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Zygfryd KOMINEK

ON A RESULT OF W.STADJE

Summary. W. Stadje has proved [7] the following theorem.
Theorem A. Let $A \subset (a,b)$ be a measurable set such that $\lambda((a,b)\setminus A) = 0$, where λ denotes the Lebesgue measure on $\mathbb{R}$. Let $f : (a,b) \longrightarrow \mathbb{R}$ be a measurable midconvex function on A i.e.

$$f\left(\frac{x+y}{2}\right) \le \frac{f(x) + f(y)}{2}$$

whenever $x, y, \frac{x+y}{2} \in A$. Then there exists a convex function $\bar{f}: (a,b) \longrightarrow \mathbb{R}$ such that $\bar{f}(x) = f(x)$ for every $x \in A$.

In this note we generalize this result to the case of X being an arbitrary real linear topological space and the set A belonging to some σ-ideal in X fulfilling some additional conditions. In particular, the assertion of Theorem A is valid if A is residual in an open and convex subset D of a real linear topological Baire space satisfying the second axiom of countability.

Dedicated to Professor Mieczysław Kucharzewski with best wishes on his 70th birthday.

In his paper [7] W. Stadje has proved the following theorem.

Theorem A. Let $-\infty \le a < b \le \infty$ and let $A \subset (a,b)$ be a measurable set such that $\lambda((a,b)\setminus A) = 0$, where λ denotes the Lebesgue measure on $\mathbb{R}$. Let $f: (a,b) \longrightarrow \mathbb{R}$ be a measurable and midconvex function on A, i.e.

$$f\left(\frac{x+y}{2}\right) \le \frac{f(x) + f(y)}{2} \tag{1}$$

whenever $x, y, \frac{x+y}{2} \in A$. Then there exists a convex function $\bar{f}: (a,b) \longrightarrow \mathbb{R}$ such that $\bar{f}(x) = f(x)$ for every $x \in A$.

In this note we shall give a different proof of Theorem A based on a result of M. Kuczma [2] (cf. also Th. 2, p.459 in [3]) concerning functions fulfilling Jensen's inequality (1) almost everywhere (in the sense of

Lebesgue measure) in $(a,b)^2$. Moreover, using the same idea, we shall prove a more general result. The results of this kind may be obtained also for additive functions.

1. FIRST WE SHALL PROVE A THEOREM BEING A GENERALIZATION OF THEOREM A.

Theorem 1. Let $-\infty \le a < b \le \infty$, and let $A \subset (a,b)$ be a Lebesgue measurable set such that $\lambda((a,b)\setminus A) = 0$. Let $f\colon (a,b) \longrightarrow R$ be a midconvex function on A. Then there exists a unique midconvex function $\bar{f}\colon (a,b) \longrightarrow \mathbb{R}$ on (a,b) such that $\bar{f}(x) = f(x)$ for each $x \in A$. If, moreover, f is measurable then $\bar{f}$ is convex.

Proof. Let $D = (a,b)$ and

$$M = \{(x,y) \in D \times D;\quad x \notin A \quad \text{or} \quad y \notin A \quad \text{or} \quad \tfrac{x+y}{2} \notin A\}.$$

Since

$$M \subset [(D\setminus A) \times D] \cup [D\times(D\setminus A)] \cup \{(x,y) \in D \times D;\ \tfrac{x+y}{2} \notin A\},$$

and the set on the right hand side has the Lebesgue measure zero, the Lebesgue measure of M is also zero. By the definition of M and from our assumptions on f we get

$$f\left(\frac{x+y}{2}\right) \le \frac{f(x) + f(y)}{2} \quad \text{for all} \quad (x,y) \in (D \times D)\setminus M,$$

i.e. f is almost midconvex on D (cf. [2]). On account of a theorem of Kuczma (Th. 5 in [2]) there exists a unique midconvex function $\bar{f}\colon D \longrightarrow \mathbb{R}$ on D such that $\bar{f}(x) = f(x)$ almost everywhere in D.

