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ON A GENERALIZATION OF THE ENTROPY EQUATION

Summary. The general solution of the generalized multi-dimensional entropy equation of multiplicative type on the open domain is given.

O UOGÓLNIENIU RÓWNANIA ENTROPII

Streszczenie. W pracy podane jest ogólne rozwiązanie uogólnionego wielowymiarowego równania entropii typu multyplikatywnego na obszarze otwartym.

ОБ ОБОБЩЕНИИ УРАВНЕНИЯ ЭНТРОПИИ

Резюме. В этой статьи мы находим общие решения обобщенного многомерного уравнения энтропии мультипликационного типа на открытой области.

1. Presentation of the result

Axiomatic characterizations of information measures (in particular, the Shannon entropy) have been studied extensively for many years. The paper [5] can be considered as the culmination of these efforts; it contains the general solution of the following equation on the open domain D_n in $\mathbf{R}^n$, defined below:

$$f_1(s) + \mu(1-s) f_2\Big(\frac{t}{1-s}\Big) = f_3(t) + \mu(1-t) f_4\Big(\frac{s}{1-t}\Big), \qquad s,t \in D_n, \tag{1}$$

where μ is a given arbitrary multiplicative function on $I_n := (0,1)^n$ (that is, $\mu(st) = \mu(s)\mu(t)$ for $s,t \in I_n$) and f_i are unknown real-valued functions on I_n for $i = 1,2,3,4$.

The domain D_n is defined in the following way:

$$D_n := \{(s,t) : \ s,t,s+t \in I_n\}.$$

Here and in the sequel vectors in $\mathbf{R}^n$ are denoted by Latin and their coordinates by the corresponding Greek characters, and operations on vectors are coordinatewise, e.g. if $t = (\tau_1, \dots, \tau_n)$ and $x = (\xi_1, \dots, \xi_n)$, then $1-t = (1-\tau_1, \dots, 1-\tau_n)$, $tx = (\tau_1\xi_1, \dots, \tau_n\xi_n)$ etc.

Equation (1) is a generalization of the so-called *fundamental equation of information*:

$$f(\xi) + (1-\xi)f\left(\frac{\eta}{1-\xi}\right) = f(\eta) + (1-\eta)f\left(\frac{\xi}{1-\eta}\right), \qquad (\xi,\eta) \in D_1, \tag{2}$$

where $f : I_1 \to \mathbf{R}^1$ is an unknown function. Equation (2) characterizes, under some additional conditions, the Shannon entropy (cf. [2], pp. 71-74, 100-101).

In [7], another description of the Shannon entropy was given by using the so-called entropy equation:

$$H(\xi,\eta,\zeta) = H(\xi+\eta,0,\zeta) + H(\xi,\eta,0), \tag{3}$$

where H is a symmetric, homogeneous (of order 1) and continuous function on the set

$$X := \{(\xi,\eta,\zeta) : \ \xi,\eta,\zeta \geq 0, \ \xi\eta + \eta\zeta + \zeta\xi > 0\}$$

or , more generally, a Schwartz distribution on an open set containing X.

Z. Daróczy in 1974 (in an unpublished manuscript) and, independently, J. Aczél in [1] have shown that the continuous (general) solution of (3) can be easily obtained from the continuous (general) solution of (2).

Note, however, that (3) and symmetry of H imply

$$F(\xi+\eta,\zeta) + F(\xi,\eta) = F(\xi+\zeta,\eta) + F(\xi,\zeta) \tag{4}$$

for $\xi,\eta,\zeta > 0$, where $F(\xi,\eta) = H(\xi,\eta,0)$ is homogeneous on $(0,\infty)^2$. It can be proved that (4) with homogeneous F is equivalent to (2). More generally, equation (1) is equivalent to

$$F_1(x+y,z) + F_2(x,y) = F_3(x+z,y) + F_4(x,z), \qquad (x,y,z) \in \bar{D}_n, \tag{5}$$

where

$$\bar{D}_n := \{(x,y,z) : \ x,y,z,x+y+z \in I_n\},$$

μ is a given multiplicative function and $F_i : \ D_n \to \mathbf{R}^1$ $(i = 1,2,3,4)$ are unknown μ-homogeneous functions, i.e.

$$F_i(tx,ty) = \mu(t)F_i(x,y), \qquad t \in I_n, \ (x,y) \in D_n. \tag{6}$$

Therefore the general solution of one of equations (1), (5) can be obtained from the general solution of the other, e.g. Theorem, formulated below, on the general solution of (5) can be derived from Theorem 3.1 in [5].

