ZESZYTY NAUKOWE POLITECHNIKI ŚLĄSKIEJ 1993

Seria: MATEMATYKA-FIZYKA Z. 68 Nr kol. 1147

Erwin KASPAREK

ON THE GEOMETRICAL METRICS OF LOBACHEVSKY'S PLANE IN KLEIN'S MODEL

Summary In this paper we found all the geometrical metrics of Lobachevsky's plane in Klein's model, i.e. in the set $p^* = \{(x,y) \in R^2 : x^2 + y^2 < 1\}$ where Lobachevsky's straight lines are the non-empty intersections of the set P^* and the Euclidean straight lines.
The group G of transformations of the set P^*, determining the congruence relation for segments, acts transitively on the set L^* of all Lobachevsky's straight lines. This fact enables first of all to find the geometrical metrics on some distinguished line l (Theorem 2) and then the formula (1) in the proof of the theorem 1 determines the geometrical metrics on the set P^*. In the corollary we give the form of an arbitrary geometrical metric on the set P^*.

INTRODUCTION

The plane Lobachevsky's geometry is an axiomatic theory based on the axioms of the absolute geometry and on the axiom which is the negation of Euclid's axiom on parallel (comp. [1]).

The primitive notions of this theory are: a non-empty set P whose elements are said to be a points, a family L of the subsets of P whose elements are called a straight lines and two relations of which the first is ternary relation for points, i.e. the betweenness relation, the second is a binary relation for segments, i.e. the congruence relation. The segment is the secondary notion which has the definition in the absolute geometry.
One of the models Lobachevsky's plane is the Klein's model which we determine below (comp. [2]). Let us define in 2-dimensional Cartesian plane R^2 a set P^* of points as follows:

$$P^* = \{(x,y) \in R^2 : x^2 + y^2 < 1 \}.$$

Let Λ be a set of straight lines in R^2 such that for each $\lambda \in \Lambda$ the intersection $\lambda \cap P^*$ is non-empty. Then the family L^* of straight lines

in P^* is defined in the following way: $l \in L^*$ if and only if there exists $\lambda \in \bigwedge$ such that $l = \lambda \cap P^*$. The betweenness relation for points of the set P^* coincides with that one for points in the Euclidean case.
To determine the congruence relation for segments we define a group of transformations of the set P^*. Let us denote by H_o the set all the orthogonal mappings of the plane R^2 which preserve the point $(0,0)$, and by H_1 we denote a one-parameter family of mappings determined as follows:

$$(x,y) \longrightarrow \left(\frac{x\sqrt{1-\gamma^2}}{1+\gamma y}, \frac{y+\gamma}{1+\gamma y}\right)$$

where $\gamma \in R$ and $|\gamma| < 1$.
Let G be the group generated by $H_o \cup H_1$. It is easy to verify that:

(a) If $g \in G$ then $g(P^*) \subset P^*$ and $g(L^*) \subset L^*$. Moreover G is transitive on the sets P^* and L^*.

(b) If $g \in G$ then it preserves the betweenness relation, i.e. if A lies between B and C then $g(A)$ lies between $g(B)$ and $g(C)$.

An ordered pair (A,B) of different points A and B is said to be a segment. The symbol $\equiv$ we will be understood as the congruence relation for segments determined as follows: $(A,B) \equiv (C,D)$ if and olny if there exists $g \in G$ such that $g(A) = C$ and $g(B) = D$.
Now we can define the concept of a geometrical metric on P^*.

Definition 1. A geometrical metric on P^* is said to be every function $f\colon P^* \times P^* \longrightarrow [0,\infty)$ satisfying the following conditions:

1^o $f(A,B) = 0 \iff A=B$,

2^o if $(A,B) \equiv (C,D)$ then $f(A,B) = f(C,D)$,

3^o if A lies between B and C then $f(A,B)+f(A,C)=f(B,C)$ for all $A,B,C,D \in P^*$.

It is easy to show that $(A,B) \equiv (B,A)$. Then from 2^o we have $f(A,B) = f(B,A)$. Moreover in the absolute geometry we can prove that every function $f\colon P \times P \longrightarrow [0,\infty)$ satisfying conditions 2^o and 3^o, for all $A,B,C,D \in P$, yields a topological metric on P (comp. [3]).

To determine all the geometrical metrics on P^* let us distinguish some straight line $l \in L^*$ as follows:

$$l = \{x, y) \in P^* : x=0, \quad y \in (-1,1)\}.$$

By G_o we denote the isotropy group of l, i.e.

$$G_o = \{g \in G : g(l) = l\}.$$

It is evident that G_o is the group genereted by $H_1 \cup \{s_x, s_y\}$, where s_x and s_y are the symetries with respect to the straight lines $y = 0$ and $x=0$ respectively.

Definition 2. A geometrical metric on l is said to be every function $f: l \times l \longrightarrow [0,\infty)$ satisfying conditions $1^o - 3^o$ the definition 1 for all A, B, C, D $\in l$. (of course, the congruence relation for segments is defined by G_o).

