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ON PSEUDO-SYMMETRIC TOTALLY UMBILICAL SUBMANIFOLDS OF RIEMANNIAN MANIFOLDS ADMITTING SOME TYPES OF GENERALIZED CURVATURE TENSORS

DEDICATED TO PROF. DR MIECZYSŁAW KUCHARZEWSKI ON HIS 70TH BIRTHDAY

Summary. Let M be a totally umbilical submanifold, with parallel mean curvature vector field H, of a manifold N admitting $(0,k)$-pseudo-symetric tensor $\tilde{T}$, where $3 \le \dim M < \dim N$ and $k \in \mathbb{N}$. Then the tensor T, the orthogonal projection of $\tilde{T}$ on M, is also pseudo-symetric. Moreover, if $\tilde{T}$ is a generalized curvature tensor on N, $\dim M \ne 4$ and the curvature tensor R of M is pseudo-symmetric, then the associated functions $L_{\tilde{T}}$, L_T and L_R of the tensors $\tilde{T}$, T and R satisfy on the set $V \subset M$ consisting of all points of M at which the tensors T and R are both non-trivial the following equality

$$L_T = L_{\tilde{T}|M} + \tilde{g}(H,H) = L_R.$$

1. INTRODUCTION

The purpose of this paper is to generalize the following result from the paper [1] (Theorem 1 (i)):

Let N ($n=\dim N$) be a Riemannian manifold, with the metric tensor $\tilde{g}$, admitting a semi-symmetric generalized curvature tensor $\tilde{T}$. Let M ($m=\dim M$) be a semi-symmetric totally umbilical submanifold of N with parallel mean curvature vector field H. If the constant $\tilde{g}(H,H)$ is non-zero and $3 \le m < n$, $m \ne 4$, then at any point of M at least one of the tensors R and T is trivial, where R is the curvaturre tensor of M and T is the orthogonal projection of the tensor $\tilde{T}$ on M.

In the present paper we prove the following theorem:

Let M, $3 \le m < n$, $m \ne 4$, be a pseudo-symmetric totally umbilical submanifold, with parallel mean curvature vector field H, of a manifold N admitting a pseudo-symmetric generalized curvature tensor $\tilde{T}$. Let V be the subset of M consisting of all points of M at which the tensors R and T are both non-trivial. Then the tensor field T is pseudo-symmetric and the following equality

$$L_T = L_{\tilde{T}|M} + \tilde{g}(H,H) = L_R$$

holds on V, where $L_{\tilde{T}}$, L_R and L_T are the associated functions of $\tilde{T}$, R and T, respectively.

The special case of the above theorem, when the ambient space is pseudo-symmetric, was considered in [4] (Corollary 1 (i); see also Theorem). Namely, we have proved:

Let M be a totally umbilical submanifold, with parallel mean curvature vector field H, of a pseudo-symmetric manifold N. Then M is also pseudo-symmetric and the equality

$$L_R = L_{\tilde{R}|M} + \tilde{g}(H,H)$$

holds on the subset U_R consisting of all points of M at which R is non-trivial, where $L_{\tilde{R}}$ is the associate function of $\tilde{R}$.

Examples of manifolds realizing the assumptions of the last result are given in [6] and [8].

Throughout this paper all manifolds are assumed to be connected paracompact manifolds of class C^{∞}.

2. PSEUDO-SYMMETRIC TENSORS

Let N be an n-dimensional $(n \geq 3)$ Riemannian manifold with not necessarily definite metric $\tilde{g}$. We denote by $\tilde{\nabla}$, $\tilde{R}$, $\tilde{S}$, $\tilde{C}$ and $\tilde{K}$ the Levi-Civita connection, the Riemann-Christoffel curvature tensor, the Ricci tensor, the Weyl conformal curvature tensor and the scalar curvature of N, respectively.

For an $(0,k)$-tensor field $\tilde{T}$ on N, $k \geq 1$, we define the tensor fields $\tilde{R}\cdot\tilde{T}$ and $Q(\tilde{g},\tilde{T})$ by the formulas

$$(\tilde{R}\cdot\tilde{T})(X_1,\dots,X_k;X,Y) = (\tilde{R}(X,Y)\cdot\tilde{T})(X_1,\dots,X_k) =$$

$$- \tilde{T}(\tilde{R}(X,Y)X_1,X_2,\dots,X_k) - \dots - \tilde{T}(X_1,\dots,X_{k-1},\tilde{R}(X,Y)X_k)$$

and

$$Q(\tilde{g},\tilde{T})(X_1,\dots,X_k;X,Y) = -((X \wedge Y)\cdot\tilde{T})(X_1,\dots,X_k) =$$

$$= \tilde{T}((X \wedge Y)X_1,X_2,\dots,X_k) + \dots + \tilde{T}(X_1,\dots,X_{k-1},(X \wedge Y)X_k)$$

respectively, where $\tilde{R}(X,Y)$ and $X \wedge Y$ are derivations of the algebra of the tensor fields on N and $X_1,\dots,X_k,X,Y,Z \in \mathfrak{X}(M)$, $\mathfrak{X}(M)$ being the Lie algebra of vector fields on N. These derivations are the extensions of the endomorphisms $\tilde{R}(X,Y)$ and $X \wedge Y$ of $\mathfrak{X}(M)$ defined by

$$\tilde{R}(X,Y)Z = \tilde{\nabla}_X\tilde{\nabla}_Y Z - \tilde{\nabla}_Y\tilde{\nabla}_X Z - \tilde{\nabla}_{[X,Y]}Z$$

and

$$(X \wedge Y)Z = \tilde{g}(Z,Y)X - \tilde{g}(Z,X)Y,$$

respectively.