We shall show that $\bar{f}(x) = f(x)$ for every $x \in A$. Let $B \subset A$ be a measurable set such that $\lambda(D\setminus B) = 0$ and

$$\bar{f}(y) = f(y) \quad \text{for each} \quad y \in B. \tag{2}$$

Fix an arbitrary $x \in A$. There exists a $h \in \mathbb{R}$ such that $x \mp \frac{1}{2^n} h \in B$ for any positive integer n. Then also

$\dfrac{x + (x + \frac{1}{2^n} h)}{2} = x + \dfrac{1}{2^{n+1}} h \in B$. Hence by the midconvexity of f on $A \supset B$, and by virtue of (2) the inequalities

$$f(x) = f\left(\frac{(x - \frac{1}{2^n} h) + (x + \frac{1}{2^n} h)}{2}\right) \le$$

$$\le \frac{f(x - \frac{1}{2^n} h) + f(x + \frac{1}{2^n} h)}{2} = \tag{3}$$

$$= \frac{\bar{f}(x - \frac{1}{2^n} h) + \bar{f}(x + \frac{1}{2^n} h)}{2}$$

and

$$\bar{f}(x + \frac{1}{2^{n+1}} h) = f(x + \frac{1}{2^{n+1}} h) = f\left(\frac{x + x + \frac{1}{2^n} h}{2}\right) \le$$

$$\le \frac{f(x) + f(x + \frac{1}{2^n} h)}{2} = \tag{4}$$

$$= \frac{f(x) + \bar{f}(x + \frac{1}{2^n} h)}{2}$$

hold for every positive integer n. If n tends to infinity then the sequences $\bar{f}(x \mp \frac{1}{2^n} h)$ both converge to $\bar{f}(x)$ ([3], Th. 1, p. 136). Therefore $\bar{f}(x) = f(x)$ by virtue of (3) and (4). This finishes the first part of the proof. The second part is a consequence of a theorem of Sierpiński [6] (cf. also [3] Th. 2 p. 218).

2. LET X BE AN ARBITRARY SET. A NON-EMPTY FAMILY $\mathcal{J}$ OF SUBSETS OF X CALLED AN IDEAL IFF IT SATISFIES THE TWO CONDITIONS

(i) if $A \in \mathcal{J}$ and $B \subset A$, then $B \in \mathcal{J}$,

(ii) if $A, B \in \mathcal{J}$, then $A \cup B \in \mathcal{J}$.

If condition (ii) is replaced by the stronger one

(iii) if $A_n \in \mathcal{J}$ for every positive integer n

then $\bigcup_{n=1}^{\infty} A_n \in \mathcal{J}$,

then $\mathcal{J}$ is called a σ-ideal. If an

ideal (σ-ideal) satisfies also the condition $X \notin \mathcal{J}$, it is called proper. If we are given a proper ideal $\mathcal{J}$ of subsets of X, then we say that a condition is satisfied $\mathcal{J}$-almost everywhere in X (written $\mathcal{J}$-(a.e.)) iff there exists a set $U \in \mathcal{J}$ such that the condition in question is satisfied for every $x \in X \setminus U$.

Now we suppose that $(X,+)$ is a group. An ideal (σ-ideal) $\mathcal{J}$ of subsets of X is linearly invariant iff, beside conditions (i), (ii), (iii)) it satisfies also (iv) for every $x \in X$ and $A \in \mathcal{J}$ the set $x-A$ belongs to $\mathcal{J}$. A proper linearly invariant ideal (σ-ideal) will in the sequel be referred to as a p.l.i. ideal (σ-ideal).

It is easily seen that in a real linear topological Baire space, the family of all subsets of the first category forms a p.l.i. σ-ideal, as well as, the family of all measurable in the Lebesgue sense subsets of $\mathbb{R}^N$ with measure equal to zero is also a p.l.i. σ-ideal.

