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Zbigniew OLSZAK

ON CONFORMALLY RECURRENT MANIFOLDS, I. SPECIAL DISTRIBUTIONS

DEDICATED TO PROFESSOR MIECZYSŁAW KUCHARZEWSKI ON THE OCCASION OF HIS 70TH BIRTHDAY.

Summary. On a pseudo-Riemannian manifold (M,g) of dimension $n \geq 4$, consider the distribution D defined in the following way

$$M \ni p \longmapsto D_p = \{u \in M_p^* \mid \underset{x,y,z}{\zeta}\, u(x)C(y,z) = 0 \text{ for any } x,y,z \in M_p\},$$

where C is the Weyl conformal curvature tensor and ζ indicates the cyclic sum. If $C_p \neq 0$, then $\dim D_p \leq 2$. Assume additionally that (M,g) is conformally recurrent. It is proved that the metric g can always be non-trivially locally conformally deformed to a certain conformally recurrent metric if $\dim D = 1$. And if $\dim D = 2$ and $n \geq 5$, then:
1) the metric g can always be locally deformed conformally to a cerain conformally symmetric metric, 2) the recurrence form of the tensor C is closed and belongs to the distribution D, 3) the Ricci tensor is generated by elements of D. In the forthcoming paper, it will be shown among others that the assertion of the last theorem does not hold when $n = 4$.

1. PRELIMINARIES

Let (M,g) be a pseudo-Riemannian maifold. We always assume that M is of class C^∞, paracompact, connected and $\dim M = n \geq 4$. ∇ is the Levi-Civita connection of (M,g) and $R(X,Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$ is the curvature operator defined for any $X,Y \in \mathfrak{X}(M)$, $\mathfrak{X}(M)$ being the Lie algebra of vector fields on M. Denote by C the Weyl's conformal curvature tensor defined by

$$C(X,Y)Z = R(X,Y)Z - \frac{1}{n-2}\Big\{\rho(Y,Z)X - \rho(X,Z)Y + g(Y,Z)\tilde{\rho}X - \tag{1}$$
$$- g(X,Z)\tilde{\rho}Y\Big\} + \frac{\tau}{(n-1)(n-2)}\Big\{g(Y,Z)X - g(X,Z)Y\Big\}$$

for $X,Y,Z \in \mathfrak{X}(M)$, ρ being the Ricci tensor, $\tilde{\rho}$ the Ricci operator ($\rho(X,Y) = g(\tilde{\rho}X,Y)$) and τ the scalar curvature. Assume additionally that $C(X,Y,Z,W) = g(C(X,Y)Z,W)$ for $X,Y,Z,W \in \mathfrak{X}(M)$.

For a point $p \in M$, let M_p (resp., M_p^*) be the tangent vector (resp., covector) space at p. Let D_p denote the linear subspace of M_p^* defined by

$$D_p = \Big\{u \in M_p^* \,\Big|\, \underset{x,y,z}{\mathfrak{S}}\; u(x)C(y,z) = 0 \quad \text{for any} \quad x,y,z \in M_p\Big\}, \tag{2}$$

where $\mathfrak{S}$ indicates the cyclic sum.

Lemma. Suppose that $C \neq 0$ at a point p of M. Then we have:

1^o. If $0 \neq u \in D_p$, then there exists a symmetric $(0,2)$-tensor S such that

$$C(x,y,z,w) = S(y,z)u(x)u(w) + S(x,w)u(y)u(z) -$$
$$- S(x,z)u(y)u(w) - S(y,w)u(x)u(z)$$

for any $x,y,z,w \in M_p$.

2^o. D_p is an isotropic subspace of M_p^*, that is, arbitrary covectors $u,v \in D_p$ are orthogonal.

3^o. $u(C(x,y)z) = 0$ for any $u \in D_p$ and $x,y,z \in M_p$.

4^o. $\dim D_p \leq 2$.