An $(0,k)$-tensor field $\tilde{T}$ is parallel (resp., semi-symmetric) if the equality $\tilde{\nabla}\tilde{T} = 0$ (resp., $\tilde{R}\cdot\tilde{T} = 0$) holds on N. Of course, any parallel tensor field is semi-symmetric. An $(0,k)$-tensor field $\tilde{T}$ is called recurrent (resp., birecurrent) if the equation

$$\tilde{\nabla}\tilde{T} = \varphi\otimes\tilde{T} \ (\text{resp., } \tilde{\nabla}^2\tilde{T} = \psi\otimes\tilde{T})$$

holds on the set $U \subset N$ consisting of all points of N at which $\tilde{T} \neq 0$, where φ is a covector field on U (resp., ψ is a tensor field of type $(0,2)$ on U). It is easy to see that any recurrent tensor field $\tilde{T}$ is semi-symmetric if and only if the covector field φ is locally a gradient. Moreover, if the metric form of N is positive definite then any recurrent tensor field $\tilde{T}$ is semi-symmetric and the covector field φ is given by

$$\varphi = \frac{1}{2}\, d(\log g(\tilde{T},\tilde{T})).$$

A birecurrent tensor field $\tilde{T}$ is semi-symmetric if and only if the tensor field ψ is symmetric. An $(0,k)$-tensor field $\tilde{T}$ is said to be pseudo-symmetric if it satisfies the condition

(*) at every point of N the tensors $\tilde{R}\cdot\tilde{T}$ and $Q(\tilde{g},\tilde{T})$ are linearly dependent.

Obviously, any semi-symmetric tensor field is pseudo-symmetric.
The converse fails in general.

An (0,4)-tensor field $\tilde{T}$ on N is said to be a generalized curvature tensor [12] if it satisfies the conditions

$$\tilde{T}(X_1,X_2,X_3,X_4) + \tilde{T}(X_1,X_3,X_4,X_2) + \tilde{T}(X_1,X_4,X_2,X_3) = 0,$$

$$\tilde{T}(X_1,X_2,X_3,X_4) = -\tilde{T}(X_2,X_1,X_3,X_4), \quad \tilde{T}(X_1,X_2,X_3,X_4) = \tilde{T}(X_3,X_4,X_1,X_2).$$

For a generalized curvature tensor field $\tilde{T}$ we define the concircular curvature tensor $Z(\tilde{T})$ by the formula

$$Z(\tilde{T})(X_1,X_2,X_3,X_4) = \tilde{T}(X_1,X_2,X_3,X_4) - \frac{K(\tilde{T})}{m(m-1)}\,\tilde{g}((X_1 \wedge X_2)X_3,X_4).$$

where $S(\tilde{T})$ is the Ricci tensor of $\tilde{T}$ and $K(\tilde{T})$ is the scalar curvature of $\tilde{T}$. A generalized curvature tensor $\tilde{T}$ is called trivial at a point $x \in N$ if its concircular tensor $Z(\tilde{T})$ vanishes at x.

Remark ([2], Lemma 1.1(iii)). For a point $x \in N$, the equality $Z(\tilde{T})(x) = 0$ is equivalent to the relation $Q(\tilde{g},\tilde{T})(x) = 0$.

Define now the set $U_{\tilde{T}}$ by

$$U_{\tilde{T}} = \{x \in N : Z(\tilde{T})(x) \neq 0\}.$$

If a generalized curvature tensor $\tilde{T}$ is pseudo-symmetric, then on the set $U_{\tilde{T}}$ we have

$$\tilde{R}\cdot\tilde{T} = L_{\tilde{T}}\, Q(\tilde{g},\tilde{T}),$$

where $L_{\tilde{T}}$ is a function defined on $U_{\tilde{T}}$. Similarly, if $\tilde{A}$ is a symmetric pseudo-symmetric (0,2)-tensor then on the set

$$U_{\tilde{A}} = \{x \in N : \tilde{A} - \frac{1}{n}\,\mathrm{tr}(\tilde{A})\,\tilde{g}\} \neq 0$$

we have

$$\tilde{R}\cdot\tilde{A} = L_{\tilde{A}}\, Q(\tilde{g},\tilde{A})$$

where $L_{\tilde{A}}$ is a function on $U_{\tilde{A}}$. The functions $L_{\tilde{T}}$ and $L_{\tilde{A}}$ are uniquely defined and called the associated functions of the tensor $\tilde{T}$ and $\tilde{A}$, respectively.