We shall call equation (5) *generalized entropy equation*, because of the mentioned connection of equations (3) and (4).

Notice that the general solution of (4) with homogeneous F follows from Theorem 6 of the known paper [6]. On the other hand, all proofs of results about various forms of (1) and (5) (including the proof of Theorem 3.1 in [5]) are based, as a matter of fact, on the same theorem and the proofs stand for rather compound combinations of transforming both equations (1) and (5) simultaneously.

Therefore it is natural to look for a proof of Theorem, independ on Theorem 3.1 and based exclusively on the generalized entropy equation.

We accomplish here this idea, giving in the next section such a proof; we avoid combined methods of solving (1) and (5) by an appropriate modification of techniques used in [8], [3] and [5].

Theorem. *Given a non zero multiplicative function on I_n, the general solution of (5) - (6) is given, for $x = (\xi_1, \ldots, \xi_n)$, $y = (\eta_1, \ldots, \eta_n)$ such that $(x, y) \in D_n$, by the formulae:*

(I) *for additive μ:*

$$
\begin{aligned}
F_1(x,y) &= \mu(x)l(x) + \mu(y)l(y) - \mu(x+y)l(x+y) + \alpha_1\mu(x) + \alpha_2\mu(y),\\
F_2(x,y) &= \mu(x)l(x) + \mu(y)l(y) - \mu(x+y)l(x+y)\\
&\qquad\qquad + (\alpha_5 - \alpha_1)\mu(x) + (\alpha_4 - \alpha_1)\mu(y),\\
F_3(x,y) &= \mu(x)l(x) + \mu(y)l(y) - \mu(x+y)l(x+y) + \alpha_3\mu(x) + \alpha_4\mu(y),\\
F_4(x,y) &= \mu(x)l(y) + \mu(y)l(y) - \mu(x+y)l(x+y)\\
&\qquad\qquad + (\alpha_5 - \alpha_3)\mu(x) + (\alpha_2 - \alpha_3)\mu(y),
\end{aligned}
$$

where l is logarithmic on I_n (i.e. $l(xy) = l(x)l(y)$ for $x, y \in I_n$) and $\mu(x) = \xi_k$ for some fixed k;

(II) *for $\mu \equiv 1$:*

$$
\begin{aligned}
F_1(x,y) &= (l_1 + l_2)\left(\frac{x}{x+y}\right) + l_3\left(\frac{y}{x+y}\right) + \alpha_1 + \alpha_2,\\
F_2(x,y) &= l_2\left(\frac{x}{x+y}\right) + l_1\left(\frac{y}{x+y}\right) + \alpha_3,\\
F_3(x,y) &= (l_2 + l_3)\left(\frac{x}{x+y}\right) + l_1\left(\frac{y}{x+y}\right) + \alpha_2 + \alpha_3,
\end{aligned}
$$

$$F_4(x,y) = l_2\left(\frac{x}{x+y}\right) + l_3\left(\frac{y}{x+y}\right) + \alpha_1,$$

where l's are logarithmic on I_n;

(III) *for $\mu \not\equiv 1$, non-additive:*

$$F_1(x,y) = b(x,y) + \alpha_1\mu(x) + \alpha_2\mu(y) - \alpha_3\mu(x+y),$$
$$F_2(x,y) = -b(x,y) + \alpha_4\mu(x) + \alpha_5\mu(y) - \alpha_1\mu(x+y),$$
$$F_3(x,y) = -b(x,y) + \alpha_6\mu(x) + \alpha_5\mu(y) - \alpha_3\mu(x+y),$$
$$F_4(x,y) = b(x,y) + \alpha_4\mu(x) + \alpha_2\mu(y) - \alpha_6\mu(x+y),$$

where b and μ have one of the following forms:

