

In the paper [5] E. Zaporowski proved some properties of the morphisms of the category OA. In the present paper we give more details.

1. PRELIMINARIES

An abstract object (object) is a triple (X,G,F) where X is a nonempty set, G is a group and $F: X \times G \longrightarrow X$ is a mapping satisfying the translation and identity conditions (cf. [2], p.12).

An equivariant mapping from the object (X,G,F) into the object (Y,H,f) is a pair (α,ϕ), where $\alpha : X \longrightarrow Y$ is a mapping, $\phi: G \longrightarrow H$ is an homomorphism which satisfies the condition:

$$\alpha(F(x,g)) = f(\alpha(x),\ \phi(g)), \quad x \in X, \quad g \in G \tag{1}$$

The composition of the equivariant mappings is also defined in a well--known manner.

The category whose objects are the abstract objects, whose morphisms are the equivariant mappings and the composition of which is the composition of equivariant mappings is called the category of abstract objects and denoted by OA (cf. [2], p. 19).

The category whose objects are Klein spaces (transitive abstract objects, homogeneous spaces, scalars), whose morphisms are equivariant mappings is called the category of Klein spaces (transitive abstract objects, homogeneous spaces, scalars) and denoted by PK (OAT, PJ, OS) (cf. [2], p. 19; [4], p. 12).

The category whose objects are geometric objects of the fixed Klein space and whose morphisms are the equivariant mappings of the from (α, id_G) and the composition is the composition of equivariant mappings is called the Klein geometry of the group G or briefly G - geometry and is denoted by OG (cf. [2]), p. 10).

2. **Let** $(\alpha, \phi) : (X,G,F) \longrightarrow (Y,H,F)$ (2)

be a morphism of OA. In the paper [5] E. Zaporowski proved

Theorem 1. Morphism (2) is a monomorphism (epimorphism, isomorphism) in the category OA iff α is an injection (surjection, bijection) and ϕ is a monomorphism (epimorphism, isomorphism) of groups.∎

From this theorem it follows that two objects are equivalent (cf. [2], p. 21-22) iff they are isomorphic as the objects of OA.

Theorem 1 is true in the categories PK, OS and OG. For the categories OAT and PJ it is true in the case of monomorphism.

Lemma. If the morphism (2) in the category OA is such that ϕ is an epimorphism and $A \subset X$ is an invariant subset of the fibre X of the object (X,G,F), then $\alpha(A)$ is an invariant subset of the fibre Y of the object (Y,H,f).

Proof. For any $y \in \alpha(A)$ and $h \in H$ from the fact that ϕ is an epimorphism we obtain that there exists $x \in X$ and $g \in G$ such that $\alpha(x)=y$ and $\phi(g) = h$. A is an invariant subset, so $F(x, g) \in A$. From (1) it follows that $f(y,h) = f(\alpha(x), \phi(g)) = \alpha(F(x,g)) \in \alpha(A)$.∎

If the object (Y,H,f) is transitive we obtain that α is a surjection.

Corollary 1. The morphism (2) is an epimorphism (isomorphism) in OAT and PJ iff ϕ is an epimorphism (isomorphism) of groups. ■

Corollary 2. For any group G and any transitive object (Y,G,F) every morphism

$$(\alpha,\ id_G) : (X,G,F) \longrightarrow (Y,G,f)$$

is an epimorphism. ■
It means that the object (Y,G,f) is a concomitant of (X,G,f) (cf. [2]), p.21).

3. Now we characterize properties of images and coimages of invariant subsets in relations of the morpisms of OA. we have

Theorem 2. [5] For every morphism (2) in OA and every $x \in X$ there is $\alpha(W(x)) \subset W(\alpha(X))$, where $W(x)$ denotes the transitive fibre of X which contains an element $x \in X$. If ϕ is an epimorphism then $\alpha(W(x)) = W(\alpha(x))$. ■

In general it follows that for every invariant subset $A \subset X$ the image $\alpha(A)$ is an invariant subset of Y. Therefore, for any morphism (2) we can define a new object:

$$(\alpha(X),\ \phi(G),\ f\big|_{\alpha(X)\ \times\ \phi(G)}) \tag{3}$$

and if α is not a surjection, also

$$(Y\backslash\alpha(X),\ \phi(G),\ f\big|_{(Y\backslash\alpha(X))\ \times\ \phi(G)}). \tag{4}$$

The object (3) is a subobject (cf. [2]), p.37) of (Y,H,f) iff α is a surjection.

If ϕ is an epimorphism of the groups then (3) and (4) are a partial objects of (Y,H,f) (cf. [2]), p.36).