The following lemmas will be used in the what follows:

Lemma 1. If $\tilde{T}$ is a pseudo-symmetric generalized curvature tensor on a manifold N, then its Ricci tensor $S(\tilde{T})$ is also pseudo-symmetric. Moreover, the equality $\tilde{R}\cdot\tilde{T} = L_{\tilde{T}}\, Q(g,T)$ implies $\tilde{R}\cdot S(\tilde{T}) = L_T\, Q(\tilde{g},S(\tilde{T}))$.

Proof. Trivial.

Lemma 2. ([11], Lemma 2.1). Let $\tilde{T}$ be the tnesor field on a manifold N defined by

$$\tilde{T}(X_1,X_2,X_3,X_4) = \frac{-\operatorname{tr}(\tilde{A})}{n-1)(n-2)}\, \tilde{g}((X_1 \wedge X_2)X_3,X_4) +$$

$$+ \frac{1}{n-2}\, (\tilde{g}(X_1,X_4)\tilde{A}(X_2,X_3) - \tilde{g}(X_1,X_3)\tilde{A}(X_2,X_4) + \tilde{g}(X_2,X_3)\tilde{A}(X_1,X_4) -$$

$$- \tilde{g}(X_2,X_4)\tilde{A}(X_2,X_3)),$$

where $\tilde{A}$ is a symmetric (0,2)-tensor field on N. Then we have:

(i) The condition $\tilde{R}\cdot\tilde{A} = L_{\tilde{A}}\, Q(\tilde{g},\tilde{A})$ implies $\tilde{R}\cdot\tilde{T} = L_{\tilde{A}}\, Q(\tilde{g},\tilde{T})$.

(ii) If the equality $Z(\tilde{T}) = 0$ holds at a point $x \in N$, then at x the relation $\tilde{A} = \frac{1}{n}\operatorname{tr}(\tilde{A})\, \tilde{g}$ is fulfilled.

Lemma 3. ([10], Lemma 2). Let $\tilde{A}$ and $\tilde{D}$ be symmetric (0,2)-tensors and $\tilde{T}$ a generalized curvature tensor at a point x of a Riemannian manifold N satisfying the relations $\tilde{R}\cdot\tilde{A} = Q(\tilde{g},\tilde{D})$ and $\tilde{R}\cdot\tilde{T} = \alpha Q(\tilde{g},\tilde{T})$, $\alpha \in \mathbb{R}$. Then the equality

$$((\tilde{D} - \alpha\tilde{A}) - \frac{1}{n}\operatorname{tr}(\tilde{D} - \alpha\tilde{A})\tilde{g})\, Z(\tilde{T}) = 0$$

holds at x.

A Riemannian manifold N is said to be pseudo-symmetric if its curvature tensor $\tilde{R}$ is pseudo-symmetric [8]. If N is pseudo-symmetric, then the equality

$$\tilde{R}\cdot\tilde{R} = L_{\tilde{R}}\, Q(\tilde{g},\tilde{R})$$

holds on the set $U_{\tilde{R}}$. Any semi-symmetric manifold ($\tilde{R}\cdot\tilde{R} = 0$, [14]) is pseudo-symmetric. Examples of non semi-symmetric pseuido-symmetric manifolds are given in [2], [6] and [8]. Recently, pseudo-symmetric manifolds were studied by various authors. For the references see [4].

A Riemannian manifold N is said to be Ricci-pseudo-symmetric if its Ricci tensor $\tilde{S}$ is pseudo-symmetric [11], [5]. If N is Ricci-pseudo-symmetric, then the equality

$$\tilde{R}\cdot\tilde{S} = L_{\tilde{S}}\, Q(\tilde{g},\tilde{S})$$

holds on the set $U_{\tilde{S}}$. In view of Lemma 1, any pseudo-symmetric manifold is Ricci-pseudo-symmetric. The converse statement fails in general (see [11], [5]).

A Riemannian manifold N is said to be Weyl-pseudo-symmetric if its Weyl conformal curvature tensor $\tilde{C}$ is pseudo-symmetric. If N is Weyl-pseudo-symmetric, then the condition

$$\tilde{R}\cdot\tilde{C} = L_{\tilde{C}}\, Q(\tilde{g},\tilde{C})$$

holds on the set $U_{\tilde{C}}$. Any pseudo-symmetric manifold N is Weyl-pseudo-symmetric. The converse statement fails in general (see [9]). Note that $U_{\tilde{C}} = \{x \in N : \tilde{C}(x) \neq 0\}$.

Let M be a submanifold of a Riemannian manifold N with a metric $\tilde{g}$. We always assume that $3 \leq m < n$. In the case when $\tilde{g}$ is not a definite metric we shall assume that the induced from $\tilde{g}$ tensor g is a metric tensor of M. Let $\nabla^{\perp}$ denotes the normal connection in the normal bundle of M. We shall say that the mean curvature vector field H of M is parallel if $\nabla^{\perp} H = 0$.

Non semi-symmetric pseudo-symmetric generalized curvature tensors arose during the study of semi-symmetric totally umbilical submanifolds, with parallel mean curvature vector field, of Riemannian manifolds admitting semi-symmetric generalized curvature tensors (see [1], Lemma 1). The following theorem is related to those results:

Theorem 1. Let M be a totally umbilical submanifold, with parallel mean curvatur vector field H, of a Riemannian manifold N admitting an $(0,k)$-pseudo-symmetric tensor field $\tilde{T}$. Then the tensor T, the orthogonal projection of $\tilde{T}$ on M, is also pseudo-symmetric.

