ZESZYTY NAUKOWE POLITECHNIKI ŚLĄSKIEJ 1993
Seria: MATEMATYKA-FIZYKA z. 68 Nr kol. 1147

Karol PETHE

ESTIMATION OF $\operatorname{Re} a_4$ OF BOUNDED UNIVALENT FUNCTIONS ON THE UNIT CIRCLE

Summary. For an arbitrary function

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in S(M), \quad M > 1,$$

with $\pm a_2 > 0$, the following estimation has been obtained for the functional $\pm \operatorname{Re} a_4$:

$$\pm \operatorname{Re} a_4 \le \begin{cases} \frac{2}{3}\left(1 - \frac{1}{M^3}\right) & \text{for} \quad 1 < M \le \frac{9}{4} \\ \frac{2}{3}\left(1 - \frac{1}{M^3}\right) + \frac{8}{3 \cdot 49}\left(4 - \frac{9}{M}\right)^3 & \text{for} \quad \frac{9}{4} \le M \le 11 \\ \frac{2}{3}\left(1 - \frac{1}{M^3}\right) + \frac{10}{3}\left(1 - \frac{1}{M}\right)^2\left(1 - \frac{4}{M}\right) & \text{for} \quad M > 11. \end{cases}$$

It has been shown, that the obtained inequalites are exact and equalities are obtained for the functions:

$$\frac{f^3(z)}{\left(1 \pm \frac{f^3(z)}{M^3}\right)^2} = \frac{z^3}{(1 \pm z^3)^2},$$

$$\frac{f^3(z)}{\left(\pm \frac{f^3(z)}{M^3} + \frac{3\lambda}{M} f^2(z) \mp 3\lambda f(z) - 1\right)^2} = \frac{z^3}{(\mp z^3 + \lambda z^2 \mp \lambda z - 1)^2}, \quad \lambda = \frac{6}{7}\left(4 - \frac{9}{M}\right).$$

$$\frac{f(z)}{\left(1 \mp \frac{f(z)}{M}\right)^2} = \frac{z}{(1 \mp z)^2}.$$

In the proof the concept applied by L.E. Ahlfors [1] by the estimation of $|a_4|$ function $f \in S$ has been used.

We shall discuss the following classes of functions:

S - the class of functions $f = z + a_2z^2 + a_3z^3 + a_4z^4 + \dots$ regular and univalent on the unit circle $K = \{z : |z| < 1\}$;

S_M - the class of functions $f \in S : |f(z)| < M,\ M > 1,\ z \in K$;

Σ_p - the class of functions $F(\varphi) = \sum_{k=-p}^{\infty} b_k\zeta^{-k}$, $b_{-p} = 1$, meromorphic and p-valent in the area $K' = \{\zeta : |\zeta| > 1\}$.

For $p = 1$ the last class is equivalent to the class Σ. The functions $F \in \Sigma$ are associated with functions $f \in S$ by the known transformation $F(\zeta) = f^{-1}(\frac{1}{\zeta})$, $\zeta \in K'$.

For the coefficients of functions of the class Σ_p the following G.M.Goluzin's inequality [2] holds:

$$\sum_{k=-p}^{\infty} k\,|b_k|^2 \le 0, \qquad b_{-p} = 1. \tag{1}$$

Theorem. If the function $f(z) = z + a_2z^2 + a_3z^3 + a_4z^4 + \dots$ belongs to S_M and $\pm a_2 > 0$, then

I. $\pm \operatorname{Re} a_4 \le \frac{2}{3}\,(1 - \frac{1}{M^3})$ for $1 < M \le \frac{9}{4}$,

with the equality for the function

$$\frac{f^3(z)}{\left(1 \pm \frac{f^3(z)}{M^3}\right)^2} = \frac{z^3}{(1 \mp z^3)^2};$$