Let $\mathcal{J}$ be a p.l.i. ideal (σ-ideal) in a group $(X,+)$ and let $\mathcal{J}_2$ be an ideal (σ-ideal) in $X \times X$. The ideals $\mathcal{J}$ and $\mathcal{J}_2$ are conjugate iff for every $A \in {}_2$ we have

$$A[x] = \{y \in X;\ (x,y) \in A\} \in \mathcal{J} \quad \mathcal{J}\text{-(a.e.)} \quad \text{in } X.$$

The family $\pi(\mathcal{J})$ defined by the formula

$$\pi(\mathcal{J}) = \{A \subset X \times X;\quad A \subset (U \times X) \cup (X \times U),\quad U \in \mathcal{J}\}$$

is a p.l.i. ideal (σ-ideal) in $X \times X$. These and some others informations concerning ideals may be found, for example, in [3] (pp. 437-443).

We have the following lemma.

Lemma 1. Let $(X,+)$ be a commutative group in which division by two is uniquely performable, let $\mathfrak{J}$ be an ideal in X such that if $S \in \mathfrak{J}$ then $2S \in \mathfrak{J}$, and assume that $U \in \mathfrak{J}$. If $\mathfrak{J}_2 \supset \pi(\mathfrak{J})$ is an ideal in $X \times X$ and condition

(v) $Z \in \mathfrak{J}_2$ implies $T_1(Z) \in \mathfrak{J}_2$,

is fulfilled, where $T_1\colon X \times X \longrightarrow X \times X$ is defined by the formula

$$T_1(x,y) = (\tfrac{x+y}{2}, \tfrac{x-y}{2}), \quad x,y \in X, \tag{5}$$

then the set

$$S_1 = \{(u,v) \in X \times X;\ \ \tfrac{u+v}{2} \in U\} \tag{6}$$

belongs to the ideal $\mathfrak{J}_2$.

Proof. Since $2U \in \mathfrak{J}$ and $\pi(\mathfrak{J}) \subset \mathfrak{J}_2$, the set $2U \times X$ belongs to $\mathfrak{J}_2$ and hence $T_1(2U \times X) \in \mathfrak{J}_2$ by virtue of (v).
But

$$T_1(2U \times X) = \{(u,v) \in X \times X;\ \ u = \tfrac{x+y}{2},\ \ v = \tfrac{x-y}{2},\ \ x \in 2U,\ y \in X\}$$

$$= \{(u,v) \in X \times X;\ \ \tfrac{u+v}{2} \in U\} = S_1.$$

Consequently $S_1 \in \mathfrak{J}_2$.

We will apply the following theorem of M.Kuczma ([3], Th.2 p.459).

Theorem B. Let D be an open, convex subset of a real linear topological space and suppose that the conjugate p.l.i. σ-ideals $\mathfrak{J}$ and $\mathfrak{J}_2$ (in X and in $X \times X$, respectively) are given. Assume that the ideals $\mathfrak{J}$ and $\mathfrak{J}_2$ satisfy the conditions

(a) if $A \in \mathfrak{J}$ and $\alpha \in \mathbb{R}$, then $\alpha A \in \mathfrak{J}$,

(b) if $A \in \mathfrak{J}_2$, then $T_1(A) \in \mathfrak{J}_2$, where T_1 is given by (5). If $f: D \longrightarrow \mathbb{R}$ is a $\mathcal{J}_2$-(a.e.) midconvex function, then there exists a unique midconvex function $\bar{f}: D \longrightarrow \mathbb{R}$ on D such that $\bar{f}(x) = f(x)$ $\mathcal{J}$-(a.e.) in D.

In fact, this theorem was formulated in the case where the real linear topological space $X = \mathbb{R}^N$, but its proof in our situation is literally the same as presented in [3].

Lemma 2. Let $(X, +)$ be a commutative uniquely 2-divisible topological group having the property: for every neighbourhood G of zero $\bigcup_{n=0}^{\infty} 2^n G = X$.