Proof. In the proof we use definitions and notations given in [13], [1] or [7]. Let the system of equations $x^r = x^r(y^a)$ be the local parametric representation of M in N, where y^a and x^r are local coordinates of M and N, respecitvely, $r, s, t, u \in \{1, 2, \ldots, n\}$ and $a, b, c, d \in \{1, 2, \ldots, m\}$. Let $N_z^{\ r}$, $z = m+1, m+2, \ldots, n$ be the local components of the pairwise orthogonal unit normals to M and let

$$B_{a_1 \ldots a_l}^{r_1 \ldots r_l} = B_{a_1}^{r_1} \ldots B_{a_l}^{r_l}, \quad B_a^r = \partial_a x^r, \quad \partial_a / \partial y^a, \quad l \in N.$$

Then at any point of M we have

$$- \tilde{g}^{rs}(\tilde{T}_{rs_2 \ldots s_k} \tilde{R}_{ss_1tu} + \ldots + \tilde{T}_{s_1 \ldots s_{k-1}r} \tilde{R}_{ss_ktu}) =$$

$$= \alpha(\tilde{g}_{s_1u}\tilde{T}_{ts_2 \ldots s_k} + \ldots + \tilde{g}_{s_ku}\tilde{T}_{s_1 \ldots s_{k-1}t} -$$

$$- \tilde{g}_{s_1t}\tilde{T}_{us_2 \ldots s_k} - \ldots - \tilde{g}_{s_kt}\tilde{T}_{s_1 \ldots s_{k-1}u}), \quad \alpha \in \mathbb{R}.$$

From this, by transvection witch $B_{a_1 \ldots a_k cd}^{s_1 \ldots s_k tu}$ and application of the relation

$$\tilde{g}^{rs} = g^{ab}B_{ab}^{rs} + \sum_z e_z N_z^{\ r} N_z^{\ s}, \quad e_z = \pm 1,$$

the Gauss equation and the equality $R_{rstu} B_{abc}^{rst} N_z^u = 0$, which is satisffied when H is parallel (cf. [13], eq. (20)), we get $R \cdot T = (\alpha + \tilde{g}(H, H))\, Q(g, T)$, completing the proof.

In the case when $\tilde{T} = \tilde{C}$, the assumption that H is parallel can be omitted in Theorem 1. Namely, we have:

Theorem 2. (cf. [7], Lemma 2). Any totally umbilical submanifold of a Weyl-pseudo-symmetric manifold N is also Weyl-pseudo-symmetric.

3. PSEUDO-SYMMETRIC TOTALLY UMBILICAL SUBMANIFOLDS OF MANIFOLDS ADMITTING PSEUDO-SYMMETRIC GENERALIZED CURVATURE TENSORS.

Let M be a submanifold of Riemannian manifold N covered by a system of coordinate neighbourhoods $\{V; y^a\}$. We denote by g_{ab}, R_{abcd}, T_{abcd}, $(R \cdot R)_{abcdef}$,

$$Q(g,R)_{abcdef} = g_{af}R_{ebcd} + g_{bf}R_{aecd} + g_{cf}R_{abed} + g_{df}R_{abce} - $$
$$- g_{ae}R_{fbcd} - g_{be}R_{afcd} - g_{ce}R_{abfd} - g_{de}R_{abcf}, \qquad (1)$$

$$Q(g,T)_{abcdef} = g_{af}T_{ebcd} + g_{bf}T_{aecd} + g_{cf}T_{abed} + g_{df}T_{abce} - $$
$$- g_{ae}T_{fbcd} - g_{be}T_{afcd} - g_{ce}T_{abfd} - g_{de}T_{abcf}, \qquad (2)$$

$$(R\cdot T)_{abcdef} = - T^{h}{}_{bcd}R_{haef} + T^{h}{}_{acd}R_{hbef} - T^{h}{}_{dab}R_{hcef} + T^{h}{}_{cab}R_{hdef}, \qquad (3)$$

the local components of the tensors g, R, T (= the orthogonal projection of a generalized curvature tensor $\tilde{T}$ from N on M), $R\cdot R$, $Q(g,R)$, $Q(g,T)$ and $R\cdot T$, respectively.

As an immediate consequence of Theorem 1 we have

Corollary 1. ([4], Proposition 1). Let M be a totally umbilical submanifold, with parallel mean curvature vector field H, of a manifold N admitting a pseudo-symmetric generalized curvature tensor field $\tilde{T}$. Then the relation

$$R\cdot T = \rho Q(g,T), \quad \rho = L_{\tilde{T}}|_M + \tilde{g}(H,H). \qquad (4)$$

holds on the set $U_{\tilde{T}} \cap M$.