(a) $b(x,y) = d(\xi_k)\cdot\eta_k - \xi_k\cdot d(\eta_k), \quad \mu(x) = \xi_k^2$ *for some k,*

(b) $b(x,y) = \alpha\,\mathrm{Im}\,[\varphi(\xi_k)\overline{\varphi(\eta_k)}], \quad \mu(x) = |\varphi(\xi_k)|^2$ *for some k,*

(c) $b(x,y) = \alpha(\xi_j\eta_k - \xi_k\eta_j), \quad \mu(x) = \xi_j\xi_k$ *for some $j,k,\ j\neq k$,*

(d) $b(x,y) = 0$

(here d is a real derivation, i.e. d is additive and $d(\xi^{-1}) = -\xi^{-2}d(\xi)$ and φ is a nontrivial field embedding of $\mathbf{R}^1$ onto $\mathbf{C}$).

All α's above are constants.

2. Proof of Theorem

In the proof of Theorem we shall need the following two statements shown in [5] (results of [2] and [8] are used in the proof). The proof of Proposition 2 (consisting of the proof of Lemma 4.1 and Theorem 4.4 in [5]) is completely free from considerations of the fundamental equation of information, used only in a fragment of the proof of Proposition 1 (Lemma 4.3 in [5]). But it can be eliminated even from that fragment by an appropriate modification of the proof. We omit here the details.

Proposition 1. *A function $\Phi : D_n \to \mathbf{R}^1$ satisfies the system of equations:*

$$\Phi(x+y,z) + \Phi(x,y) = \Phi(x+z,y) + \Phi(x,z), \qquad (x,y,z)\in \bar{D}_n; \tag{7}$$
$$\Phi(x,y) = \Phi(y,x), \qquad (x,y,)\in D_n; \tag{8}$$
$$\Phi(tx,ty) = \mu(t)\Phi(x,y), \qquad (x,y)\in D_n,\ t\in I_n; \tag{9}$$

where $\mu \not\equiv 0$ is multiplicative on I_n, if and only if

$$\Phi(x,y) = \beta[\mu(x) + \mu(y) + \mu(x+y)], \qquad (x,y)\in D_n \tag{10}$$

in case μ is not additive, and

$$\Phi(x,y) = \mu(x)l(x) + \mu(y)l(y) - \mu(x+y)l(x+y), \qquad (x,y) \in D_n \tag{11}$$

in case μ is additive, where l is logarithmic on I_n and $\beta \in \mathbf{R}^1$; μ and l are extendable to $P_n := (0,\infty)^n$ so that μ is multiplicative and l is logarithmic on P_n and Φ, defined by (10) - (11) on P^2, satisfies (7) - (9) for $x,y,z,t \in P_n$.

Proposition 2. *Let μ be a multiplicative function on P_n. If Φ is μ-homogeneous on P_n^2 and*

$$\Phi(x,y) = f_1(x) + f_2(y) - f_3(x+y), \qquad x,y \in P_n,$$

then

$$f_i(x) = a(x) + \beta_i\mu(x) + r_i(x), \qquad x \in P_n$$

in case μ is not additive, and

$$f_i(x) = a(x) + \alpha_i\mu(x) + \mu(x)l(x) + r_i(x), \qquad x \in P_n$$

in case μ is additive, for $i = 1,2,3$, where α's are constants, the function a is additive, l is logarithmic on P_n and r's satisfy the equation:

$$r_i(xy) = \mu(x)r_i(y) + r_i(x) \tag{12}$$

for $x,y \in P_n$ and are of the form:

$$r_i(x) = l_i(x) \text{ in case } \mu \equiv 1; \quad r_i(x) = \alpha_i - \alpha_i\mu(x) \text{ in case } \mu \not\equiv 1 \tag{13}$$

for $x \in P_n$ and $i = 1,2,3$. Moreover, $r_1 + r_2 = r_3$ on P_n.

