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ON AN EXTENSION OF THE SPACE OF TEMPERED DISTRIBUTIONS

Summary. In the paper, a class of distributions wider than the space of all tempered distributions is introduced so that the Fourier transform can be defined for all distributions of this class.

O ROZSZERZENIU PRZESTRZENI DYSTRYBUCJI TEMPEROWANYCH

Streszczenie. W pracy wprowadza się klasę dystrybucji szerszą od przestrzeni wszystkich dystrybucji temperowanych o tej własności, że dla wszystkich dystrybucji z tej klasy można określić transformaty Fouriera.

О РАСШИРЕНИИ ПРОСТРАНСТВА ОБОБЩЕННЫХ ФУНКЦИЙ МЕДЛЕННОГО РОСТА

Резюме. В работе вводится класс обобщенных функций шире пространства всех обобщенных функций медленного роста такой, что для всех функций этого класса можно определить преобразование Фурье.

1. Preliminaries

It is very well known (see [3]) that the Fourier transforms $\mathcal{F}(f)$ can be defined for all $f \in \mathcal{S}'$ so that $\mathcal{F}(f) \in \mathcal{S}'$ or for all $f \in \mathcal{D}'$ in such a way that $\mathcal{F}(f) \in \mathcal{Z}'$ (see e.g. [1]). But $\mathcal{Z}' \not\subset \mathcal{D}'$, which means that the Fourier transforms $\mathcal{F}(f)$ so defined for $f \in \mathcal{D}'$ are not distributions, in general. It is natural to ask whether the definition of the Fourier transform may be extended to a subspace $\mathcal{T}$ of $\mathcal{D}'$, larger than $\mathcal{S}'$ (i.e. $\mathcal{S}' \subset \mathcal{T} \subset \mathcal{D}'$), in such a way that $\mathcal{F}(f) \in \mathcal{D}'$ for $f \in \mathcal{T}$. In [4], the affirmative answer to this question is given, but the proof shown there contains an essential gap which cannot be removed. We are going to present in this note a completely different construction using a simple idea based on the notion of delta-sequence. The subspace $\mathcal{T}$ constructed in this way will be called the space of transformable distributions.

We shall also consider a subspace $\mathcal{T}_0$ of $\mathcal{T}$ consisting of all so-called regularly transformable distributions. In section 5, we prove that the standard formula $\mathcal{F}\mathcal{F}(f) = f^-$ is true for all regularly transformable distributions. Here and in the sequel f^- denotes the distribution such that $< f^-, \phi > := < f, \phi^- >$, where $\phi^-(t) = \phi(-t)$ for $\phi \in \mathcal{D}$.

All functions and distributions in the paper are assumed to be complex-valued and defined on the real line, but all considerations can be easily transfered to the case of n-dimensional Euclidean space.

The notation used in this paper will be mostly standard, e.g. $\mathbf{R}$ will denote the reals, $\mathbf{N}$ the set of all positive integers, $\mathbf{N}_0$ the set $\mathbf{N} \cup \{0\}$, and the symbols $\mathcal{D}, \mathcal{S}, \mathcal{Z}$ will denote the respective spaces of test functions while $\mathcal{D}', \mathcal{S}', \mathcal{Z}'$ their duals endowed with the standard topologies (see [3] and [1]). By $\mathcal{C}$ and $\mathcal{C}^\infty$ we denote the spaces of all continuous and all smooth (i.e. diferentiable infinitely many times) functions, respectively, with the topology defined by the almost uniform convergence of (F_n) in case of $\mathcal{C}$ and of $(F_n^{(k)})$ for $k \in \mathbf{N}_0$ in case of $\mathcal{C}^\infty$.

Given an $F \in \mathcal{C}$ define the following sequence of norms:

$$\|F\|_k := \sup\{(1 + |t|)^{-k} |f(t)| : \ t \in \mathbf{R}\}, \quad k \in \mathbf{N}_0. \tag{1}$$

Instead of the Schwartz space O_M of all slowly increasing smooth functions we shall consider the space

$$O(M) := \{F \in \mathcal{C} : \ \|F\|_k < \infty \ \text{ for } \ k \in \mathbf{N}_0\},$$

i.e.

$$O(M) = \bigcap_{k \in \mathbf{N}_0} O(k),$$

where

$$O(k) := \{F \in \mathcal{C} : \ \|F\|_k < \infty\}, \qquad k \in \mathbf{N}_0.$$

Let the space $O(M)$ be endowed with the topology defined by the norms (1).