Using the algebraic properties of generalized curvature tensors and (2) we can easily prove the following

Lemma 4. Let T be a generalized curvature tensor at a point of a Reimannian manifold M. Then the relations

$$Q(g,T)_{abcdef} = - Q(g,T)_{bacdef} = Q(g,T)_{cdabef} = - Q(g,T)_{abcdfe}, \qquad (5)$$

$$T_{hcak}R^{hk}{}_{ef} + T_{hkac}R^{h}{}_{ef}{}^{k} = 0, \qquad (6)$$

$$(T_{hkca} - 2\, T_{hack})R^{hk}{}_{ef} = 0, \qquad (7)$$

$$(T_{hkla} - 2\, T_{hlka})R^{hkl}{}_{f} +0, \qquad (8)$$

hold at this point. Moreover, if at this point we have

$$S(T) = \frac{1}{n} K(T) g$$

then the equality

$$Q(g,T)^{k}{}_{bcdek} = (m-1) Z(T)_{ebcd} \tag{9}$$

is satisfied.

As an immediate consequence of Lemma 3 and Corollary 1 we obtain

Lemma 5. Let M be a pseudo-symmetric totally umbilical submanifold, with parallel mean curvature vector field H, of a Riemannian manifold N admitting a pseudo-symmetric generalized curvature tensor $\tilde{T}$. Then the relations

$$\mu(S - \frac{1}{m} K g) = 0, \tag{10}$$

$$\mu(S(T) - \frac{1}{m} K(T) g) = 0, \tag{11}$$

hold on the set $U_R \cap U_T$, where

$$\mu = L_R - \rho \tag{12}$$

and ρ is defined by (4).

Lemma 6. Let M be a pseudo-symmetric totally umbilical submanifold, with parallel mean curvature vector field H, of a Riemannian manifold N admitting a pseudo-symmetric generalized curvature tensor $\tilde{T}$. If at a point $x \in U_R \cap U_T$ the function μ is non-zero then the relations

$$Z(T)_{hcak} R^{hk}{}_{ef} = \rho Z(T)_{caef}, \tag{13}$$

$$\rho = \frac{K}{m(m-1)} \tag{14}$$

hold on some open set $V \subset U_R \cap U_T$, $x \in V$.

Proof. Let $V \subset U_R \cap U_T$ be a neighbourhood of the point x such that the function μ is non-zero at every point of V. Now, in view of Lemma 5, we obtain on V the equalities

$$S = \frac{1}{m} K g, \tag{15}$$

$$S(T) = \frac{1}{m} K(T)\, g. \tag{16}$$

The relation (4), in virtue of (3), can be written in the form

$$-T_{hbcd}R^{h}{}_{aef} + T_{hacd}R^{h}{}_{bef} - T_{hdab}R^{h}{}_{cef} + T_{hcab}R^{h}{}_{def} = \rho Q(g,T)_{abcdef}. \tag{17}$$

From this we obtain

$$\begin{aligned}
&-(R\cdot T)_{hbcdij}R^{h}{}_{aef} + (R\cdot T)_{hacdij}R^{h}{}_{bef}\\
&-(R\cdot T)_{hdabij}R^{h}{}_{cef} + (R\cdot T)_{hcabij}R^{h}{}_{def}\\
&-T^{h}{}_{bcd}(R\cdot R)_{haefij} + T^{h}{}_{acd}(R\cdot R)_{hbefij}\\
&-T^{h}{}_{dab}(R\cdot R)_{hcefij} + T^{h}{}_{cab}(R\cdot R)_{hdefij} = \rho(R\cdot Q(g,T))_{abcdefij}.
\end{aligned} \tag{18}$$

Applying the relations (4), (2) and

$$R\cdot R = \tau Q(g,R), \quad \tau = L_{R}, \tag{19}$$

to the above equality we get

$$\begin{aligned}
&\rho(- Q(g,T)_{hbcdij}R^{h}{}_{aef} + Q(g,T)_{hacdij}R^{h}{}_{bef} -\\
&- Q(g,T)_{hdabij}R^{h}{}_{cef} + Q(g,T)_{hcabij}R^{h}{}_{def}) +\\
&+ \tau(- T^{h}{}_{bcd}Q(g,R)_{haefij} + T^{h}{}_{acd}Q(g,R)_{hbefij} -\\
&- T^{h}{}_{dab}Q(g,R)_{hcefij} + T^{h}{}_{cab}Q(g,R)_{hdefij}) =\\
&\rho^{2}(g_{af}Q(g,T)_{ebcdij} - g_{ae}Q(g,t)_{fbcdij} - g_{bf}Q(g,T)_{eacdij} +\\
&+ g_{be}Q(g,T)_{facdij} + g_{cf}Q(g,T)_{edabij} - g_{ce}Q(g,T)_{fdabij} -\\
&- g_{df}Q(g,T)_{ecabij} - g_{de}Q(g,T)_{fcabij}).
\end{aligned} \tag{20}$$