The following lemma will be used in the proof of Theorem and will allow us to avoid considering separately the three different domains: P_n, $[1,\infty)^n$ and $[2,\infty)^n$, appearing in the proofs given in [5].

Lemma. *Let $\Phi : P_n \to \mathbf{R}^1$ be a given function and suppose that for every $\sigma \in (0,1]$ there exist functions f_i: $[\sigma,\infty)^n \to \mathbf{R}^1$ $(i = 1,2,3)$ such that*

$$\Phi(x,y) = f_1^\sigma(x) + f_2^\sigma(y) - f_3^\sigma(x+y), \qquad (x,y) \in [\sigma,\infty)^n, \tag{14}$$

then there exist functions f_i: $P_n \to \mathbf{R}^1$ (i = 1,2,3) such that

$$\Phi(x,y) = f_1(x) + f_2(y) - f_3(x+y), \qquad (x,y) \in P_n. \tag{15}$$

Proof. Take arbitrary $\sigma, \tau \in (0,1]$, $\sigma < \tau$ and define the functions $g_i := g_i^{\sigma,\tau} := f_i^{\sigma} - f_i^{\tau}$ $(i = 1,2,3)$. By (14), g_i satisfy the Pexider equation on the domain $[\tau, \infty)^{2n}$. Using appropriate substitutions and Theorem (0. 3. 3) from [2] on the general solution of the Cauchy equation on $[\tau, \infty)^2$, one can easily deduce that

$$f_i^{\sigma}(x) = f_i^{\tau}(x) + a^{\sigma,\tau}(x) + \alpha_i^{\sigma,\tau}, \qquad x \in [3\tau, \infty)^n \tag{16}$$

for $i = 1,2,3$, where $a^{\sigma,\tau}$ is an additive function on $\mathbf{R}^n$ and $\alpha_i^{\sigma,\tau}$ are constants such that $\alpha_1^{\sigma,\tau} + \alpha_2^{\sigma,\tau} = \alpha_3^{\sigma,\tau}$. In particular, for every $\rho \in I_n$,

$$f_i^{\rho}(x) = f_i^{1}(x) + a^{\rho}(x) + \alpha_i^{\rho}, \qquad x \in [3, \infty)^n \tag{17}$$

where $a^{\rho} := a^{\rho,1}$ and $\alpha_i^{\rho} := \alpha_i^{\rho,1}$, i.e. $\alpha_1^{\rho} + \alpha_2^{\rho} = \alpha_3^{\rho}$.

Fix $\sigma, \tau \in (0,1)$, $\sigma < \tau$. By (16) and (17) with $\rho = \sigma$ and $\rho = \tau$, we get

$$a^{\sigma,\tau}(x) + \alpha_i^{\sigma,\tau} = a^{\sigma}(x) - a^{\tau}(x) + \alpha_i^{\sigma} - \alpha_i^{\tau}, \qquad x \in [3, \infty)^n \tag{18}$$

for $i = 1,2,3$. Of course, each $z \in P_n$ can be represented in the form $z = x - y$, where $x, y \in [3, \infty)^n$.Therefore additivity of the a's and (18) imply

$$a^{\sigma,\tau}(z) = a^{\sigma}(z) - a^{\tau}(z), \quad z \in P_n; \tag{19}$$

$$\alpha_i^{\sigma,\tau} = \alpha_i^{\sigma} - \alpha_i^{\tau}, \quad i = 1,2,3. \tag{20}$$

Now, define

$$f_i(x) := f_i^{\sigma}(x) - a^{\sigma}(x) - \alpha_i^{\sigma},$$

whenever $x \in [3\sigma, \infty)^n$ for $\sigma \in (0,1)$ and $i = 1,2,3$. The functions f_i are well defined on P_n, according to (16), (19), and (20). Clearly, (15) holds.