5°. dim $D_p = 2$ if and only if at the point p it holds that $C = \varepsilon\, \omega \otimes \omega$, where $\varepsilon = \pm 1$ and ω is certain skew-symmetric (0,2)-tensor. The 2-form ω occurring in the above can always be choosing as $\omega = u_1 \wedge u_2$ for some linearly independent $u_1, u_2 \in D_p$.

The assertions of the above lemma are pure algebraic consequences of the definition of the subspace D_p and seem to be known. Therefore, we shall omit the details of the proof.

Remark 1. If dim $M = 4$, then the subspace D_p can be equivalently defined as

$$D_p = \left\{ u \in M_p^* \,\middle|\, u(C(x,y)z) = 0 \quad \text{for any} \quad x,y,z \in M_p \right\}. \tag{3}$$

Indeed, in view of Lemma, p. 3°, it is sufficient to show that in this case the subspace defined by (3) is just that subspace defined by (2). For, recall that any Riemannian as well as pseudo-Riemannian 4-dimensional manifold fulfils the identity (cf. [PA]. eq. (8.7))

$$\mathop{\mathfrak{S}}_{x_1,x_2,x_3} \mathop{\mathfrak{S}}_{y_1,y_2,y_3} g(x_1,y_1)C(x_2,x_3,y_2,y_3) = 0$$

for any $x_i, y_i \in M_p$, $i=1,2,3$. Consequently, in this dimension we have

$$\mathop{\mathfrak{S}}_{x_1,x_2,x_3} \left\{ u(x_1)C(x_2,x_3,y,z) + g(x_1,y)u(C(x_2,x_3)z) - g(x_1,z)u(C(x_2,x_3)y) \right\} = 0$$

for any $x_i, y, z \in M_p$, $i=1,2,3$, and $u \in M_p^*$. Hence the assertion follows.

In this paper we consider conformally recurrent manifolds, i.e., pseudo--Riemannian manifolds (M,g) whose Weyl's conformal curvature tensors C satisfy the following condition: At any $p \in M$, the tensors $\nabla_x C$ and C are linearly dependent for any $x \in M_p$. Let $\hat{M}$ be the set of the points of M at which $C \neq 0$. If (M,g) is conformally recurrent and $\hat{M} \neq \phi$, then it holds on $\hat{M}$ that $\nabla C = A \otimes C$ for certain (uniquely defined) 1-from A, which will be called the recurrence form of C. A conformally recurrent manifold for which $\nabla C = 0$ will be called coformally symmetric (cf. [CG]).

The Roter's paper $[RO]_1$ on conformally related conformally recurrent metrics has stimulated the author to study the subspaces D_p and the distribution $M \ni p \longmapsto D_p$, which will be denoted by D. The following proposition shows that the distribution D has nice properties in certain situations.

Proposition. Let the Weyl's conformal curvature tensor C of a pseudo-Riemannian manifold (M,g) satisfy the condition $\nabla C = A \otimes C$ on the whole of M for certain 1-form A. Then the distribution D has constant dimension on M, and moreover, it is smooth and parallel.

Proof. Since $\nabla C = A \otimes C$, the tensor field C vanishes everywhere or nowhere on M (cf. $[WO]_{1,2}$). We shall concentrate on the second case only, otherwise the assertion is trivial.

At first, let us make the following observation. For arbitrary fixed points $p,q \in M$, let $[0,1] \ni t \longrightarrow \gamma(t) \in M$ be a (smooth) curve joining p and q, $\gamma(0) = p$, $\gamma(1) = q$. Let $(E_1,\dots,E_n)$ be an orthonormal frame consisting parallel vector fields along γ. Define $C_{ijk}{}^l$, $1 \le i,j,k,l \le n$, to be the (smooth) functions on $[0,1]$ such that $C(E_i,E_j)E_k = \sum_{l=1}^{n} C_{ijk}{}^l E_l$. There is a positive function f on $[0,1]$ such that $C_{ijk}{}^l(t) = C_{ijk}{}^l(0) \cdot f(t)$ for $t \in [0,1]$ (cf. $[WO]_2$, p. 477). Denote by $(\omega^1,\dots,\omega^n)$ the dual frame to $(E_1,\dots,E_n)$ along γ. We claim that if $u_0 \in D_p$ and $u_0 = \sum_{i=1}^{n} \lambda_i \omega^i(0)$, then $u = \sum_{i=1}^{n} \lambda_i \omega^i$ the parallel covector field along γ such that $u(0) = u_0$, and $u(t) \in D_{\gamma(t)}$, $t \in [0,1]$.