II. $\pm \operatorname{Re} a_4 \le \frac{2}{3}\,(1 - \frac{1}{M^3}) + \frac{1}{3.49}\,(4 - \frac{9}{M})^3$ for $\frac{9}{4} \le M < 11$

with the equality for the function

$$\frac{f^3(z)}{(\pm \frac{f^3(z)}{M^3} + \frac{3\lambda}{M}\, f^2(z) \mp 3\lambda f(z) - 1)^2} = \frac{z^3}{(\mp z^3 + \lambda z^2 \mp \lambda z - 1)^2},\ \lambda = \frac{6}{7}\,(4 - \frac{9}{M});$$

III. $\pm \operatorname{Re} a_4 \le \frac{2}{3}(1-\frac{1}{M^3}) + \frac{10}{3}(1-\frac{1}{M})^2(1-\frac{4}{M}), \quad M \ge 11$

with the equality for the function

$$\frac{f(z)}{(1 \mp \frac{f(z)}{M})^2} = \frac{z}{(1 \mp z)^2}.$$

If $a_2 = 0$, then

IV. $|a_4| \le \frac{2}{3}(1 - \frac{1}{M^3})$ for $M > 1$.

Proof. Let $f \in S_M$ with $a_2 \ne 0$ and define

$$g(z)= \frac{f(z)}{(1- \frac{e^{i\Theta}}{M} f(z))^2}, \quad F(\zeta)= \left[g(\frac{1}{\zeta^2})\right]^{-1/2} = f^{-1/2}(\frac{1}{\zeta^2}) - \frac{e^{i\Theta}}{M} \cdot f^{1/2}(\frac{1}{\zeta^2}),$$

where Θ is a real number clearly, $g \in S$. The expansion of the function $F(\zeta)$ is

$$F(\zeta) = \zeta\,(1 + \frac{b_1}{\zeta^2} + \frac{b_3}{\zeta^4} + \frac{b_5}{\zeta^6} + \dots),$$

where

$$b_1 = - \frac{a_2}{2} - \frac{e^{i\Theta}}{M},$$

$$b_3 = - \frac{1}{2} a_3 + \frac{3}{8} a_2^2 - \frac{e^{i\Theta}}{M} a_2, \tag{2}$$

$$b_5 = - \frac{1}{2} a_4 + \frac{3}{4} a_2 a_3 - \frac{5}{16} a_2^3 - \frac{e^{i\Theta}}{M}\frac{a_3}{2} - \frac{e^{i\Theta}}{M}\frac{a_2^2}{8}.$$

Let us consider now the polynomial $W(\omega) = \omega^3 + \alpha\,\omega$ and the following function:

$$\Phi(\zeta) = F^3(\zeta) + \alpha F(\zeta), \quad \alpha \text{ - complex parameter}, \tag{3}$$

which belongs to the class $\sum_3$. We have

$$F^3(\zeta) = \zeta^3\left(1 + \frac{c_{-1}}{\zeta^2} + \frac{c_1}{\zeta^4} + \frac{c_3}{\zeta^6} + \dots\right),$$

where

$$c_{-1} = 3\, b_1,$$

$$c_1 = 3\, b_3 + 3\, b_1^2, \tag{4}$$

$$c_3 = 3\, b_5 + 6\, b_1 b_3 + b_1^3.$$

Aplying inequality (1) to function (3), we obtain

$$(1 - |b_1|^2 - 3|b_3|^2)|\alpha|^2 + (\bar{c}_{-1} - \bar{c}_1 b_1 - 3\bar{c}_3 b_3)\alpha + (c_{-1} - c_1\bar{b}_1 - 3c_3\bar{b}_3)\,\bar{\alpha} +$$

$$+ (3 + |c_{-1}|^2 - |c_1|^2 - 3\,|c_3|^2) \geq 0.$$

Substituting $\alpha = \frac{\alpha_1}{\alpha_2}$, where α_1, α_2 are complex numbers, and denoting

$$A_{11} = 1 - |b_1|^2 - 3\,|b_3|^2, \qquad A_{12} = \bar{c}_{-1} - \bar{c}_1 b_1 - 3\,\bar{c}_3 b_3$$

$$A_{21} = c_{-1} - c_1\bar{b}_1 - 3\,c_3\bar{b}_3, \qquad A_{22} = 3 + |c_{-1}|^2 - |c_1|^2 - 3\,|c_3|^2,$$

the last inequality may be written in the from

$$A_{11}|\alpha_1|^2 + A_{12}\alpha_1\alpha_2 + A_{21}\bar{\alpha}_1\alpha_2 + A_{22}|\alpha_2|^2 \geq 0. \tag{5}$$