By the *Fourier transform* of a function $\phi \in \mathcal{S}$ we mean

$$\hat{\phi}(t) := \mathcal{F}(\phi)(t) := (2\pi)^{-1/2} \int_{-\infty}^{\infty} e^{-itx} \phi(x)\, dx \tag{2}$$

and by the *inverse Fourier transform*

$$\check{\phi}(t) := \mathcal{F}^{-1}(\phi)(t) := (2\pi)^{-1/2} \int_{-\infty}^{\infty} e^{itx} \phi(x)\, dx. \tag{3}$$

It will be convenient for us to use both symbols introduced in (2) and (3) for the Fourier transform as well as for its inverse. It is well known (cf. [2], p. 180) that

$$\mathcal{F}^{-1}\mathcal{F}(\phi) = \mathcal{F}\mathcal{F}^{-1}(\phi) = \phi; \qquad \mathcal{F}\mathcal{F}(\phi) = \phi^{-}$$

for $\phi \in \mathcal{S}$. For $f \in \mathcal{S}'$, we define traditionally

$$< \hat{f}, \phi > := < f, \hat{\phi} >; \qquad < \check{f}, \phi > := < f, \check{\phi} > \qquad\qquad \phi \in \mathcal{S}.$$

By τ_λ for $\lambda \in \mathbf{R}$ we denote the *shift-operator*, i.e.

$$(\tau_\lambda \phi)(t) := \phi(t - \lambda); \qquad < \tau_\lambda f, \phi > := < f, \tau_\lambda \phi >$$

for $f \in \mathcal{D}'$ and $\phi \in \mathcal{D}$.

By a *delta-sequence* we mean a sequence (δ_n) of smooth functions such that there are a sequence (α_n) of positive numbers with $\alpha_n \to 0$ as $n \to \infty$ and a positive constant K so that the following conditions are satisfied:

1° $\operatorname{supp} \delta_n \subset [-\alpha_n, \alpha_n]$ for $n \in \mathbf{N}$;

2° $\int_{-\infty}^{\infty} \delta_n(t)dt = 1$ for $n \in \mathbf{N}$;

3° $\int_{-\infty}^{\infty} |\delta_n(t)|dt < K$ for $n \in \mathbf{N}$.

2. Transformable distributions

Let us start with the following definition:

Definition 1. *A distribution $f \in \mathcal{D}'$ is said to be transformable if there exists a delta-sequnce (δ_n) such that*

$$f * \delta_n \in \mathcal{S}' \qquad \text{for } n \in \mathbf{N}. \tag{4}$$

The set of all transformable distributions will be denoted by $\mathcal{T}$.

Remark 1. *Evidently, $f \in \mathcal{T}$ if and only if there exists a delta-sequence (δ_n) such that $f * \delta_n \in O(M)$ for $n \in \mathbf{N}$.*

Given an $f \in \mathcal{T}$, denote

$$\mathcal{D}(f) := \{\phi \in \mathcal{D} : \ f * \phi \in \mathcal{S}'\}.$$

Lemma 1. *If $f \in \mathcal{T}$, then there is a unique distribution g such that*

$$\mathcal{F}(f * \phi) = (2\pi)^{1/2} g \cdot \mathcal{F}(\phi) \tag{5}$$

for every $\phi \in \mathcal{D}(f)$.

Proof. Let (δ_n) be a delta-sequence satisfying (4). We shall prove that the sequence $(\mathcal{F}(f * \delta_n))$ is convergent in $\mathcal{D}'$.

Fix $\phi \in \mathcal{D}$ assuming that $\operatorname{supp} \phi \subset [-T, T]$. Since $\hat{\delta}_n \to (2\pi)^{-1/2}$ as $n \to \infty$ in $\mathcal{C}^\infty$, there exists an $n_0 \in \mathbf{N}$ such that

$$|\hat{\delta}_n(t)| > \frac{1}{2\sqrt{2\pi}} \qquad \text{for } n \geq n_0, \; t \in [-T, T].$$

Fixing $n, m \geq n_0$, we have

$$\begin{aligned} < \mathcal{F}(f * \delta_n), \phi \cdot \hat{\delta}_n^{-1} > &= (2\pi)^{-1/2} < \mathcal{F}(f * \delta_n * \delta_m), \phi \cdot \hat{\delta}_n^{-1} \cdot \hat{\delta}_m^{-1} > \\ &= < \mathcal{F}(f * \delta_m), \phi \cdot \hat{\delta}_m^{-1} > . \end{aligned}$$