With the help of the above observation one can deduce that $\dim D_p$ does not depend on $p \in M$, and D is parallel. Of course, D is smooth. Indeed, taking a normal coordinate neighborhood P around a point $p \in M$, fixing a basis of D_p and propagating the basis to each point of P along geodesics starting from p we obtain a smooth basis for D on P. QED.

Now, we formulate the main results of the paper $[RO]_1$ since they will be applied in the present paper.

Theorem A. Let g and $\bar{g} = e^{2f} g$ be two conformally related conformally recurrent metrics on a manifold M, f being a function on M. Denote by A and $\bar{A}$ the recurrence forms of C and $\bar{C}$, respectively.

Then

$$\underset{x,y,z}{\mathfrak{S}} \; df(x)C(y,z) = 0 \tag{4}$$

for any $x,y,z \in M_p$ and any $p \in M$. Moreover, $\bar{A} = A - 4df$ a points at which $C \neq 0$.

Theorem B. Suppose that (M,g) is a conformally recurrent manifold and f a function satisfying (4) on M. Then the metric $\bar{g} = e^{2f}g$ is conformally recurrent too.

2. A CONFORMAL INTERPRETATION OF THE DISTRIBUTION D

Using Theorem A, Lemma and Proposition, we can deduce the following: If (M,g) is a conformally recurrent manifold with $C \neq 0$ at each point of M, and the metric g admits a non-homothetic conformal transformation into a new conformally recurrent metric, then $\dim D = 1$ or 2.

However, it should be added that there are conformally recurrent manifolds for which $D = 0$. Taking the product $M = V_1 \times V_2$ of two surfaces V_1, V_2 such that the sum of the Gauss cuvatures $K_i + K_2$ never vanishes on M we get a 4-dimensional pseudo-Riemannian manifold with the desired property.

In this section we prove that a conformally recurrent metric g for which $\dim D = 1$ or 2 can always be locally conformally transformed into some conformally recurrent metric. The transformations are non-trivial in general.

Theorem 1. Let (M,g) be a conformally recurrent manifold for which the distribution D has constant dimension equal to 1. Then for any $p \in M$ there exists a neighborhood P of p and a function $f: P \longrightarrow R$ such that $df \neq 0$ at each point of P and the metric $\bar{g} = e^{2f}g$ is conformally recurrent on P.

Following Roter (cf. $[RO]_3$) an analytic conformally recurrent manifold (M,g) is said to be special if its metric is locally non-trivially conformal to a non-conformally flat conformally recurrent one. From Theorem 1 we obtain immediately:

Corollary. An analytic conformally recurrent manifold for which the distribution D has constant dimension equal to 1 is special.

We also need another definition given by Roter in $[RO]_2$. A conformally recurrent manifold is called simple if its metric is locally conformal to a non-conformally flat conformally symmetric one.

Theorem 2. Let (M, g) be a conformally recurrent manifold of dimension $n \geq 5$ for which the distribution D has constant dimension equal to 2. Then:

(a) for the recurrence form A we have $A \in D$ and $dA = 0$,

(b) for any $p \in M$ there exists a neighborhood P of p and a function $f: P \longrightarrow \mathbb{R}$ such that the conformally related metric $\bar{g} = e^{2f}g$ is conformally symmetric (this is of special interest at points at which $A \neq 0$), i.e., (M, g) is simple conformally recurrent,

(c) the Ricci tensor ρ of the manifold (M, g) is generated by elements of the distribution D.

Remmark 2. In case of $\dim M = 4$, the assertion of Theorem 2 does not hold in general. An example of 4-dimensional conformally recurrent metrics for which $\dim D = 2$ and $dA \neq 0$ will be constructed in our forthcoming paper.

