

Jolanta LIPIŃSKA

## THE CATEGORY OF PSEUDOGRUPPS

**Summary.** In this paper we propose to define a notion of a category of pseudogroups.

We show a covariant functor from this category to the category of topological spaces and a contravariant functor from this category to the category of generalised inverse semigroups which were introduced in [2].

For a given pseudogroup we define the set of its local diffeomorphisms and a subpseudogroup induced to an open set. We also introduce the notion of a pseudogroup closed with respect to the set of its diffeomorphisms. We show two examples:

the set of all homeomorphisms of a given topological space is a pseudogroup closed with respect to the set of its diffeomorphisms while the set which consists of identities on all open sets of a given topological space is not.

We adopt the following notations. For any function  $g$  we denote the domain of  $g$  by  $D_g$ . If  $f$  is a function, then  $f^{-1}$  stands for the inverse relation and  $g \circ f$  stands for the function defined on the set  $f^{-1}(D_g)$  being the inverse image of  $D_g$  given by  $f$ , i.e.  $f^{-1}(D_g)$  is the set of all  $x$  of  $D_f$  such that  $f(x) \in D_g$ . If  $S$  is a set, then  $\text{id}|S$  stands for the identity function defined on  $S$ .

If  $X$  is a topological space, then by  $\underline{X}$  and  $\omega(X)$  we denote the set of all points and the family of all open sets of  $X$ , respectively.

If  $\Gamma$  is a non-empty set of functions then we denote  $\bigcup_{f \in \Gamma} D_f$  by  $\underline{\Gamma}$ ,

$\{f^{-1}; f \in \Gamma\}$  by  $\Gamma^{-1}$ ,  $\{D_f\}_{f \in \Gamma} \cup \{\emptyset\}$  by  $\omega(\Gamma)$ ,  $(\underline{\Gamma}, \omega(\Gamma))$  by  $X_\Gamma$  and  $\{\Gamma'; \emptyset \neq \Gamma' \subset \Gamma \wedge \bigcup \Gamma'$

is a function  $\wedge \bigcup (\Gamma')^{-1}$  is a function} by  $\langle \Gamma \rangle$ .

The following definition of a pseudogroup was introduced in [1].

**Definition 1.** A non-empty set  $\Gamma$  of functions whose domains are non-empty, will be called a pseudogroup of functions if it satisfies the following conditions:

$$1) \bigwedge_{f,g \in \Gamma} (f(D_f) \cap D_g \neq \emptyset \implies g \circ f \in \Gamma),$$

$$2) \bigwedge_{f \in \Gamma} (f^{-1} \in \Gamma),$$

$$3) \bigwedge_{\Gamma' \in \langle \Gamma \rangle} (\cup \Gamma' \in \Gamma).$$

It was found in [1] that if  $\Gamma$  is a pseudogroup of functions then  $X_\Gamma$  is a topological space and  $\Gamma$  is Ehresmann's pseudogroup of transformations on the topological space  $X_\Gamma$ . It is obvious that Ehresmann's pseudogroup also satisfies the axioms of Definition 1.

**Definition 2.** We say that the triple  $(f, \Gamma_1, \Gamma_2)$  is a morphism of the category of pseudogroups (we also write  $f: \Gamma_1 \rightarrow \Gamma_2$  and say that  $f$  is smooth) iff  $f$  is a bijection from  $\underline{\Gamma_1}$  onto  $\underline{\Gamma_2}$  and the condition

$$\bigwedge_{\alpha \in \Gamma_2} f^{-1} \circ \alpha \circ f \in \Gamma_1$$

is satisfied.

We say that  $(f, \Gamma_1, \Gamma_2)$  is diffeomorphism (we write  $f: \Gamma_1 \xrightarrow{\approx} \Gamma_2$ ) iff  $f: \Gamma_1 \rightarrow \Gamma_2$  and  $f^{-1}: \Gamma_2 \rightarrow \Gamma_1$ .

**Proposition 1.** If  $(f, \Gamma_1, \Gamma_2)$  is a morphism of the category of pseudogroups then  $(f, X_{\Gamma_1}, X_{\Gamma_2})$  is a morphism of the category of topological spaces.

**Proof:** Let  $\emptyset \neq V \in \omega(X_\Gamma)$ . Then there exists  $\alpha \in \Gamma_2$  such that  $D_\alpha = V$ .

It follows that  $\text{id}|V \in \Gamma_2^2$  so  $f^{-1} \circ \text{id}|V \circ f \in \Gamma_1$  and  $f^{-1} \circ \text{id}|V \circ f = f^{-1} \circ f \circ \text{id}|f^{-1}(V) = \text{id}|D_f \circ \text{id}|f^{-1}(V) = \text{id}|f^{-1}(V)$ .

Therefore  $f$  is continuous.

Hence we have the covariant functor from the category of pseudogroups to the category of topological spaces. We will show the contravariant functor from the category of pseudogroups to the category of generalised inverse semigroups. The notion of a generalised inverse semigroup was introduced in [2]. It was found in [3] that if  $\Gamma$  is Ehresmann's pseudogroup then  $(\Gamma, \circ_\Gamma)$  is a generalised inverse semigroup, where  $\circ_\Gamma((g, f)) = g \circ f$  for  $(g, f) \in D \circ_\Gamma$

$D_{\circ}\Gamma = \{(g, f) \in \Gamma \times \Gamma; f(D_f) \cap D_g \neq \emptyset\}$ . Therefore for any object of the category of pseudogroups we have the object of the category of generalised inverse semigroups. With any morphism  $(f, \Gamma_1, \Gamma_2)$  of the category of pseudogroups we can assign the triple  $(\Phi(f), (\Gamma_2, \circ_{\Gamma_2}), (\Gamma_1, \circ_{\Gamma_1}))$  where  $\Phi(f)(\alpha) = f^{-1} \circ \alpha \circ f$  for  $\alpha \in \Gamma_2$ .