This means that the distribution g defined by

$$< g, \phi > := (2\pi)^{-1/2} < \mathcal{F}(f * \delta_n), \phi \cdot \hat{\delta}_n^{-1} > \tag{6}$$

does not depend on the index n (provided that it is chosen sufficiently large). Let $(\bar{\delta}_n)$ be another delta-sequence satisfying (4). Then the interlaced sequence (ϱ_n) defined by

$$\varrho_{2n-1} := \delta_n; \qquad \varrho_{2n} := \bar{\delta}_n \qquad \text{for } n \in \mathbf{N}$$

is also a delta-sequence satisfying (4). Now replacing (δ_n) by (ϱ_n) in (6) and using the consistency property we have just proved, we infer that the distribution g does not depend on a delta-sequence (δ_n) satisfying (4), either.

Since, for an arbitrary $\phi \in \mathcal{D}$,

$$< \mathcal{F}(f * \delta_n), \phi > = < \mathcal{F}(f * \delta_m) \cdot \hat{\delta}_n, \phi \cdot \hat{\delta}_m^{-1} > = (2\pi)^{1/2} < g, \phi \cdot \hat{\delta}_n >,$$

we infer that $\mathcal{F}(f * \delta_n) \to g$ in $\mathcal{D}'$ as $n \to \infty$.

Now, if $\phi \in \mathcal{D}(f)$, then

$$\mathcal{F}(f * \phi) \cdot \hat{\delta}_n = \mathcal{F}(f * \delta_n) \cdot \hat{\phi} \to g \cdot \mathcal{F}(\phi) \qquad \text{in } \mathcal{D}'$$

as $n \to \infty$. On the other hand, $\hat{\delta}_n \to (2\pi)^{-1/2}$ in $\mathcal{C}^\infty$ as $n \to \infty$. Since $\mathcal{F}(f * \phi) \in \mathcal{S}'$, this implies that $\mathcal{F}(f * \phi) \cdot \hat{\delta}_n \to (2\pi)^{-1/2} \mathcal{F}(f * \phi)$ in $\mathcal{S}'$ and the more in $\mathcal{D}'$. Consequently, (5) is valid for g defined above.

Suppose now that

$$\mathcal{F}(f * \phi) = (2\pi)^{\frac{1}{2}} g_i \mathcal{F}(\phi) \qquad (i = 1, 2) \tag{7}$$

for certain distributions g_1, g_2 and for all $\phi \in \mathcal{D}(f)$. Let (δ_n) be a delta-sequence such that $\delta_n \in \mathcal{D}(f)$ for $n \in \mathbf{N}$. By (7), $g_1 \cdot \hat{\delta}_n = g_2 \cdot \hat{\delta}_n$ for $n \in \mathbf{N}$, so passing to the limit as $n \to \infty$ we get $g_1 = g_2$ and this completes the proof. □

Definition 2. *If $f \in \mathcal{T}$, then the unique distribution g satisfying (5) will be called the Fourier transform of f and denoted by $\mathcal{F}(f)$ or by $\hat{f}$.*

It is clear that the set $\mathcal{T}$ is a subspace of $\mathcal{D}'$ which contains the space $\mathcal{S}'$ of all tempered distributions. Moreover, the Fourier transform $\mathcal{F}(f)$ defined above for $f \in \mathcal{T}$ coincides with the classical definition for $f \in \mathcal{S}'$, so we can use the traditional notation $\mathcal{F}$ for the Fourier transform of distributions of the class $\mathcal{T}$.

To show an example of a distribution $f \in \mathcal{T}$ which is not a tempered distribution we need some definitions.

Let $\mathbf{M} = (M_n)$ be a sequence of positive numbers such that

$$M_0 = 1; \qquad M_n^2 \leq M_{n-1} M_{n+1} \quad (n \in \mathbf{N}). \tag{8}$$

Define

$$\mathcal{D}(\mathbf{M}) := \{\phi \in \mathcal{D} : \ \exists \lambda > 0 \ \ p_{\lambda,\mathbf{M}}(\phi) < \infty\},$$

where

$$p_{\lambda,\mathbf{M}}(\phi) := \sup\{\lambda^{-k} M_k^{-1} \|\phi^{(k)}\| : \ k \in \mathbf{N}_0\}.$$

Definition 3. *Let $\mathbf{M} = (M_k)$ be a sequence of positive numbers satisfying (8) such that $\mathcal{D}(\mathbf{M})$ is not quasi-analytic. A distribution f is said to be $\mathbf{M}$-transformable if $f * \phi \in \mathcal{S}'$ for every $\phi \in \mathcal{D}(\mathbf{M})$.*

Remark 2. *It is known (see e.g. [2], p. 376) that $\mathcal{D}(\mathbf{M})$ is not quasi-analytic (i.e. $\mathcal{D}(\mathbf{M}) \neq \emptyset$) if and only if*

$$\sum_{k=1}^{\infty} \frac{M_{k-1}}{M_k} < \infty.$$

One can easily prove that if $\mathcal{D}(\mathbf{M})$ is not quasi-analytic, then $\mathcal{D}(\mathbf{M})$ contains a delta-sequence. Therefore every $\mathbf{M}$-transformable distribution is transformable.