Let D be an open subset of X such that $(D-x) \subset 2^n (D-x)$ for every $x \in D$ and each positive integer n and assume that in X we have a p.l.i. σ-ideal $\mathcal{J}$ fulfilling the condition

(vi) if $U \in \mathcal{J}$, then $\frac{1}{2^n} U \in \mathcal{J}$ and $2^n U \in \mathcal{J}$

for every positive integer n. If $V \in \mathcal{J}$ and $x \in D$, then the set

$\bigcap_{n=1}^{\infty} [2^n(B-x) \cap 2^n(x-B)]$, where $B = D \setminus V$, is non-empty.

Proof. Let $V \in \mathcal{J}$, $B = D \setminus V$ and fix an $x \in D$. The set $(D-x) \cap (x-D)$ is a non-empty neighbourhood of zero. Therefore

$$(D-x) \cap (x-D) \notin \mathcal{J} \tag{7}$$

because $\mathcal{J}$ is a proper σ-ideal fulfilling (vi) (cf. also [3], Lemma 1 p. 452). Assume that

$$\bigcap_{n=1}^{\infty} [2^n(B-x) \cap 2^n(x-B)] = \emptyset.$$

We have

$$\begin{aligned}(D-x) \cap (x-D) &= (D-x) \cap (x-D) \setminus \bigcap_{n=1}^{\infty} [2^n(B-x) \cap 2^n(x-B)] = \\ &= \bigcup_{n=1}^{\infty} \Big([(D-x) \cap (x-D)] \cap [2^n(B-x) \cap 2^n(x-B)]' \Big) \subset \\ &\subset \bigcup_{n=0}^{\infty} \big[2^n(D-x) \cap 2^n(x-D) \cap [2^n(B-x)' \cup 2^n(x-B)'] \big] \subset \\ &\subset \bigcup_{n=0}^{\infty} 2^n(V-x) \cup \bigcup_{n=0}^{\infty} 2^n(x-V),\end{aligned}$$

where C' denotes the complement of the set C. Since $V \in \mathfrak{J}$ and $\mathfrak{J}$ is a p.l.i. σ-ideal, $(D-x) \cap (x-D)$ belongs to $\mathfrak{J}$, which contradicts (7). This completes the proof.

Theorem 2. Let D be an open and convex subset of a real linear topological space X, and suppose that in X we have a p.l.i. σ-ideal fulfilling condition (a) and in $X \times X$ we have a p.l.i. σ-ideal $\mathfrak{J}_2 \supset \pi(\mathfrak{J})$ fulfilling condition (b) such that $\mathfrak{J}$ and $\mathfrak{J}_2$ are conjugate. If $f: D \longrightarrow \mathbb{R}$ satisfies (1) whenever $x, y, \frac{x+y}{2} \in D\setminus U$, where $U \in \mathfrak{J}$, then there exists a unique function $\bar{f}: D \longrightarrow \mathbb{R}$ midconvex on D such that $\bar{f}(x) = f(x)$ for every $x \in D\setminus U$.

Proof. Let us put

$$M = (U \times D) \cup (D \times U) \cup S_1,$$

where S_1 is defined by (6). It follows from inclusion $\mathfrak{J}_2 \supset \pi(\mathfrak{J})$ and Lemma 1 that $M \in \mathfrak{J}_2$. If $x, y \in (D \times D)\setminus M$, then $x, y, \frac{x+y}{2} \in D\setminus U$ and hence f is $\mathfrak{J}_2$-almost everywhere midconvex function. On account of Theorem B there exists a unique midconvex function $\bar{f}: D \longrightarrow \mathbb{R}$ on D and a subset $V \in \mathfrak{J}$, $V \supset U$, such that

$$\bar{f}(y) = f(y) \quad \text{for every} \quad y \in D\setminus V = B.$$

Fix an arbitrary $x \in D\setminus U$. By Lemma 2 there exists $h \in X$ such that $x \mp \frac{1}{2^n} h \in B$ for every positive integer n. The rest of the proof runs as a suitable part of the proof of Theorem 1.