**Proposition 2.** If  $(f, \Gamma_1, \Gamma_2)$  is a morphism of the category of pseudogroups then  $(\Phi(f), (\Gamma_2, \circ_{\Gamma_2}), (\Gamma_1, \circ_{\Gamma_1}))$  is a morphism of the category of generalised inverse semigroups.

**Proof.** Let  $\alpha, \beta \in \Gamma_2$  and  $\alpha(D_\alpha) \cap D_\beta \neq \emptyset$ . Then  $\Phi(f)(\beta \circ \alpha) = f^{-1} \circ \beta \circ \alpha \circ f = f^{-1} \circ \beta \circ \text{id}|_{\Gamma_2} \circ \alpha \circ f = f^{-1} \circ \beta \circ \text{id}|_{f(D_f)} \circ \alpha \circ f = f^{-1} \circ \beta \circ f \circ f^{-1} \circ \alpha \circ f = \Phi(f)(\beta) \circ \Phi(f)(\alpha)$ .

The proof now follows by a direct verification.

If  $f$  is the identity function then  $\Phi(f)$  is also the identity function. The domain and the range change themselves, the order of composition changes too:

$$\Phi(f_2 \circ f_1) = \Phi(f_1) \circ \Phi(f_2) \text{ because}$$

$$\Phi(f_2 \circ f_1)(\alpha) = (f_2 \circ f_1)^{-1} \circ \alpha \circ (f_2 \circ f_1) = f_1^{-1} \circ f_2^{-1} \circ \alpha \circ f_2 \circ f_1 =$$

$$f_1^{-1} \circ (f_2^{-1} \circ \alpha \circ f_2) \circ f_1 = \Phi(f_1) \circ \Phi(f_2).$$

Hence we have the contravariant functor from the category of pseudogroups to the category of generalised inverse semigroups.

Let  $\emptyset \neq U \in \omega(\Gamma)$  where  $\Gamma$  is a pseudogroup. Then the set  $\{f \in \Gamma; D_f \subset U \wedge f(D_f) \subset U\}$  we denote by  $\Gamma|U$  and call a subpseudogroup induced to an open set. The set  $\{f; f : \Gamma|U \xrightarrow{\sim} \Gamma|V \wedge U, V \in \omega(\Gamma)\}$  we denote by  $\text{Diff } (\Gamma)$ . We say that a pseudogroup is closed with respect to the set of its local diffeomorphisms if  $\text{Diff } (\Gamma) \subset \Gamma$ .

**Example 1.** Let  $\Gamma$  be the set of all homeomorphisms of a given topological space. From Proposition 1 it follows that  $\text{Diff } (\Gamma) \subset \Gamma$ .

**Example 2.** Let  $X$  be a topological space and  $f$  a homeomorphism  $X$  onto  $X$  but not identity. Let  $\Gamma$  be the set  $\{\text{id}|U; U \in \omega(X)\}$ . It is obvious that  $\Gamma$  is a pseudogroup,  $f \in \text{Diff } (\Gamma)$  but  $f \notin \Gamma$ .

## REFERENCES

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## KATEGORIA PSEUDOGRUP

## S t r e s z c z e n i e

W artykule tym proponujemy pojęcie kategorii pseudogrup. Przyjmujemy definicję pseudogrupy tak, jak to było podane w [1], i jest ona równoważna definicji Ehresmanna. Definiujemy morfizmy tej kategorii jako funkcje  $f$ , które są bijekcjami i spełniają następujący warunek: dla każdego  $\alpha$ , które należy do pseudogrupy  $\Gamma_2$ , złożenie  $f^{-1} \circ \alpha \circ f$  należy do  $\Gamma_1$ . Podajemy funktor kowariantny z tej kategorii do kategorii przestrzeni topologicznych i funktor kontrawariantny z tej kategorii do kategorii uogólnionych półgrup inwersyjnych, które były wprowadzone w [2]. Dla danej pseudogrupy definiujemy zbiór jej dyfeomorfizmów i podpseudogrupę indukowaną do zbioru otwartego. Wprowadzamy pojęcie pseudogrupy zamkniętej ze względu na zbiór swoich dyfeomorfizmów. Podajemy dwa przykłady: zbiór wszystkich homeomorfizmów danej przestrzeni topologicznej jest pseudogrupą zamkniętą ze względu na zbiór swoich dyfeomorfizmów, a zbiór identyczności na wszystkich otwartych zbiorach danej przestrzeni topologicznej nie jest.

## КАТЕГОРИЯ ПСЕВДОГРУПП

**Резюме.** В работе введено понятие категории псевдогрупп. Принимается определение псевдогруппы, как это было сделано в [1]. Это определение эквивалентно определению Эресмана. Морфизмы этой категории определены как биективные преобразования  $f$ , которые обладают следующим свойством: для всех  $\alpha$ , которые принадлежат псевдогруппе  $G_2$  сложное преобразование  $f^{-1} \circ \alpha \circ f$  принадлежит псевдогруппе  $G_1$ . Указан ковариантный функтор переводящий эту категорию в категорию топологических пространств и контравариантный функтор переводящий в категорию обобщенных инверсных полугрупп. Введено понятие подпсевдогруппы индуцированной в открытом множестве и понятие псевдогруппы замкнутой по отношению к множеству своих диффеоморфизмов. Представлены два примера.