Contracting this with g^{jd} and g^{ib} and applying (1),(2),(5),(6),(9),(15),(16) we find

$$
\begin{aligned}
&- 2(m-1)\,\mu\, Z(T)_{hcak}R^{hk}{}_{ef} - 2(m-1)\,\rho^2\, Z(T)_{caef} = \\
&= \tau\big(2\, T_{hcak}R^{hk}{}_{ef} - \frac{2}{m} K\, T_{caef} - \\
&- T^{h}{}_{ec}{}^{k}R_{hfak} + T^{h}{}_{fc}{}^{k}R_{heak} - T^{h}{}_{fa}{}^{k}R_{heck} + T^{h}{}_{ea}{}^{k}R_{hfck}\big).
\end{aligned}
\tag{21}
$$

On the other hand, the contraction of (17) with g^{af}, in virtue of (15) and (9), yields

$$
R^{k}{}_{ec}{}^{h}T_{kbdh} - R^{k}{}_{ed}{}^{h}T_{kbch} = (m-1)\rho Z(T)_{ebcd} - \frac{K}{m}\, T_{ebcd} - T_{hkcd}R^{h}{}_{be}{}^{k}.
$$

Hence we obtain

$$
R^{h}{}_{fa}{}^{k}T_{heck} - R^{h}{}_{fc}{}^{k}T_{heak} = -\,(m-1)\rho Z(T)_{ebcd} - \frac{K}{m}\, T_{acef} - T_{hkac}R^{h}{}_{ef}{}^{k}.
$$

This, by the alternation in (e,f), gives

$$
\begin{aligned}
&R^{h}{}_{fa}{}^{k}T_{heck} - R^{h}{}_{ea}{}^{k}T_{hfck} - R^{h}{}_{fc}{}^{k}T_{heak} + R^{h}{}_{ec}{}^{k}T_{hfak} = \\
&= -\,2(m-1)\rho Z(T)_{acef} + \frac{2K}{m}\, T_{acef} - 2\, T_{hkac}R^{h}{}_{ef}{}^{k}.
\end{aligned}
\tag{22}
$$

Substituting this into (21) we obtain

$$
-(m-1)\mu Z(T)_{hcak}R^{hk}{}_{ef} + (m-1)\rho Z(T)_{caef} = \tau\big(T_{hcak}R^{hk}{}_{ef} + T_{hkac}R^{h}{}_{ef}{}^{k}\big)
$$

which, by (6), turns into (13). We prove now that the condition (14) holds on V. Contracting (20) with g^{ae} and g^{bf} and making use of (2), (5) and (9) we can get

$$
\begin{aligned}
&2\big(- R^{k}{}_{ic}{}^{l}T_{kjdl} + R^{h}{}_{jc}{}^{l}T_{kidl} - R^{k}{}_{jd}{}^{l}T_{kicl} + R^{k}{}_{id}{}^{l}T_{kjcl}\big) - \\
&- R^{kl}{}_{ic}T_{kljd} + R^{kl}{}_{jc}T_{klid} - R^{kl}{}_{jd}T_{klic} + R^{kl}{}_{id}T_{kljc} + \\
&+ g_{dj}R^{lkh}{}_{c}T_{lkhi} - g_{di}R^{lkh}{}_{c}T_{lkhj} + g_{ci}R^{lkh}{}_{d}T_{lkhj} - g_{cj}R^{lkh}{}_{d}T_{lkhi} = 0.
\end{aligned}
\tag{23}
$$

From (13), by contraction with g^{ec} and making use of (15) and (16), it follows

$$R^{hkl}{}_{f}T_{hlka} = \frac{K(T)K}{m^2(m-1)} g_{af}.$$

Further, applying this, (7) and (8) in (23) we obtain

$$R^{h}{}_{fc}{}^{k}T_{heak} - R^{h}{}_{fa}{}^{k}T_{heck} + R^{h}{}_{ae}{}^{k}T_{hcfk} - R^{h}{}_{ce}{}^{k}T_{hafk} =$$

$$= R^{hk}{}_{ec}T_{hafk} - R^{hk}{}_{fc}T_{haek} + R^{hk}{}_{fa}T_{heck} - R^{hk}{}_{ea}T_{hcfk} +$$

$$+ \frac{2\ K(T)\ K}{m^2(m-1)} (g_{ea}g_{cf} - g_{fa}g_{ce}).$$

But the last relation, in view of (22), (6) and (13), turns into

$$(\frac{K}{m(m-1)} - \rho)\ Z(T) = 0.$$

which completes the proof.

Lemma 7. Let M be a pseudo-symmetric totally umbilical submanifold, with parallel mean curvature vector field H, of a Riemannian manifold N admitting a pseudo-symmetric generalized curvature tensor $\tilde{T}$. If at a point $x \in U_R \cap U_T$ the function μ is non-zero then the relation

$$(m-4)(R_{hicd}T^{h}{}_{eab} - \frac{K(T)}{m(m-1)} (g_{ea}R_{bicd} - g_{eb}R_{aicd}) -$$
$$- \rho(g_{di}Z(T)_{abec} - g_{ci}Z(T)_{abed})) = 0 \qquad (24)$$

holds at x.