Corollary 1. Let X be a real N-space $\mathbb{R}^N$, and let $\mathfrak{J}$ be the ideal of all subsets which have Lebesgue measure zero. Assume that D is an open and convex subset of $\mathbb{R}^N$ and that $f: D \longrightarrow \mathbb{R}$ fulfils condition (1) whenever $x, y, \frac{x+y}{2} \in D\setminus U$, where U is an element of $\mathfrak{J}$. Then there exists a unique midconvex function $\bar{f}: D \longrightarrow \mathbb{R}$ on D such that $\bar{f}(x) = f(x)$ for every $x \in D\setminus U$. If moreover, f is Lebesgue measurable then $\bar{f}$ is continuous and convex.

Proof. It is well known that the ideal $\mathfrak{J}$ of all subsets of $\mathbb{R}^N$ of measure zero satisfies condition (a) and the ideal $\mathfrak{J}_2$ of all subsets of $\mathbb{R}^N \times \mathbb{R}^N$ of measure zero fulfils condition (b) and $\mathfrak{J}_2 \supset \pi(\mathfrak{J})$. The first part of our assertion follows from Theorem 2, and the second part is a consequence of the first part and a theorem of Sierpiński [6] (cf. also Th. 2 p.218 in [3]).

Similarly we can obtain

Corollary 2. Let X be a real linear topological Baire space satisfying the second axiom of countability and let $\mathfrak{J}$ be the ideal of all first category subsets of X. Assume that $D \subset X$ is an open and convex set and $f: D \longrightarrow \mathbb{R}$ fulfils (1) whenever $x, y, \frac{x+y}{2} \in D \setminus U$, where U is a set from $\mathfrak{J}$. Then there exists a unique midconvex function $\bar{f}: D \longrightarrow \mathbb{R}$ on D such that $\bar{f}(x) = f(x)$ for each $x \in D \setminus U$. If, moreover, f satisfies the condition of Baire (i.e. the inverse image $f^{-1}(G)$ is a Baire subset of X for each open subset G of $\mathbb{R}$), then $\bar{f}$ is continuous and convex.

Proof. By a theorem of Oxtoby [5] the ideal $\mathfrak{J}_2$ of all first category subsets of $X \times X$ contains the ideal $\pi(\mathfrak{J})$. The first part of our assertion follows from Theorem 2, and the second part is a consequence of a theorem of a theorem of Mehdi [4].

3. THE FOLLOWING LEMMA WILL BE USED IN THE PROOF OF A THEOREM ANALOGUE TO THEOREM 2 FOR ADDITIVE FUNCTION

Lemma 3. ([3], Lemma 4 p. 441). Let (X,+) be a group and let $\mathfrak{J}$ be a p.l.i. ideal in X. If $U \in \mathfrak{J}$ then

$$S_2 = \{(x,y) \in X \times X; \quad x+y \in U\} \in \Omega(\mathfrak{J}), \tag{8}$$

where

$$\Omega(\ ) = \{A \subset X \times X; \quad A[x] \in \mathfrak{J} \qquad \mathfrak{J}\text{-(a.e.)} \ \text{ in } \ X\}.$$

Remark. It is well known fact (Lemma 3 in [3], p. 441) that if $\mathfrak{J}$ is a p.l.i. ideal in X then $\Omega(\mathfrak{J})$ is a p.l.i. ideal in $X \times X$ and, moreover, $\mathfrak{J}$ and $\Omega(\mathfrak{J})$ are conjugate.

Theorem 3. Let (X,+) and (Y,+) be groups (not necessarily commutative) and suppose that there is a given p.l.i. ideal $\mathfrak{J}$ in X. If $f: X \longrightarrow Y$ satisfies the following relation

$$f(x+y) = f(x) + f(y) \tag{9}$$

whenever $x, y, x+y \in X\setminus U$, where $U \in \mathfrak{J}$, then there exists exactly one additive function $\bar{f}: X \longrightarrow Y$ (i.e. such that $\bar{f}(x+y) = \bar{f}(x) + \bar{f}(y)$ for all $x, y \in X$) such that $\bar{f}(x) = f(x)$ for every $x \in X\setminus U$.