3. PROOFS OF THE THEOREMS

Proof of Theorem 1. By the assumptions, $C \neq 0$ at each point of M $(\hat{M} = M)$, and the recurrence form A is defined on the whole of M. Fix a point $p \in M$ and choose a 1-form $U \in D$ which is nonzero on some neighborhood, say P_1, of p. Thus we have

$$\underset{X,Y,Z}{\mathfrak{S}} U(X)C(Y,Z) = 0 \tag{5}$$

for $X, Y, Z \in \mathfrak{X}(P_1)$. Differentiating (5) covariantly and using $\nabla_W C = A(W)C$, we find

$$\underset{X,Y,Z}{\mathfrak{S}} (\nabla_W U)(X)C(Y,Z) = 0$$

for $W, X, Y, Z \in \mathfrak{X}(P_1)$. Hence, by the assumption $\dim D = 1$, we must have

$$\nabla_X U = V(X)U \tag{6}$$

for $X \in \mathfrak{X}(P_1)$, where V is certain 1-form on P_1. Consider the distribution ker U defined on P_1. Using (6) and

$$dU(X,Y) = \frac{1}{2}\left\{(\nabla_X U)(Y) - (\nabla_Y U)(X)\right\},$$

we see that $dU = V \wedge U$. By Frobenius theorem, the distribution ker U is integrable. Therefore, we can choose a neighborhood P of p ($P \subset P_1$) and two functions $f, h\colon P \longrightarrow R$ such that $hU = df$ and $h \neq 0$ at each point of P. Now, we have on P: $df \neq 0$ and $df \in D$. To finish the proof it is sufficient to use Theorem B. QED.

Proof of Theorem 2. As in the previous proof, $C \neq 0$ at each point of M and the recurrence form A is defined on M. In the sequel we shall prove that under our assumptions the following two facts hold for (M,g)

$$A \in D, \tag{7}$$

$$dA = 0. \tag{8}$$

This is just the point (a) of our theorem. In virtue of the above facts, to obtain the assertion (b) it is sufficient to take a function f defined on a neighborhood of a point $p \in M$ and such that $A = 4df$. By Theorem B the metric $\bar{g} = e^{2f}g$ is conformally recurrent, and by Theorem A we have $\bar{A} = A - 4df = 0$. Thus, the metric $\bar{g}$ is conformally symmetric. The assertion (c) follows from the formula (23) which will be proved below. QED.

Proof of (7). Fix a point $p \in M$ and choose a neighborhood Q of p with 1-forms $U_1, U_2 \in D|Q$ such that on Q we have $C = \varepsilon\, \omega \otimes \omega$, $\varepsilon = \pm 1$, $\omega = U_1 \wedge U_2$ (cf. Lemma, p. 5°). As it is well-known, the Weyl's conformal curvature tensor C satisfies the identity (cf., e.g., [EI], p. 91)

$$\underset{X,Y,Z}{\zeta}\left\{(\nabla_X C)(Y,Z,W_1W_2) - \frac{1}{n-3}\sum_{n=1}^{n}\varepsilon_i\left\{(\nabla_{E_i} C)(X,Y,W_1,E_i)g(Z,W_2) - (\nabla_{E_i} C)(X,Y,W_2,E_i)g(Z,W_1)\right\}\right\} = 0. \tag{9}$$

for any $X,Y,Z,W_1,W_2 \in \mathfrak{X}(M)$, where $(E_1,\dots,E_n)$ is an orthonormal frame and $\varepsilon_i = g(E_i,E_i)$, $1 \leq i \leq n$. Let U'_α $(\alpha = 1,2)$ denote the contravariant field of U_α, i.e., $g(X,U'_\alpha) = U_\alpha(X)$ for $X \in \mathfrak{X}(Q)$. From (9) we get

$$\underset{X,Y,Z}{\mathfrak{S}}\left\{\omega(W_1,W_2)A(X) + \frac{1}{n-3}\left\{g(W_1,X)\omega(W_2,A') - g(W_2,X)\omega(W_1,A')\right\}\right\}\omega(Y,Z) = 0 \quad (10)$$