Example. Let $\mathbf{M}$ be as in Definition 3. and (α_n) be a sequence of complex numbers such that $|\alpha_n| \leq M_k^{-1} k^{-k}$ for $k \in \mathbf{N}$. We shall prove that the distribution defined by

$$f := \sum_{k=0}^{\infty} \alpha_k \delta^{(k)},$$

where δ is the Dirac delta-distribution, is an M-transformable distribution.

Let ϕ be an arbitrary function in $\mathcal{D}(\mathbf{M})$. Then there exist $\lambda > 0$ and $C > 0$ such that

$$\|\phi^{(k)}\| \le C\lambda^k M_k, \qquad \text{i.e.} \qquad \|\alpha_k \phi^{(k)}\| \le C\lambda^k k^{-k}$$

for $k \in \mathbf{N}_0$. Consequently, the series $\sum_{k=0}^{\infty} \alpha_k \phi^{(k)}$ converges uniformly to a continuous, bounded function. This means that

$$f * \phi = \sum_{k=0}^{\infty} \alpha_k \phi^{(k)} \in \mathcal{S}'$$

and hence f is an M-transformable distribution.

On the other hand, $f \notin \mathcal{S}'$, because f is a distribution of infinite order.

3. Properties of the space $\mathcal{T}$

We begin this section with the definition:

Definition 4. *Let $f \in \mathcal{T}$, $F \in \mathcal{C}$ and $\phi \in \mathcal{S}$. We say that the convolution $f * \phi$ exists and equals F if*

$$(f * \psi) * \phi = F * \psi$$

for each $\psi \in \mathcal{D}(f)$. We shall write $f * \phi \in \mathcal{C}$ if there exists an $F \in \mathcal{C}$ such that $f * \phi$ exists and equals F.

Given an $f \in \mathcal{T}$, denote

$$\mathcal{S}(f) := \{\phi \in \mathcal{S} : \ f * \phi \in \mathcal{C}\}.$$

Notice that if $f \in \mathcal{T}$ and $\phi \in \mathcal{S}(f)$, then $f * \phi \in \mathcal{T}$ and

$$\mathcal{F}(f * \phi) = \mathcal{F}(f)\,\mathcal{F}(\phi).$$

Lemma 2. *Let $f \in \mathcal{T}$. Then $\mathcal{S}(f)$ is a shift-invariant subspace of $\mathcal{S}$ containing the spaces $\mathcal{D}$ and $\mathcal{Z}$.*

Proof. That $\mathcal{S}(f)$ is a shift-invariant subspace of $\mathcal{S}$ and $\mathcal{D} \subset \mathcal{S}(f)$ is obvious.

Let $\phi \in \mathcal{Z}$. Then $\hat{f} \cdot \hat{\phi}$ is a distribution of bounded support, so $F := (2\pi)^{1/2}\mathcal{F}^{-1}(\hat{f} \cdot \hat{\phi})$ is a continuous function. For an arbitrary $\psi \in \mathcal{D}(f)$, we have

$$F * \psi = \mathcal{F}^{-1}(\hat{f} \cdot \hat{\phi} \cdot \hat{\psi}) = (2\pi)^{1/2}\mathcal{F}^{-1}(\hat{f} \cdot \hat{\phi}) * \psi = (f * \psi) * \phi,$$

which means that $f * \phi = F$, by Definition 4.. Consequently, $\phi \in \mathcal{S}(f)$ and the inclusion $\mathcal{Z} \subset \mathcal{S}(f)$ is proved. □

We endow the space $\mathcal{S}(f)$ for an arbitrary $f \in \mathcal{T}$ with the topology defined by the convergence described as follows:

Definition 5. *Let $\phi_n, \phi \in \mathcal{S}(f)$ for $n \in \mathbf{N}$. We say that $\phi_n \to \phi$ in $\mathcal{S}(f)$ if the two conditions are valid:*

1° $\phi_n \to \phi$ *in* $\mathcal{S}$ *as* $n \to \infty$;

2° $f * \phi_n \to f * \phi$ *in* $\mathcal{C}$ *as* $n \to \infty$.