Proof. The equality (13), by (4) and (19), leads to

$$\rho R^{hk}{}_{ef}Q(g,T)_{hcakij} + \tau T^{h}{}_{ca}{}^{k}Q(g,R)_{hkefij} = Q(g, \frac{K(T)}{m(m-1)} R + \rho^2\ T)_{caefij}.$$

This, in view of (2), (13) and (12), yields

$$- R^{h}{}_{ief}T_{hjac} + R^{h}{}_{jef}T_{hiac} =$$

$$= \frac{K(T)}{m(m-1)} (g_{cj}R_{aief} - g_{ci}R_{ajef} + g_{ai}R_{cjef} - g_{aj}R_{cief}) +$$

$$+ \rho(-g_{ej}Z(T)_{acif} + g_{ei}Z(T)_{acjf} - g_{fi}Z(T)_{acje} + g_{fj}Z(T)_{acie}).$$

The last equality can be rewritten in the form

$$R^h{}_{jcd}T_{hiab} - R^h{}_{icd}T_{hjab} =$$

$$= \frac{K(T)}{m(m-1)} (g_{bj}R_{aicd} - g_{bi}R_{ajcd} + g_{ai}R_{bjcd} - g_{aj}R_{bicd}) + \tag{25}$$

$$+ \rho(-g_{cj}Z(T)_{abid} + g_{ci}Z(T)_{abjc} - g_{di}Z(T)_{abjc} + g_{dj}Z(T)_{abic}).$$

Now, in the same way as above, from (25) we obtain the following relation

$$\tau(Q(g,R)_{hjcdef}T^h{}_{iab} - Q(g,R)_{hicdef}T^h{}_{jab})$$

$$+ \rho(Q(g,T)_{hiabef}R^h{}_{jcd} - Q(g,T)_{hjabef}R^h{}_{icd})$$

$$= \frac{\tau\ K(T)}{m(m-1)} (g_{bj}Q(g,R)_{aicdef} - g_{bi}Q(g,R)_{ajcdef} \tag{26}$$

$$+ g_{ai}Q(g,R)_{bjcdef} - g_{aj}Q(g,R)_{bicdef})$$

$$+\rho^2(-g_{cj}Q(g,T)_{abidef} + g_{ci}Q(g,T)_{abjdef} - g_{di}Q(g,T)_{abjcef} + g_{dj}Q(g,T)_{abicef})$$

From the last equality, by making use of (1),(2) and (12), we find

$$\mu(T_{fiab}R_{ejcd} - T_{eiab}R_{fjcd} + T_{ejab}R_{ficd} - T_{fjab}R_{eicd})$$

$$+ \tau(g_{cf}(R^h{}_{jed}T_{hiab} - R^h{}_{ied}T_{hjab}) - g_{ce}(R^h{}_{jfd}T_{hiab} - R^h{}_{ifd}T_{hjab})$$

$$+ g_{df}(R^h{}_{jce}T_{hiab} - R^h{}_{ice}T_{hjab}) - g_{de}(R^h{}_{jcf}T_{hiab} - R^h{}_{icf}T_{hjab}))$$

$$+ \rho(g_{af}(R^h{}_{jcd}T_{hieb} - R^h{}_{icd}T_{hjeb}) - g_{ae}(R^h{}_{jcd}T_{hifb} - R^h{}_{icd}T_{hjfb})$$

$$g_{bf}(R^h{}_{jcd}T_{hiae} - R^h{}_{icd}T_{hjae}) - g_{be}(R^h{}_{jcd}T_{hiaf} - R^h{}_{icd}T_{hjaf}))$$

$$+ \tau(g_{jf}R^h{}_{ecd}T_{hiab} - g_{je}R^h{}_{fcd}T_{hiab} + g_{ie}R^h{}_{fcd}T_{hjab} - g_{if}R^h{}_{ecd}T_{hjab})$$

$$- \rho(g_{jf}R^{h}{}_{icd}T_{heab} - g_{je}R^{h}{}_{icd}T_{hfab} + g_{ie}R^{h}{}_{jcd}T_{hfab} - g_{if}R^{h}{}_{jcd}T_{heab})$$

$$= \frac{\tau\ K(T)}{m(m-1)} (g_{bj}Q(g,R)_{aicdef} - g_{bi}Q(g,R)_{ajcdef} + g_{ai}Q(g,R)_{bjcdef}$$

$$- g_{aj}Q(g,R)_{bicdef} - \rho^{2} (g_{cj}Q(g,T)_{abidef}$$

$$- g_{ci}Q(g,R)_{abjdef} + g_{di}Q(g,R)_{abjcef} - g_{dj}Q(g,R)_{abicef}).$$

Applying in this (25) and contracting the resulting relation with g^{jf} we get

$$(m-4)\mu(R_{hicd}T^{h}{}_{eab} - \frac{K(T)}{m(m-1)} (g_{ea}R_{bicd} - g_{eb}R_{aicd})$$

$$- \rho(g_{di}Z(T)_{abec} - g_{ci}Z(T)_{abed}))$$

$$+ \mu g_{ei}(R^{hk}{}_{cd}T_{hkab} + \frac{2\ K(T)}{m(m-1)} R_{abcd} + 2\ \rho\ T_{abcd}$$

$$- \frac{2\rho\ K(T)}{m(m-1)} (g_{ad}g_{bc} - g_{ac}g_{bd})) = 0.$$

But the last equality. together with (7) and (13), leads to (24), which completes the proof.