for $X, Y, Z, W_1, W_2 \in \mathfrak{X}(Q)$, where A' is the contravariant field of A. Putting $X = U'_\alpha$ into (10) and using $\omega(U'_\alpha, .) = 0$ (this is a consequence of Lemma, p. 2°), we find

$$\omega(W_1,W_2)A(U'_\alpha) + \frac{1}{n-3}\left\{g(W_1,U'_\alpha)\omega(W_2,A') - g(W_2,U'_\alpha)\omega(W_1,A')\right\} = 0 \quad (11)$$

for $W_1, W_2 \in \mathfrak{X}(Q)$. Putting now $W_2 = A'$ into (11) we obtain $(n-4)A(U'_\alpha)\omega(W_1, A') = 0$ for $W_1 \in \mathfrak{X}(Q)$. Since $n \geq 5$, comparing the last relation with (11) one has $U_\alpha(A') = A(U'_\alpha) = 0$. Therefore $\omega(., A') = 0$, which applied to (10) yields

$$\underset{X,Y,Z}{\mathfrak{S}} A(X)\omega(Y,Z) = 0$$

for $X, Y, Z \in \mathfrak{X}(Q)$. Hence it follows that $A \in D$. QED.

Proof of (8). p, Q, ω, U'_α are the same as in the above. From $\nabla C = A \otimes C$ and $C = \varepsilon\, \omega \otimes \omega$ we get $\nabla_X\omega = \frac{1}{2} A(X)\omega$ for $X \in \mathfrak{X}(Q)$. Hence $\nabla^2_{XY}\omega = \frac{1}{2}\left\{(\nabla_X A)(Y) + \frac{1}{2} A(X)\, A(Y)\right\}\omega$ for $X, Y \in \mathfrak{X}(Q)$, and consequently

$$\begin{aligned}&-\omega(R(X,Y)Z,W)-\omega(Z,R(X,Y)W) = (R(X,Y)\omega)(Z,W) = \\ &= (\nabla^2_{XY}\omega - \nabla^2_{YX}\omega)(Z,W) = dA(X,Y)\omega(Z,W).\end{aligned} \quad (12)$$

For convenience, we restricte our considerations to the point p. And so x, y, z, w will be arbitrary vectors tangent to M at p. Moreover, suppose $u_\alpha = (U_\alpha)_p$, $\alpha = 1, 2$. Since $\omega_p = u_1 \wedge u_2$, from (12) we find

$$\begin{aligned}&- u_1(R(x,y)z)u_2(w) + u_2(R(x,y)z)u_1(w) - u_1(z)u_2(R(x,y)w) \\ &+ u_2(z)u_1(R(x,y)w) = (dA)_p(x,y)\{u_1(z)u_2(w) - u_2(z)u_1(w)\}.\end{aligned}$$

This shows that we must have

$$u_\alpha(R(x,y)z) = \frac{1}{n-2}\sum_{\beta=1}^{2} b_\alpha^\beta(x,y)u_\beta(z), \quad (13)_\alpha$$

where $b_\alpha^\beta (1 \le \alpha, \beta \le 2)$ are certain skew-symmetric (0,2)-tensors such that

$$(dA)_p = -\frac{1}{n-2}(b_1^1 + b_2^2). \tag{14}$$

Applying (1) and the third part of Lemma to the identities (13) we get

$$\begin{aligned}&\rho(y,z)u_\alpha(x) - \rho(x,z)u_\alpha(y) + g(y,z)\rho(x,u'_\alpha) - g(x,z)\rho(y,u'_\alpha)\\ &-\frac{\tau}{n-1}\Big\{g(y,z)u_\alpha(x) - g(x,z)u_\alpha(y)\Big\} = \sum_{\beta=1}^{2} b_\alpha^\beta(x,y)u_\beta(z),\end{aligned} \tag{15$_\alpha$}$$

where $u'_\alpha = (U'_\alpha)_p$. Taking $z = u'_2$ into $(15)_1$ we obtain

$$\begin{aligned}&\rho(y,u'_2)u_1(x) - \rho(x,u'_2)u_1(y) + \rho(x,u'_1)u_2(y) - \rho(y,u'_1)u_2(x) =\\ &= \frac{\tau}{n-1}\Big\{u_1(x)u_2(y) - u_2(x)u_1(y)\Big\}.\end{aligned}$$