Remark 3. *It is easy to see that the convergence in $\mathcal{D}$ and $\mathcal{Z}$ implies the convergence in $\mathcal{S}(f)$ for $f \in \mathcal{T}$.*

Lemma 3. *Let $f \in \mathcal{T}, \phi_n, \phi \in \mathcal{S}(f)$ and $\phi_n \to \phi$ in $\mathcal{S}(f)$ as $n \to \infty$. Then*

(a) there exists a delta sequence (δ_k) such that $\phi_n * \delta_k \to \phi * \delta_k$ in $\mathcal{S}(f)$ as $n \to \infty$ for each $k \in \mathbf{N}$;

(b) there exists a delta-sequence (ϱ_n) such that $\phi_n * \varrho_n \to \phi$ in $\mathcal{S}(f)$ as $n \to \infty$.

Proof. Let (δ_k) be a delta-sequence such that $g_k := f * \delta_k \in \mathcal{S}'$ for $k \in \mathbf{N}$. Then

$$f * (\phi_n * \delta_k) \;=\; g_k * \phi_n \;\;\to\;\; g_k * \phi \;=\; f * (\phi * \delta_k) \qquad \text{in } \mathcal{C}$$

as $n \to \infty$ for every $k \in \mathbf{N}$. On the other hand, $\phi_n * \delta_k \to \phi * \delta_k$ in $\mathcal{S}$ as $n \to \infty$. Consequently, property (a) is proved. Property (b) follows easily from (a). □

The following assertion is a direct consequence of Lemma 3..

Corollary. *Let $f \in \mathcal{T}$ and let $E \subset \mathcal{S}(f)$ be a dense subspace of $\mathcal{S}$. Assume that E has the following property: if $\phi \in E$ and $\psi \in \mathcal{D}$, then $\phi * \psi \in E$. Then E is dense in $\mathcal{S}(f)$.*

In particular, the space $\mathcal{D}$ amd $\mathcal{Z}$ are dense in $\mathcal{S}(f)$.

Given an $f \in \mathcal{T}$, define

$$< f, \phi > := (f * \phi^{-})(0)$$

for every $\phi \in \mathcal{S}(f)$.

Remark 4. *Let $f \in \mathcal{T}$ and let (δ_n) be a delta-sequence such that $f * \delta_n \in \mathcal{S}'$ for $n \in \mathbf{N}$. Then, obviously,*

$$< f * \delta_n, \phi > \to < f, \phi >$$

as $n \to \infty$ for every $\phi \in \mathcal{D}$.

Lemma 4. *Let $f \in \mathcal{T}$. Then*

$$< \mathcal{F}(f), \phi > \; = \; < f, \mathcal{F}(f) >$$

for each $\phi \in \mathcal{D}$.

Proof. Let $\phi \in \mathcal{D}$ and let (δ_n) be a delta-sequence such that $f * \delta_n \in \mathcal{S}'$ for $n \in \mathbf{N}$. Then

$$< \mathcal{F}(f * \delta_n), \phi > \; \to \; < \mathcal{F}(f), \phi >$$

as $n \to \infty$. On the other hand,

$$< \mathcal{F}(f * \delta_n), \phi > = < f * \delta_n, \mathcal{F}(\phi) > \; \to \; < f, \mathcal{F}(\phi) >$$

as $n \to \infty$ and the assertion follows. □

4. Regularly transformable distributions

First let us introduce the convergence in $\mathcal{T}$.

Definition 6. *Suppose that $f_n, f \in \mathcal{T}$ for $n \in \mathbf{N}$. We say that $f_n \to f$ in $\mathcal{T}$ as $n \to \infty$ if the following two conditions are fulfilled:*

1° $f_n \to f$ *in $\mathcal{D}'$ as $n \to \infty$;*

2° *there exists a delta-sequence (δ_k) such that $f_n * \delta_k \to f * \delta_k$ in $\mathcal{S}'$ for every $k \in \mathbf{N}$ as $n \to \infty$.*

For an arbitrary $\lambda \in \mathbf{R}$ denote

$$e_\lambda(t) := e^{i\lambda t}, \qquad t \in \mathbf{R}.$$

It is easy to see that $f \in \mathcal{T}$ implies $e_\lambda f \in \mathcal{T}$ for any $\lambda \in \mathbf{R}$.