Since $A_{12} = A_{21}$, the left side of inequality (5) has the Hermitian form, i.e.

$$A_{11} \geq 0, \qquad \begin{vmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{vmatrix} \geq 0,$$

and thus

$$|b_1|^2 + 3\,|b_3|^2 \leq 1$$

$$|c_{-1} - \bar{c}_1 b_1 - 3\bar{c}_3 b_3|^2 \leq (1 - |b_1|^2 - 3|b_3|^2)(3 + |c_{-1}|^2 - |c_1|^2 - 3|c_3|^2).$$

We can write the second inequality in the form

$$c_{-1}\bar{c}_1 b_1 - \bar{c}_{-1} c_1 \bar{b}_1 - 3c_{-1}\bar{c}_3 b_3 - 3c_{-1} c_3 \bar{b}_3 + 3\bar{c}_1 b_1 c_3 \bar{b}_3 + 3c_1 \bar{b}_1 \bar{c}_3 b_3 \le$$

$$3(1- |b_1|^2- -3|b_3|^2) - |c_1|^2 - 3|c_3|^2 - |c_{-1}|^2|b_1|^2 + 3|c_3|^2|b_1|^2 -$$

$$- 3|c_{-1}|^2|b_3|^2 + 3|c_1|^2 |b_3|^2,$$

and, taking into account (4), we obtain

$$|c_3|^2(1-|b_1|^2) - 3\bar{b}_1\bar{b}_3 c_3(1-|b_1|^2)- 3b_1 b_3\bar{c}_3(1-|b_1|^2) + 3b_1\bar{b}_3^{\,2} c_3 +$$

$$+ 3\bar{b}_1 b_3^{\,2}\, \bar{c}_3 \le 1-|b_1|^2 - 6\, |b_3|^2- 9|b_1|^4|b_3|^2 + \tag{6}$$

$$+ 9|b_3|^4 + 9|b_3|^2\, b_1^2\, \bar{b}_3 + 9\, |b_3|^2\, \bar{b}_1^{2} b_3.$$

Let us assume now that $|b_1|<1$, (if $|b_1| = 1$, we have $b_3 = 0$ and $b_5 = 0$). Taking a_2 real and $\Theta = 0$, then considering (2) and $b_3 = b_5 = 0$, a_3 and a_4 will be real too; in this case the theorem was proved by V. Singh [3]). Inequality (6) after multiplying both sides by the positive expression $1 - |b_1|^2$ we transform into

$$|c_3(1- |b_1|^2)- 3\, b_1 b_3 + 3\, \bar{b}_1 b_3^2|^2 \le (1 - |b_1|^2 - 3\, |b_3|^2)^2,$$

hence

$$\left| c_3 - 3b_1 b_3 + \frac{3\bar{b}_1 b_3^2}{1 - |b_1|^2} \right| \le 1 - \frac{3|b_3|^2}{1 - |b_1|^2} .$$

Using (4) and (2) we obtain succesively

$$\left| - \frac{3}{2}\, a_4 + \frac{3\bar{b}_1 b_3^{\,2}}{1 - b_1^{\,2}} + 12b_3(b_1+ \frac{e^{i\Theta}}{M})-5b_1^3 - 30\, \frac{e^{i\Theta}}{M}\, b_1^2 - 45\, \frac{e^{2i\Theta}}{M^2}\, b_1 - 21\, \frac{e^{3i\Theta}}{M^3}\right| \le$$

$$\le 1 - \frac{3|b_3|^2}{1 - |b_1|^2},$$

$$\left| a_4 - \frac{2\bar{b}_1 b_3^2}{1-|b_1|^2} - 8b_3(b_1 + \frac{e^{i\Theta}}{M}) + \frac{10}{3} b_1^3 + 20 \frac{e^{i\Theta}}{M} b_1^2 + \frac{30e^{2i\Theta}}{M^2} b_1 + 14 \frac{e^{3i\Theta}}{M^3} \right| \leq \tag{7}$$