Hence we see it must hold that

$$\rho(x,u'_\alpha) = u_\alpha(\bar{\rho}\, x) = \sum_{\beta=1}^{2} \lambda_\alpha^\beta\, u_\beta(x), \tag{16$_\alpha$}$$

where λ_α^β are constants such that

$$\lambda_1^1 + \lambda_2^2 = \frac{\tau}{n-1}. \tag{17}$$

From $(16)_\alpha$ and Lemma, p. 2°, it follows easily that

$$\rho(u'_\alpha, u'_\beta) = 0 \quad \text{for} \quad \alpha,\beta = 1,2. \tag{18}$$

Supposing $y = u'_\beta$ into $(15)_\alpha$ for $\alpha,\beta = 1,2$ and using $(16)_\alpha$, (17), (18) and Lemma it is possible to derive the relations

$$\begin{cases} b_1^1(x,u'_1) = (\lambda_1^1 - \lambda_2^2)u_1(x) + \lambda_1^2\, u_2(x),\\ b_1^2(x,u'_2) = \lambda_1^2\, u_2(x), \qquad b_2^1(x,u'_1) = \lambda_2^1\, u_1(x),\\ b_2^2(x,u'_2) = \lambda_2^1\, u_1(x) + (\lambda_2^2 - \lambda_1^1)\, u_2(x). \end{cases} \tag{19}$$

On the other hand, from $(13)_\alpha$ it follows that

$$\rho(x,u'_\alpha) = \frac{1}{n-2}\sum_{\beta=1}^{2} b_\alpha^\beta\,(x,u'_\beta).$$

Hence and from (19) one finds

$$\begin{cases} \rho(x,u'_1) = \frac{1}{n-2}\left\{(\lambda_1^1 - \lambda_2^2)u_1(x) + 2\,\lambda_1^2\,u_2(x)\right\}, \\ \rho(x,u'_2) = \frac{1}{n-2}\left\{2\,\lambda_2^1\,u_1(x) + (\lambda_2^2 - \lambda_1^1)u_2(x)\right\}. \end{cases} \tag{20}$$

Comparing the relations (20) and $(16)_\alpha$, since $n \geq 5$, we get easily $\lambda_\alpha^\beta = 0$ for $1 \leq \alpha,\beta \leq 2$. Therefore, from (17) we have $\tau = 0$, and from (20) $\rho(x,u'_\alpha) = 0$ for $\alpha = 1,2$.

Thus, the equalities $(15)_\alpha$ can be rewritten as follows

$$\rho(y,z)u_\alpha(x) - \rho(x,z)u_\alpha(y) = \sum_{\beta=1}^{2} b_\alpha^\beta(x,y)u_\beta(z). \tag*{$(21)_\alpha$}$$

It can be deduced from $(21)_1$ that

$$b_1^\alpha = 2\,u_1 \wedge v_\alpha \tag{22}$$

for certain covectors v_α, $\alpha = 1,2$. Using the first Bianchi identity it can be deduced from $(13)_1$ that $b_1^1 \wedge u_1 + b_1^2 \wedge u_2 = 0$. Hence, in view of (22), it follows that $u_1 \wedge u_2 \wedge v_2 = 0$. Therefore we have $v_2 = \lambda_1 u_1 + \lambda_2 u_2$ for certain $\lambda_1, \lambda_2 \in \mathbb{R}$. The substitution of the last relation and (22) into $(21)_1$ gives