Definition 7. *We say that a distribution f is regularly transformable if $f \in \mathcal{T}$ and the mapping*

$$\mathbf{R} \ni \lambda \mapsto e_\lambda f \in \mathcal{T}$$

is $\mathcal{T}$-continuous, i.e. $\lambda_n \to \lambda$ implies $e_{\lambda_n} f \to e_\lambda f$ in $\mathcal{T}$ as $n \to \infty$.

Remark 5. *One can prove that every* M*-transformable distribution is regularly transformable.*

In the two lemmas below the following notation will be used:

$$\Delta(\varepsilon) := \{\phi \in \mathcal{D} : \ \mathrm{supp}(\phi) \subset [-\varepsilon, \varepsilon], \ \int_{-\infty}^{\infty} \phi(t)dt = 1, \ \int_{-\infty}^{\infty} |\phi(t)|dt \leq 2\}$$

for an arbitrary $\varepsilon > 0$.

Lemma 5. *Let X be a metric space, $T : X \to \mathcal{T}$ be a continuous function. Then, given an arbitrary λ_0 in X and $\varepsilon > 0$, there exist a neighbourhood U of λ_0, an integer $k_0 \in \mathbf{N}$ and a positive constant K_0 such that, for arbitrary $n \in \mathbf{N}$ and $\lambda_1, \ldots, \lambda_n \in U$, there is a function $\psi \in \Delta(\varepsilon/2)$ such that*

$$T(\lambda_i) * \psi \in O(k_0) \qquad \text{and} \qquad \|T(\lambda_i) * \psi\|_{k_0} \leq K_0$$

for $i = 1, \ldots, n$.

Proof. Assume that the assertion is not true. Then there are a sequence (λ_n) in X, $\lambda_n \to \lambda_0$ as $n \to \infty$, and an increasing sequence (p_n) in $\mathbf{N}$ with the following property: for every $k \in \mathbf{N}$ there is an $i \in \{p_k, p_k + 1, \ldots, p_{k+1} - 1\}$ such that

$$\|T(\lambda_i) * \psi\|_i \geq i, \tag{9}$$

whenever $\psi \in \Delta(\varepsilon/2)$ and $T(\lambda_n) * \psi \in O(M)$ for $n \in \mathbf{N}$. By the continuity of T, there are a function $\psi \in \Delta(\varepsilon/2)$ and an integer $m_0 \in \mathbf{N}$ such that the sequence $(T(\lambda_n) * \psi)$ is convergent in $O(m_0)$, so the sequence $(T(\lambda_n) * \psi_{m_0})$ is bounded. This contradicts (9). □

Lemma 6. *Let X be a metric space and $T: X \to \mathcal{T}$ be a continuous function. Then for an arbitrary λ_0 in X there exists a delta-sequence (δ_k) and a sequence (U_k) of neighbourhoods of λ_0 such that the two conditions are satisfied:*

(*a*) $T(\lambda) * \delta_k \in \mathcal{S}'$ *for* $\lambda \in U_k$, $k \in \mathbf{N}$;

(*b*) *the mapping* $U_k \ni \lambda \mapsto T(\lambda) * \delta_k \in \mathcal{S}'$ *is* $\mathcal{S}'$*-continuous for* $k \in \mathbf{N}$.

Proof. It is enough to prove that for every $\varepsilon > 0$ there exist a $\gamma \in \Delta(\varepsilon)$ and a neighbourhood U of λ_0 such that

(a') $T(\lambda) * \gamma \in \mathcal{S}'$ for $\lambda \in U$;

(b') the mapping $U \ni \lambda \mapsto T(\lambda) * \gamma \in \mathcal{S}'$ is $\mathcal{S}'$- continuous.

Fix $\varepsilon > 0$ and let U, k_0 and K_0 be as in Lemma 5.. In addition, fix $\gamma_0 \in \Delta(\varepsilon/2)$. Given an arbitrary $\lambda \in U$, denote

$$A_\lambda := \{\psi \in \Delta(\varepsilon/2) : \ T(\lambda) * \psi \in O(k_0), \ \ \|T(\lambda) * \psi\|_{k_0} \le K_0\}.$$

The family $\{A_\lambda : \ \lambda \in U\}$ has the finite intersection property.

Notice that there is a positive K such that $\|T(\lambda) * \psi\|_{k_0} \le K$ for all $\psi \in A_\lambda * \gamma_0$ and the family $\{B_\lambda : \lambda \in U\}$, where B_λ is the closure of the set $A_\lambda * \gamma_0$ in the topology of $\mathcal{D}$, has also the finite intersection property. By the Ascoli theorem, the sets B_λ are compact, so

$$B := \bigcap\{B_\lambda : \ \lambda \in U\} \ne \emptyset.$$

Fix $\psi \in B$ and define $\gamma := \gamma_0 * \psi$. We are going to prove that γ satisfies (a') and (b').