Proof. Let us put

$$M = (U \times X) \cup (X \times U) \cup S_2,$$

where S_2 is given by (8). It is easy to check that $M \in \Omega(\mathfrak{J})$. It $(x,y) \notin M$ then (9) is fulfilled. So f is $\Omega(\mathfrak{J})$ - a.e. additive function. In view of a theorem of Ger [1], there exists exactly one additive function $\bar{f}: X \longrightarrow Y$ such that $\bar{f}(x) = f(x)$ $\mathfrak{J}$ - (a.e.) in X.

Let $V \supset U$ be an element of $\mathfrak{J}$ such that

$$\bar{f}(y) = f(y) \quad \text{for every} \quad y \in X\setminus V. \tag{10}$$

Take an arbitrary $x \in X\setminus U$. It is easily seen that $(X\setminus V) \cap [-(X\setminus V) + x] \neq \emptyset$. If $h \in (X\setminus V) \cap [-(X\setminus V) + x]$ then also $x - h \in X\setminus V$ and hence

$$f(x) = f(x-h+h) = f(x-h) + f(h) = \bar{f}(x-h) + \bar{f}(h) = \bar{f}(x)$$

by virtue of (10) and the additivity of $\bar{f}$. This ends the proof.

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Institute of Mathematics
Silesian University
Bankowa 14
40-007 Katowice

O PEWNYM WYNIKU W. STADJE

S t r e s z c z e n i e

W. Stadje udowodnił następujące twierdzenie.

Twierdzenie A. Niech $A \subset (a,b)$ będzie takim mierzalnym zbiorem, że $\lambda((a,b)\setminus A) = 0$, gdzie λ oznacza miarę Lebesgue'a na $\mathbb{R}$. Niech $f\colon (a,b) \longrightarrow \mathbb{R}$ będzie mierzalną J-wypukłą funkcją na A, tzn. f spełnia nierówność

$$f\left(\frac{x+y}{2}\right) \le \frac{f(x) + f(y)}{2}$$

dla wszelkich $x, y, \frac{x+y}{2} \in A$. Wtedy istnieje taka wypukła funkcja $\bar{f}\colon (a,b) \longrightarrow \mathbb{R}$, że $\bar{f}(x) = f(x)$ dla każdego $x \in A$.

W tej pracy uogólniamy ten wynik na przypadek gdy X jest rzeczywistą przestrzenią liniowo-topologiczną, a zbiór A należy do pewnego σ-ideału w X. W szczególności, teza Twierdzenia A jest spełniona, gdy A jest rezydualnym podzbiorem pewnego otwartego i wypukłego podzbioru D rzeczywistej przestrzeni liniowo-topologicznej Baire'a spełniającej II postulat przeliczalności.

О НЕКОТОРОМ РЕЗУЛЬТАТЕ В.ШТАДЕ

Резюме. В.Штаде (W.Stadje) доказал [1] следующую теорему.

Теорема А. Пусть $A \subset (a,b)$ будеть измеримым множеством таким, что $\lambda((a,b)/A) = 0$, где λ обозначает лебегобую меру в R. Пусть f: (a,b) → R измеримая J-вынуклая жункция A, т.е.

$$f(\frac{x+y}{2}) \le \frac{f(x) + f(y)}{2}$$

Для всех x, y, $\frac{x+y}{2} \in A$. Тогда существует такая выпуклая жункция f: (a,b) → R, что f(x) для всех x ∈A. В настоящей статье обобщаем этот результат на случай когда X является вещественным линейно-монологическим пространством, а множество A принадлежит некоторомуδ σ- идеалу в X исполняющему некоторые дополнительные условия. В гастности тезис Теоремы A сохраняется если A - выметное множество в некотором открытом и выпуклом подмножестве D дещественного линейно-монологического пространства бера, исполняющего вторую аксиому перечисленности.