$$\left\{\rho(y,z) - v_1(y)u_1(z) - \lambda_2 u_2(y)u_2(z)\right\}\, u_1(x) =$$

$$\left\{\rho(x,z) - v_1(x)u_1(z) - \lambda_2 u_2(x)u_2(z)\right\}\, u_1(y).$$

This and the symmetry of ρ imply

$$\rho = \lambda u_1\otimes u_1 + v_1\otimes u_1 + u_1\otimes v_1 + \lambda_2 u_2\otimes u_2,$$

where $\lambda \in \mathbb{R}$. Substituting the last identity into $(21)_2$, we can show that v_1 depends linearly of u_1 and u_2. Consequently, the Ricci tensor ρ has the form

$$\rho = \lambda u_1 \otimes u_1 + \mu(u_1 \otimes u_2 + u_2 \otimes u_1) + \nu u_2 \otimes u_2 \tag{23}$$

for some $\lambda, \mu, \nu \in \mathbb{R}$. Finally, from $(21)_\alpha$ by (23) it follows that

$$b_1^1 = 2\mu\ u_1 \wedge u_2 = 2\mu\omega_p, \quad b_2^2 = -\ 2\mu\omega_p,$$

which used in (14) give $(dA)_p = 0$. Since p was taken as an arbitrary point of M, we have $dA = 0$. QED.

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O ROZMAITOŚCIACH KONFOREMNIE REKURENCYJNYCH, I. DYSTRYBUCJE SPECJALNE

S t r e s z c z e n i e

Na rozmaitości pseudoriemannowskiej (M,g) wymiaru $n \geq 4$ rozważmy dystrybucję D zdefiniowaną następująco:

$$M \ni p \longrightarrow D_p = \{u \in M_p^* \,|\, \underset{x,y,z}{\mathfrak{S}}\ u(x)C(y,z) = 0 \text{ dla dowolnych } x,y,z \in M_p\},$$

gdzie C jest tensorem krzywizny konforemnej Weyla a $\mathfrak{S}$ oznacza sumę cykliczną. Jeśli $C_p \neq 0$, to $\dim D_p \leq 2$. Załóżmy dodatkowo, że rozmaitość (M,g) jest konformnie rekurencyjna. Dowodzi się, że jeśli $\dim D = 1$, to metryka g może być w sposób nietrywialny lokalnie zdeformowana konforemnie do pewnej metryki konforemnie rekurencyjnej. A jeśli $\dim D = 2$ i $n \geq 5$, to: 1) metryka g może być lokalnie zdeformowana konforemnie do pewnej metryki konforemnie symetrycznej, 2) forma rekurencji tensora C jest zamknięta i leży w dystrybucji D, 3) tensor Ricciego jest generowany przez elementy dystrybucji D. W następnej mojej pracy wykazuję m.in., że teza ostatniego twierdzenia nie jest prawdziwa w przypadku, gdy $\dim M = n = 4$.

О КОНФОРМНО РЕКУРРЕНТНЫХ МНОГООБРАЗИЯХ, I. СПЕЦИАЛЬНЫЕ РАСПРЕДЕЛЕНИЯ

Резюме. На псевдоримановом многообразии (M,g) размерности $n \geq 4$ рассмотривается распределение D определенное следующим образом:

$$M \ni p \longmapsto D_p = \{u \in M_p^* \,|\, \underset{x,y,z}{\mathfrak{S}}\ u(x)C(y,z) = 0 \text{ for any } x,y,z \in M_p\},$$

где C - тензор конформной кривизны Вейла, а ζ обозначает циклическую сумму. dim $D_p \le 2$ если $C_p \ne 0$. Кроме того, мы предполагаем, что (M,g) конформно рекуррентное многообразие. Доказывается, что метрика g может быть нетривиальным способом локально конформно деформированная в некаторую конформно рекуррентную метрику в случае когда dim $D = 1$. В случае когда dim $D = 2$ и $n \ge 5$: 1) метрика дможет быть локально конформно деформированная в некоторую конформно симметрическую метрику, 2) форма рекуррентности тензора C замкнута и приналежит к распределению D, 3) злементы распределения D генерируют тензор Риччи метрики д. В следующей работе доказывается между прочим, что тезис последней теоремы неверен, когда $n = 4$.