$$\leq \frac{2}{3} - \frac{2|b_3|^2}{1-|b_1|^2}$$

As the number Θ is arbitrary, we choose such a Θ that

$$\operatorname{atan} a_2 = \Theta,$$

Then taking into account (2) and (8) the coefficients b_1 and b_3 may be expressed in the form

$$b_1 = -(\frac{|a_2|}{2} + \frac{1}{M})\, e^{i\Theta} = x\, e^{i\Theta}, \qquad \frac{1}{M} \leq x < 1,$$

$$b_3 = (-\frac{1}{2} a_3 e^{-2i\Theta} + \frac{3}{8} |a_2|^2 - \frac{1}{M} \frac{|a_2|}{2})\, e^{2i\Theta} = t\, e^{2i\Theta}. \tag{9}$$

Substituting now (9) into (7), we obtain

$$\left| a_4 e^{-3i\Theta} + \frac{2x\, t^2}{1-x^2} - 8t\,(-x + \frac{1}{M}) - \frac{10}{3} x^3 + \frac{20}{M} x^2 - \frac{30}{M^2} x + \frac{14}{M^3} \right| \leq$$

$$\leq \frac{2}{3} - \frac{2\,|t|^2}{1-x^2}$$

and hence

$$\operatorname{Re}(a_4 e^{-3i\Theta}) \leq \tag{10}$$

$$\leq \operatorname{Re}(-\frac{2x\, t^2}{1-x^2} + 8t\,(-x + \frac{1}{M}) + \frac{10}{3} x^3 - \frac{20}{M} x^2 + \frac{30}{M^2} x - \frac{14}{M^3}) + \frac{2}{3} - \frac{2|t|^2}{1-x^2}.$$

If we put $t = u + i\, v$, in (10) then

$$\operatorname{Re}(a_4 e^{-3i\Theta}) \leq -\frac{2u^2}{1-x} + 8u\,(-x + \frac{1}{M}) + \frac{2v^2}{1+x} + \frac{10}{3} x^3 - \frac{20}{M} x^2 + \frac{30}{M^2} x - \frac{14}{3}) + \frac{2}{3}. \tag{11}$$

We can observe that when x is constant, right hand side of (11) attains its maximum for

$$u = 2\left(-x + \frac{1}{M}\right)(1 - x),$$

$$v = 0. \tag{12}$$

Substituting (12) into (11), we obtain successively

$$\mathrm{Re}(a_4 e^{-3i\theta}) \le 8\left(-x + \frac{1}{M}\right)(1-x) + \frac{10}{3}x^3 - \frac{20}{M}x^2 + \frac{30}{M^2}x - \frac{14}{3}) + \frac{2}{3},$$

$$\mathrm{Re}(a_4 e^{-3i\theta}) \le -\frac{14}{3}x^3 + 4\left(2 - \frac{1}{M}\right)x^2 - 2\left(\frac{8}{M} - \frac{11}{M^2}\right) + \frac{2}{3} + \frac{8}{M^2} - \frac{14}{M^3}. \tag{13}$$

We shall seek now the maximum value of the right hand side of (13). Denoting

$$h(x,M) = -\frac{14}{3}x^3 + 4\left(2 - \frac{1}{M}\right)x^2 - 2\left(\frac{8}{M} - \frac{11}{M^2}\right)x + \frac{2}{3} + \frac{8}{M^2} - \frac{14}{M^3},$$

we have

$$\frac{dh(x,M)}{dx} = -14x^2 + 8\left(2 - \frac{1}{M}\right)x - 2\left(\frac{8}{M} - \frac{11}{M^2}\right), \tag{14}$$

and

$$\frac{dh(x,M)}{dx} = 0 \quad \text{for}$$

$$x_1 = \frac{1}{M} \quad \text{and} \quad x_2 = \frac{1}{7}\left(8 - \frac{11}{M}\right). \tag{15}$$

Let us consider now three cases.