Theorem 1. Every geometrical metric on the straight line l is the restriction to $l \times l$ exactly a one geometrical metric on P^*.

Proof. Let F be a geometrical metric on l and A, B an arbitrary distinct points of P^*. By l_{AB} we denote the straight line passing through A and B. Designating as g the element of G such that $g(l_{AB}) = l$ (this follows form (a)) we put

$$F_g(A,B) = F(g(A), g(B)).$$

Note that if $h \in G$ and $h(l_{AB}) = l$ then $F_h(A,B) = F_g(A,B)$.
Indeed, we can write $h = kg$ where $k = hg^{-1}$.
Now we have the following equalities:

$$F_h(A,B) = F(h(A), h(B)) = F(k(g(A)), k(g(B))).$$

Since $k \in G_o$ and F is the geometrical metric on l then we get

$$F = F(k(g(A)), k(g(B))) = F(g(A), g(B)),$$

of which follows that $F_h(A,B) = F_g(A,B)$.

This fact permits us to determine a function $f\colon P^* \times P^* \longrightarrow [0,\infty)$ as follows:

$$f(A,B) = F(g(A),g(B)) \tag{1}$$

where g is an arbitrary element of G such that $g(1_{A,B}) = 1$.
It is easy to verify that f is the geometrical metric on P^* and $F = f_{|1\times 1}$ (the restriction of f to 1×1).
Now if f_1 is a geometrical metric on P^* which is not equal to f then there is a pair (A,B) of points A and B such that $f_1(A,B) \neq f(A,B)$.
By 2° we have

$$f_1(A,B) = f_1(g(A),g(B)) \quad \text{and} \quad f(A,B) = f(g(A),g(B))$$

where $g \in G$ and $g(1_{AB}) = 1$. Since $(g(A),g(B)) \in 1 \times 1$ thus $f_{1|1\times 1} \neq f_{|1\times 1}$ which completes the proof.
Now the line 1 we shall identify with the interval $(-1,1)$.
The function $(0,y) \longrightarrow y$ establishes a one-to-one correspondence between the points $(0,y)$ of the line 1 and the points y of the interval $(-1,1)$. We shall prove, taking into account the above identification, the following
Theorem 2. Every geometrical metric f on 1 has the form

$$f(y_1,y_2) = c\,\left|\ln\frac{(1-y_1)(1+y_2)}{(1-y_2)(1+y_1)}\right| \tag{2}$$

where c is a positive real number. Conversely, for an arbitrary positive real number c the function given by (2) is a geometrical metric on 1.

Proof. Form 2° we have

$$f(y_1,y_2) = f\left(\frac{y_1+\gamma}{1+\gamma y_1},\ \frac{y_1+\gamma}{1+\gamma y_1}\right) \tag{3}$$

for an arbitrary real number γ such that $|\gamma| < 1$ and for all $y_1, y_2 \in (-1,1)$.
If we substitute $\gamma = -y_1$ into formula (3) we obtain,

$$f(y_1,y_2) = f\left(0,\ \frac{y_2-y_1}{1-y_1y_2}\right). \tag{4}$$

Moreover from 2° we have, too

$$f(y_1,y_2) = f(s_x(y_1),\ s_x(y_2)) = f(-y,-y_2). \tag{5}$$

From this fact and (4) we get the equality

$$f\left(0, \frac{y_2-y_1}{1-y_1y_2}\right) = f\left(0, -\frac{y_2-y_1}{1-y_1y_2}\right)$$

of which we conclude that

$$f(y_1, y_2) = f\left(0, \left|\frac{y_2-y_1}{1-y_1y_2}\right|\right). \tag{6}$$

The function given by (6) .satisfies condition 2° because the function $\left|\frac{y_2-y_1}{1-y_1y_2}\right|$ satisfies this condition as well. By 1° we obtain

$$f(0,0) = 0. \tag{7}$$

From 3°, taking into account the formula (6), we get the following implication:

$$-1 < y_1 < t < y_2 < 1 \Longrightarrow f\left(0, \frac{t-y_1}{1-ty_1}\right) + \left(\frac{y_2-t}{1-ty_2}\right) = f\left(0, \frac{y_2,-y_1}{1-y_1y_2}\right). \tag{8}$$