Theorem 3. Let M, $\dim M \neq 4$, be a pseudo-symmetric totally umbilical submanifold, with parallel mean curvature vector field H, of a Reimannian manifold N admitting a pseudo-symmetric generalized curvature tensor $\tilde{T}$. Then the equality

$$L_{R} = L_{\tilde{T}|M} + \tilde{g}(H,H)$$

holds on the set $U_{R} \cap U_{T}$.

Proof. Suppose that $\mu \neq 0$ at a point $x \in U_{R} \cap U_{T}$. In view of Lemma 7 we have

$$R_{hjcd}T^{h}{}_{iab} = \frac{K(T)}{m(m-1)} (g_{bi}R_{jacd} - g_{ai}R_{jbcd})$$

$$+ \rho(g_{dj}Z(T)_{abic} - g_{cj}Z(T)_{abid}),$$

whence

$$\tau(Q(g,R)_{hjcdef}T^{h}{}_{iab} - \rho Q(g,T)_{hiabef}R^{h}{}_{jcd})$$

$$= \frac{\tau\ K(T)}{m(m-1)}(g_{bi}Q(g,R)_{jacdef} - g_{ai}Q(g,R)_{jbcdef})$$

$$+ \rho^{2}(f_{dj}Q(g,T)_{abicef} - g_{cj}Q(g,T)_{abidef}) \quad \text{and}$$

$$U_{efjcdiab} = U_{fejcdiab}, \tag{27}$$

where $U_{efjcdiab} = Z(R)_{ejcd}Z(T)_{fiab}$. Now (27), together with the relation $U_{efjcdiab} = -U_{jfecdiab}$, yields ([3], Lemma 1) $U_{jfjcdiab} = 0$, i.e. $Z(R) = 0$ or $Z(T) = 0$, a contradiction.

Combining Theorem 3 with Lemma 2 we obtain

Theorem 4. Let M, $\dim M \neq 4$, be a pseudo-symmetric totally umbilical submanifold, with parallel mean curvature vector field H, of a Riemannian manifold N admitting a Rici-pseudo-symmetric tensor field $\tilde{A}$. Then the equality

$$L_{R} = L_{\tilde{A}|M} + \tilde{g}(H,H)$$

holds on the set $U_{R} \cap U_{A}$, where A is the orthogonal projection of $\tilde{A}$ on M.

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O PSEUDOSYMETRYCZNYCH CAŁKOWICIE UMBILIKALNYCH PODROZMAITOŚCIACH ROZMAITOŚCI RIEMANNOWSKICH, DOPUSZCZAJĄCYCH PEWNE TYPY UOGÓLNIONYCH TENSORÓW KRZYWIZNY

S t r e s z c z e n i e

Niech M będzie całkowicie umbilikalną podrozmaitością, z równoległym wektorem krzywizny średniej H, rozmaitości N dopuszczającej (0,k) - pseudosymetryczny tensor $\tilde{T}$, przy czym $3 \le \dim M < \dim N$ i $k \in \mathbb{N}$. Wówczas tensor T, będący rzutem ortogonalnym tensora $\tilde{T}$ na M, jest również pseudosymetryczny. Ponadto, jeśli $\tilde{T}$ jest uogólnionym tensorem krzywizny na N,

dim $M \neq 4$ i tensor krzywizny R podrozmaitości M jest pseudosymetryczny, to funkcje stowarzyszone $L_{\tilde{T}}, L_T$ i L_R tensorów $\tilde{T}, T$ i R spełniają na zbiorze $V \subset M$, składającym się ze wszystkich punktów M, w których tensory T i R są jednocześnie nietrywialne, następującą równość

$$L_T = L_{\tilde{T}|M} + \tilde{g}(H,H) = L_R.$$

О ПСЕВДОСИММЕТРИЧЕСКИХ МОЛНОСТЬЮ ОМБИЛИЧЕСКИХ ПОДМНОГООБРАЗИЯХ РИМАНОВЫХ МНОГООБРАЗИИ ДОПУСКАЕМЫХ НЕКОТОРЫЕ ТИПЫ ОБОБЩЕННЫХ ТЕНЗОРОВ КРИВИЗНЫ

Резюме. Пусть M будет полностью омбилическим подмногообразием, с параллельным вектором средней кривизны H, многообразия N допускающего $(0,k)$ - псевдосимметрический тензор $\tilde{T}$ где $3 \leq \dim M < \dim N$ и $k \in \mathbb{N}$. Тогда тензор T, являющийся ортогональной пройекцей тензора $\tilde{T}$ на M, есть также псевдосимметрический тензор. Кроме того, если $\tilde{T}$ есть обобщенный тензор кривизны на N, $\dim M \neq 4$ и тензор кривизны R подмногообразия M псевдосимметрический, тогда объединенные функции $L_{\tilde{T}}, L_T$ и L_R тензоров $\tilde{T}, T$ и R исполняют на множестве $V \subset M$, состоящим из всех точек M, в которых тензоры T и R одновременно нетривиальные, следующее равенство

$$L_T = L_{\tilde{T}|M} + \tilde{g}(H,H) = L_R.$$