1°. If $1 < M \le \frac{9}{4}$

then, (14) and (15), $x_1 \ge x_2$.

Since $\frac{dh(x,M)}{dx} < 0$ for $x \in \langle \frac{1}{M}, 1)$, $h(x,M)$ attains the maximum value for $x = x_1$. If we substitute (14) into (13), then we obtain

$$\mathrm{Re}(a_4 e^{-3i\theta}) \le \frac{2}{3}\left(1 - \frac{1}{M^3}\right). \tag{16}$$

2°. If $\frac{9}{4} \le M \le 11$,

then $x_1 \le x_2 < 1$ and $\frac{dh\ (x,M)}{dx} > 0$ for $x_1 < x < x_2$; $\frac{dh\ (x,M)}{dx} < 0$ for $x < x_1$, $x > x_2$, which means that h(x,M) attains the maximum value at $x = x_2$. Substituting (15) into (13), we obtain

$$\mathrm{Re}(a_4 e^{-3i\Theta}) \le \frac{2}{3}\left(1 - \frac{1}{M^3}\right) + \frac{8}{3.49}\left(4 - \frac{9}{M}\right)^3 . \tag{17}$$

3°. If $M \ge 11$,

then $x_2 > x_1$, $x_2 > 1$ and $\frac{dh\ (x,M)}{dx} > 0$ for $x \in (x_1, x_2)$.
Hence $x < 1$ and therefore $h(x,M)$ attains its maximum value for $x = 1$; consequently

$$\mathrm{Re}(a_4 e^{-3i\Theta}) \le \frac{2}{3}\left(1 - \frac{1}{M^3}\right) + \frac{10}{3}\left(1 - \frac{1}{M}\right)^2\left(1 - \frac{4}{M}\right). \tag{18}$$

Putting $\Theta = 0, \pi$ in equalities (16), (17) and (18), we obtain inequalities I, II and III of the theorem. Let us consider now the case $a_2 = 0$.
By (2), it follows in that case

$$b_1 = -\frac{1}{M} e^{i\Theta}, \quad b_3 = -\frac{1}{2} a_3$$

and inequality (7) takes the form

$$\left| a_4 + \frac{2}{3}\frac{e^{3i\Theta}}{M^3} + \frac{\frac{1}{2} M a_3^2}{M^2 - 1} e^{-i\Theta} \right| \le \frac{2}{3} - \frac{\frac{1}{2} M^2 |a_3|^2}{M^2 - 1}.$$

Hence

$$\left| a_4 + \frac{2}{3}\frac{e^{3i\Theta}}{M^3} \right| \le \frac{2}{3} + \frac{\frac{1}{2} M |a_3|^2}{M^2 - 1} - \frac{\frac{1}{2} M^2 |a_3|^2}{M^2 - 1} = \frac{2}{3} - \frac{\frac{1}{2} M |a_3|^2}{M + 1} \le \frac{2}{3},$$

that is, $\left| a_4 + \frac{2}{3}\frac{e^{3i\Theta}}{M^3} \right| \le \frac{2}{3}$.

Substituting $\Theta = \frac{1}{3} \arg a_4$, we obtain

$$\left| |a_4| e^{i \arg a_4} + \frac{2}{3} \frac{e^{i \arg a_4}}{M^3} \right| \le \frac{2}{3}$$

and

$$|a_4| \le \frac{2}{3} (1 - \frac{1}{M^3}),$$

that is, inequality IV of the theorem.

REFERENCES

[1] **Ahlfors L.E.**: Nerawenstwo meżdu koefficientami a_2 i a_4 odnolistnoj funkcji. Nekotorye problemy matematiki i mechaniki. Izd. Nauka, Leningrad 1970, 71-74.

[2] **Gołuzin G.M.**: O p-listnych funkcijach. Matem. Sb. 8 (50): 2, 277-284.

[3] **Singh V.**: Grunsky inequalities and coefficients of bounded schlicht functions, Ann. Acad. Sci. Fenn. Ser. A I, 310, 1962.