Since, for each $\lambda \in U$,

$$T(\lambda) * \psi \in O(k_0), \qquad \|T(\lambda) * \psi\|_{k_0} \le K,$$

we infer that $T(\lambda) * \psi \in O(M)$ for $\lambda \in U$, i.e. (a') holds.

Assume that $\lambda_n \to \lambda$ as $n \to \infty$. To prove (b') it suffices to show that an arbitrary increasing sequence (p_n) in $\mathbf{N}$ contains a subsequence (r_n) such that

$$T(\lambda_{r_n}) * \gamma \ \to \ T(\lambda) * \gamma \qquad \text{in } \ \mathcal{S}' \tag{10}$$

as $n \to \infty$. Since every bounded sequence in $O(M)$ contains a subsequence convergent in $\mathcal{S}'$, there exists a subsequence (r_n) of (p_n) and a distribution $f \in \mathcal{S}'$ such that $T(\lambda_{r_n}) * \gamma \to f$ in $\mathcal{S}'$ as $n \to \infty$. On the other hand, $T(\lambda_{r_n}) * \gamma \to T(\lambda) * \gamma$ in $\mathcal{D}'$ as $n \to \infty$. Therefore (10) holds true and thus γ satisfies (b'). □

5. Main result

We are going now to prove the main result of this note.

Theorem. *Let f be a regularly transformable distribution. If its Fourier transform $\mathcal{F}(f)$ is transformable, then*

$$\mathcal{F}\mathcal{F}(f) = f^- \tag{11}$$

or, equivalently, $\mathcal{F}^{-1}\mathcal{F}(f) = f$.

Proof. To prove (11) it suffices to show that

$$< f, \phi^- > = < \mathcal{F}(f), \mathcal{F}(\phi) > \tag{12}$$

for all $\phi \in \mathcal{D}$.

Fix $\phi \in \mathcal{D}$ and let (ψ_n) be a sequence in $\mathcal{Z}$ such that

$$\psi_n \to \phi \qquad \text{in } \mathcal{S}(f) \tag{13}$$

as $n \to \infty$. In view of Lemma 6., there exist a sequence (α_k) of positive numbers in $(0,1)$ and a delta-sequence (δ_k) such that

$$(e_\lambda f) * \delta_k \in \mathcal{S}' \qquad \text{for } \lambda \in [-\alpha_k, \alpha_k], \ k \in \mathbf{N}$$

and the following mapping is $\mathcal{S}'$-continuous:

$$[-\alpha_k, \alpha_k] \ni \lambda \ \mapsto \ (e_\lambda f) * \delta_k \in \mathcal{S}'. \tag{14}$$

For arbitrary $n, k \in \mathbf{N}$ and $\lambda \in [-\alpha_k, \alpha_k]$ define

$$g_k(\lambda) := < (e_\lambda f) * \delta_k, \phi >; \quad g(\lambda) := < e_\lambda f, \phi >;$$

$$h_k := \hat{f} * \mathcal{F}(\phi * \delta_k^-); \quad h := \hat{f} * \mathcal{F}(\phi); \quad h_{n,k} := \hat{f} * \mathcal{F}(\psi_n * \delta_k^-).$$

Notice that g_k, g, h_k, h and $h_{n,k}$ are continuous functions.

Let (ρ_k) be a delta-sequence such that

$$\text{supp } \rho_k \subset [-\alpha_k/2, \alpha_k/2]; \quad \mathcal{F}(f) * \rho_k \in \mathcal{S}' \qquad (k \in \mathbf{N}).$$

By (13),

$$g_{nk} * \rho_k \to h_k * \rho_k \qquad \text{almost uniformly on } \mathbf{R} \tag{15}$$

as $n \to \infty$ for each $k \in \mathbf{N}$. Also

$$h_k \to h \quad \text{and} \quad h_k * \rho_k \to h \qquad \text{almost uniformly on } \mathbf{R} \tag{16}$$

as $k \to \infty$ (see Remark 5.). Further we have

$$\begin{aligned} h_{n,k}(\lambda) &= < \tau_\lambda \left(\hat{f}\right)^{-}, \mathcal{F}(\psi_n * \delta_k^-) > = < \mathcal{F}(\tau_\lambda \left(\hat{f}\right)^{-}, \psi_n * \delta_k^- > \\ &= < e_\lambda f, \psi_n * \delta_k^- > = < (e_\lambda f) * \delta_k, \psi_n > \end{aligned}$$

for $n, k \in \mathbf{N}$ and, since the mapping (14) is $\mathcal{S}'$-continuous, it follows that

$$h_{n,k} \to g_k \qquad \text{uniformly on } [-\alpha_k, \alpha_k] \tag{17}$$

as $n \to \infty$ for each $k \in \mathbf{N}$, in view of Lemma 6..