In φ is not an epimorphism then $\alpha(x)$ in general is not an invariant subset of Y.

For any morphism (2) we define

$$H_\alpha := \bigcup_{x \in X} W(\alpha(X)). \tag{5}$$

X_α is an invariant subset of Y, so we can define a partial object

$$(X_\alpha\ H,\ f\big|_{X_\alpha \times H}). \tag{6}$$

Theorem 3. The object (6) is identically equal to $(Y,\ H,\ f)$ iff $\alpha(X)$ has nonempty intersection with every transitive fibre of Y.∎

In the case of coimages we have

Theorem 4. For every morphism (2) the coimage $\alpha^{-1}(B)$ of any invariant subset $B \subset Y$ is an invariant subset of the fibre X of the object $(X,\ G,\ F)$.

Proof. Let $x \in \alpha^{-1}(B)$. For every $g \in G$ we have $\alpha(F(x,g)) = f(\alpha(x), \phi(g)) \in B$, so $F(x,g) \in \alpha^{-1}(B)$.∎

In the definition of concomitant α is said to be a surjection. Sometimes we can prove the existence of a concomitant if there exists an equivariant mapping where α is an injection.

Theorem 5. If there exists one-element trasitive fibre of (X,G,F) and exists a monomorphism

$$(\alpha,\ id_G) : (X,G,F) \longrightarrow (Y,G,f)$$

then the object (X,G,F) is a concomitant of (Y,G,f).

Proof. If α is also a surjection then the theorem is ture. Assume that α is an injection but not a surjection. Let $\{x_o\} \subset X$ denotes one element transitive fibre of (X,G,F). A surjection $\beta : Y \longrightarrow X$ we define as follows:

$$\beta(y) := \begin{cases} \alpha^{-1}(y), & \text{for } y \in \alpha(X), \\ x_o, & \text{for } y \in Y \setminus \alpha(X). \end{cases}$$

It is easy to verify that the pair $(\beta,\ id_G)$ satisfies the condition (1).∎

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PODSTAWOWE WŁASNOŚCI MORFIZMÓW KATEGORII OBIEKTÓW ABSTRAKCYJNYCH

S t r e s z c z e n i e

Kategoryjne podejście do teorii obiektów abstrakcyjnych zostało przedstawione przez M. Kucharzewskiego w pracy [1].

W niniejszej pracy rozważamy własności morfizmów kategorii obiektów abstrakcyjnych i jej podkategorii.

Pewne fakty dotyczące tego problemu zostały podane przez E. Zaporowskiego w artykule [5].

W niniejszej pracy podajemy postaci monomorfizmów, epimorfizmów i izomorfizmów w podkategoriach kategorii OA (wnioski 1 i 2).
Pokażemy także pewną konstrukcję obiektu abstrakcyjnego, generowanego przez odwzorowanie ekwiwariantne. Stwierdzimy, kiedy obiekt ten będzie się pokrywał z kodziedziną odwzorowania, które go generuje (twierdzenie 3).

Udowodnimy także, że dla każdego morfizmu (α,ϕ) kategorii OA przeciwobraz $\alpha^{-1}(B)$ dowolnego podzbioru niezmienniczego B zawartego we włóknie Y kodziedziny (α,ϕ) jest podzbiorem niezmienniczym włókna X dziedziny tego morfizmu (twierdzenie 4).

W twierdzeniu 5 udowodnimy pewną własność monomorfizmów kategorii OA związaną z komitantami.

ОСНОБНЫЕ СВОЙСТВА МОРФИЗМОВ КАТЕГОРИИ АБСТРАКТНЫХ ОБЪЕКТОВ

Резюме. Категорный подход к теории абстрактных объектов представлен М.Кухаржевским в работе [1].В настоящей работе рассматриваются свойства морфизмов категории абстрактных объектов и ее подкатегорий. Некоторые факты, касающиеся этой проблемы, были приведены Е.Запоровским в статьи [5]. Внастоящей работе мы приводим формы мономорфизмов, эпиморфизмов и изоморфизмов в подкатегориях категории OA (следствия 1 и 2). Представляем также некоторую конструкцию абстрактного объекта, порожденного зквивариантным отображением. Определим, в каком случае этот объект будет совпадать с кообластью порождающего его отображения (теорема 3). Докажем также, что для любого морфизма (α,ϕ) категории OA прообраз $\alpha^{-1}(B)$ произвольного инвариантного подмножества B. содержащегося в слое Y кообласти (α,ϕ), есть инвариантное нодмножество слоя X области этого морфизма (теорема 4). В теореме 5 мы докажем некоторое свойство мономорфизмов категории OA, связанное с конкомитантами.