OSZACOWANIE Re a_4 FUNKCJI OGRANICZONYCH I JEDNOLISTNYCH W KOLE JEDNOSTKOWYM

S t r e s z c z e n i e

Dla dowolnej funkcji

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in S(M), \quad M > 1,$$

przy $\pm a_2 > 0$, otrzymano dla funkcjonału $\pm \mathrm{Re}\, a_4$ oszacowanie

$$\pm \mathrm{Re}\, a_4 \le \begin{cases} \frac{2}{3} (1 - \frac{1}{M^3}) & \text{for} \quad 1 < M \le \frac{9}{4} \\ \frac{2}{3} (1 - \frac{1}{M^3}) + \frac{8}{3 \cdot 49} (4 - \frac{9}{M})^3 & \text{for} \quad \frac{9}{4} \le M \le 11 \\ \frac{2}{3} (1 - \frac{1}{M^3}) + \frac{10}{3} (1 - \frac{1}{M})^2 (1 - \frac{4}{M}) & \text{for} \quad M > 11. \end{cases}$$

Wykazano, że uzyskane nierówności są dokładne, przy czym równości są osiągnięte przez funkcje:

$$\frac{f^3(z)}{\left(1 \pm \frac{f^3(z)}{M^3}\right)^2} = \frac{z^3}{(1 \pm z^3)^2},$$

$$\frac{f^3(z)}{\left(\pm \frac{f^3(z)}{M^3} + \frac{3\lambda}{M} f^2(z) \mp 3\lambda f(z) - 1\right)^2} = \frac{{}^3z}{(\mp z^3 + \lambda z^2 \mp \lambda z - 1)^2}, \quad \lambda = \frac{6}{7}\left(4 - \frac{9}{M}\right),$$

$$\frac{f(z)}{\left(1 \mp \frac{f(z)}{M}\right)^2} = \frac{z}{(1 \mp z)^2}.$$

W dowodzie został wykorzystany pomysł, jaki stosował L.E. Ahlfors [1] przy szacowaniu $|a_4|$ dla funkcji $f \in S$.

ОЦЕНКА $\operatorname{Re} a_4$ ОДНОЛИСТНЫХ И ОГРАНИЧЕННЫХ ФУНКЦИЙ В ЕДИНИЧНОМ КРУГЕ

Резюме. В работе для призвольной функции

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in S(M), \quad M > 1,$$

получена, при условии $\pm a_2 > 0$, точная оценка функционала $\operatorname{Re} a_4$ в виде:

$$\pm \operatorname{Re} a_4 \le \begin{cases} \frac{2}{3}\left(1 - \frac{1}{M^3}\right) & \text{for} \quad 1 < M \le \frac{9}{4} \\ \frac{2}{3}\left(1 - \frac{1}{M^3}\right) + \frac{8}{3 \cdot 49}\left(4 - \frac{9}{M}\right)^3 & \text{for} \quad \frac{9}{4} \le M \le 11 \\ \frac{2}{3}\left(1 - \frac{1}{M^3}\right) + \frac{10}{3}\left(1 - \frac{1}{M}\right)^2\left(1 - \frac{4}{M}\right) & \text{for} \quad M > 11. \end{cases}$$

Доказано, что при этом равенства достигаются соответственно для функций:

$$\frac{f^3(z)}{\left(1 \pm \frac{f^3(z)}{M^3}\right)^2} = \frac{z^3}{(1 \pm z^3)^2}$$

$$\frac{f^3(z)}{\left(\pm \frac{f^3(z)}{M^3} + \frac{3\lambda}{M} f^2(z) \mp 3\lambda f(z) - 1\right)^2} = \frac{z^3}{(\mp z^3 + \lambda z^2 \mp \lambda z - 1)^2}, \quad \lambda = \frac{6}{7}\left(4 - \frac{9}{M}\right),$$

$$\frac{f(z)}{\left(1 \mp \frac{f(z)}{M}\right)^2} = \frac{z}{(1 \mp z)^2}.$$

В доказательстве использована идея применима Альфорсом при оценке $|a_4|$ для функции $f \in S$.