It is easy to verify that the mapping $(x,y) \longrightarrow \frac{x-y}{1-xy}$ maps the set $[(-1,1) \times (-1,1)] \times \{(x,y) \in R^2 : x > y\}$ onto the open interval $(0,1)$. Therefore, if the premise of the implication (8) is satisfied then the real numbers u and v introduced by the formulas

$$u = \frac{t-y_1}{1-ty_1} \quad \text{and} \quad v = \frac{y_2-t}{1-ty_2} \tag{9}$$

belong to the interval (0,1) and satisfy the equality

$$\frac{u+v}{1+uv} = \frac{y_2-y_1}{1-y_1y_2}. \tag{10}$$

Conversely, if $u,v \in (0,1)$ then there exist the real numbers t, y_1 and y_2 satistying equalities (9), (10) and the premise of the implication (8).
Let us also put

$$g(x) = f(0,x), \quad x \in (0,1). \tag{11}$$

Now the conclusion of the implication (8) can be written, taking into account (7), as a functional equation for the function g

$$g(u) + g(v) = g\left(\frac{u+v}{1+uv}\right) \tag{12}$$

for all u, v [0,1).
To solve the equation (12) let us note that the function tanh x (hyperbolic tangent) is invertible and it maps the interval $[0,\infty)$ onto the interval $[0,1)$. Moreover it satisfies the equation

$$\tanh(x+y) = \frac{\tanh x + \tanh y}{1 + \tanh x \tanh y}.$$

Putting $k = g\cdot\tanh$ then the equation (12) is equivalent to the equation $k(x+y) = k(x) + k(y)$ for all $x, y \in [0,\infty)$.
The function k is positive then the solutions of this equation have the form $k(x) = ax$, $a > 0$ (comp. [4]).
From this we find

$$g(x) = \frac{a}{2} \ln \frac{1+x}{1-x}.$$

Finally, by (6) and (11), we get

$$f(y_1, y_2) = \frac{a}{2}\left|\ln \frac{(1-y_1)(1+y_2)}{(1-y_2)(1+y_1)}\right|. \tag{13}$$

It is easy to show that the function given by (13) is a geometrical metric on l.
Corollary. Every geometrical metric r on P* has the form

$$r(p,q) = \frac{a}{2} \ln \frac{1 + m(p,q)}{1 - m(p,q)}$$

where

(i) $a > 0$ and $m(p,q) = \sqrt{1 - \frac{(1-p^2)(1-q^2)}{(1-pq)^2}}$,

(ii) $p^2 = p_1{}^2 + p_2{}^2$, $q^2 = q_1{}^2 + q_2{}^2$, if $p = (p_1 p_2)$ and $q = (q_1, q_2)$,

(iii) $pq = p_1 q_1 + p_2 q_2$.

Proof. From Theorem 1 follows that it suffices to find an element $g \in G$ such that $g(p) = (0,0)$ and $g(q) = (0, s(p,q))$ then the formula (1) in the proof of this theorem determines the metric r as follows: $r(p,q) = f(g(p), g(q))$

where f is the geometrical metric on l given by (13) (the points $g(p)$ and $g(q)$ we identify with the real numbers 0 and $s(p,q)$ respectively).
It can be shown (we omit the calculations) that $s(p,q)$ is one of the numbers $m(p,q)$ or $-m(p,q)$. Now putting in (13) $y_1 = 0$ and $y_2 = m(p,q)$ we obtain required thesis.

REFERENCES

[1] Н.В.Ефимов: Высшая геометрия, Москва 1971,

[2] А.В.Погорелов: Основания геометрии, Москва 1968,

[3] Korsuk K., Szmielew W.: Podstawy geometrii, PWN Warszawa 1975.

[4] Aczel J.: Ein Blick auf Funftionalgleichungen und ihre Anwendungen, Berlin 1962.

METRYKI GEOMETRYCZNE W MODELU KLEINA PŁASKIEJ GEOMETRII ŁOBACZEWSKIEGO

S t r e s z c z e n i e

W pracy zostały wyznaczone wszystkie metryki geometryczne płaszczyzny Łobaczewskiego w modelu Kleina, tzn. w zbiorze $P^* = \{(x,y) \in R^2 : x^2 + y^2 < 1\}$, gdzie proste Łobaczewskiego są niepustą częścią wspólną zbioru P^* i prostych euklidesowych. Grupa G przekształceń zbioru P^*, określająca przystawanie odcinków, działa tranzytywnie na zbiorze L^* wszystkich prostych Łobaczewskiego. Ten fakt pozwala nam najpierw wyznaczyć metryki geometryczne na pewnej wyróżnionej prostej l (twierdzenie 2) a potem poprzez wzór (1) znajdujący się w dowodzie twierdzenia 1 rozszerzyć je na zbiór P^*. We wniosku po twierdzeniu 2 jest podana postać dowolnej metryki geometrycznej w zbiorze P^*.

Резюме: В работе определены все геометрические метрики на плоскости Лобачевского в модели Клейна, т.е. в множестве

$$P^{*} = \{(x,y) \in R^2 : x^2 + y^2 < 1\},$$

где прямые Лобачевского определены как непустые пересечения множества P^{*} с евклидовыми прямыми. Группа G преобразований множества P^{*}, определяющая конгрузнтность отрезков, транзитивно действует на множество всех прямых Лобачевского. Это позволяет нам сначала определить геометрические метрики на некоторой выделенной прямой l (теорема 2), а потом через формулу (1), которая находится в доказательстве теоремы 1 распостранить эти метрики на множество P^{*}. В предложении после теоремы 2 определен вид произвольной геометрической метрики в множестве P^{*}.