Since

$$g_k \to g \qquad \text{uniformly on } [-1, 1] \tag{18}$$

as $k \to \infty$, we have $\varepsilon_k \to 0$ as $k \to \infty$, where

$$\varepsilon_k := 2 \sup\{|g_k(\lambda) - g(\lambda)| : \lambda \in [0, 1]\}.$$

By virtue of (15), (16), (17) and (18), there exists an increasing sequence (p_n) in $\mathbf{N}$ such that

$$h_{p_n,n} * \rho_n \to h \qquad \text{almost uniformly on } \mathbf{R} \tag{19}$$

as $n \to \infty$ and

$$|h_{p_n,n}(\lambda) - g(\lambda)| < \varepsilon_k \qquad \text{for } \lambda \in [-\alpha_k, \alpha_k], \ k \in \mathbf{N}.$$

Hence

$$|(h_{p_n,n} * \rho_n)(0) - (g * \rho_n)(0)| \le \varepsilon_n \int_{-\infty}^{\infty} |\rho_n(t) dt| \to 0$$

as $n \to \infty$. On the other hand, $(g * \rho_n)(0) \to g(0)$ as $n \to \infty$, so $g(0) = h(0)$. Consequently, $< f, \phi > = < \mathcal{F}(f), \left(\hat{\phi}\right)^{-} >$ for every $\phi \in \mathcal{D}$, i.e. (12) holds. □

The following problem is open:

Problem. *Does identity* (11) *remain true for an arbitrary* $f \in \mathcal{T}$ *such that* $\mathcal{F}(f) \in \mathcal{T}$ *?*

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Streszczenie

W pracy wprowadza się klasę dystrybucji, szerszą od przestrzeni wszystkich dystrybucji temperowanych, taką że dla wszystkich dystrybucji tej klasy można określić transformaty Fouriera. Podane są przykłady takich dystrybucji. Dowodzi się przy pewnych założeniach twierdzenia o transformacie odwrotnej.

Mówimy, że dystrybucja $f \in \mathcal{D}'$ jest transformowalna, jeżeli istnieje ciąg deltowy (δ_n), taki że $f * \delta_n \in \mathcal{S}'$ dla $n \in \mathbb{N}$. Klasę wszystkich dystrybucji transformowalnych oznaczamy przez $\mathcal{T}$. Dla dowolnej dystrybucji $f \in \mathcal{T}$ wprowadzamy oznaczenie $\mathcal{D}(f) := \{\phi \in \mathcal{D} : f * \phi \in \mathcal{S}'\}$. Jeżeli $f \in \mathcal{T}$, to istnieje dokładnie jedna dystrybucja g taka, że dla dowolnej funkcji $\phi \in \mathcal{D}(f)$ zachodzi równość $\mathcal{F}(f * \phi) = (2\pi)^{1/2} g \cdot \mathcal{F}(\phi)$. Dystrybucję g spełniającą tę równość nazywamy transformatą Fouriera dystrybucji f i oznaczamy przez $\mathcal{F}(f)$. Definicja ta pokrywa się ze zwykłą definicją transformaty Fouriera w zakresie dystrybucji temperowanych i zachowuje jej podstawowe własności. Przestrzeń dystrybucji transformowalnych jest istotnie szersza od przestrzeni dystrybucji temperowanych i zawiera między innymi podprzestrzeń tzw. dystrybucji M-transformowalnych.

Wprowadza się zbieżność w przestrzeni $\mathcal{T}$ i definicję podklasy $\mathcal{T}_0 \in \mathcal{T}$ dystrybucji regularnie transformowalnych, dla której zachodzi następujące, główne twierdzenie pracy, pozwalające zdefiniować klasę dystrybucji niezmienniczą ze względu na transformatę Fouriera i istotnie szerszą od przestrzeni dystrybucji temperowanych:

Twierdzenie. Jeśli $f \in \mathcal{T}_0$ oraz $\mathcal{F}(f) \in \mathcal{T}$, to $\mathcal{F}\mathcal{F}(f) = f^-$.