Proof of Theorem. First notice that $\mu \colon I_n \to \mathbf{R}^1$ and $F_i \colon D_n \to \mathbf{R}^1$ $(i = 1,2,3,4)$ can be uniquely extended to P_n and P_n^2, respectively, so that equations (5) - (6) are satisfied for $t, x, y, z \in P_n$ (cf. [3]), pp. 8-9). If we replace y by z and conversely in (6), then add and subtract the resulting equation to and from the original equation in an appropriate way, we obtain the following equations:

$$\Phi_1(x+y, z) + \Phi_2(x, y) = \Phi_1(x+z, y) + \Phi_2(x, z), \tag{21}$$

$$\Phi_3(x+y, z) + \Phi_4(x, y) + \Phi_3(x+z, y) + \Phi_4(x, z) = 0 \tag{22}$$

for $x, y, z \in P_n$, where

$$\Phi_1 := F_1 + F_3, \quad \Phi_2 := F_2 + F_4, \quad \Phi_3 := F_1 - F_3, \quad \Phi_4 := F_2 - F_4 \tag{23}$$

are μ-multiplicative on P_n.

Consider first (21) and put $z = z_0 = (\zeta_0, \dots, \zeta_0) \in P_n$. This gives

$$\Phi_2(x,y) = \Phi_1(x+z_0, y) - \Phi_1(x+y, z_0) + \Phi_2(x, z_0). \tag{24}$$

Substituting (24) into (21), letting $x = u + z_0$, $y = v + z_0$, $z = w + z_0$ and denoting

$$\Psi_1(u,v) := \Phi_1(u+2z_0, v+z_0) - \Phi_1(u+2z_0, z_0) \tag{25}$$

for $u, v \in \bar{P}_n := [0,\infty)^n$, we obtain the generalized entropy equation:

$$\Psi_1(u+v, w) + \Psi_1(u,v) = \Psi_1(u+w, v) + \Psi_1(u,w) \tag{26}$$

for $u, v, w \in \bar{P}_n$. Defining

$$\Psi_2(u,v) := \Psi_1(u,v) - \Psi_1(0,v), \qquad u, v \in \bar{P}_n, \tag{27}$$

we see that Ψ_2 satisfies the generalized entropy equation and is symmetric, in view of (26) with $u = 0$. By Theorem 5 in [6],

$$\Psi_2(u,v) = \varphi(u) + \varphi(v) - \varphi(u+v), \qquad u, v \in \bar{P}_n$$

for some function $\varphi\colon \bar{P}_n \to \mathbf{R}^1$. Hence, by (27) and (25),

$$\Phi_1(x,y) = \varphi_1(x) + \varphi_2(y) - \varphi_3(x+y), \qquad x, y \in [3\zeta_0, \infty)^n,$$

where $\varphi_1(t) := \varphi(t-2z_0) + \Phi_1(t, z_0)$, $\varphi_2(t) := \varphi(t-z_0) + \Psi_1(0, t-z_0)$ and $\varphi_3(t) := \varphi(t-3z_0)$ are functions defined for $t \in [3\zeta_0, \infty)^n$.

By Lemma,

$$\Phi_1(x,y) = f_1(x) + f_2(y) - f_3(x+y), \qquad x, y \in P_n$$

for some functions $f_i : P_n \to \mathbf{R}^1$ $(i = 1,2,3)$. In view of Proposition 2, we have

$$\begin{aligned} \Phi_1(x,y) &= \beta_1\mu(x) + \beta_2\mu(y) - \beta_3\mu(x+y) + \\ &\quad + r_1(x) + r_2(y) - (r_1 + r_2)(x+y) \end{aligned} \tag{28}$$

in case μ is not additive, and

$$\begin{aligned} \Phi_1(x,y) &= \alpha_1\mu(x) + \alpha_2\mu(y) - \alpha_3\mu(x+y) + \\ &\quad + r_1(x) + r_2(y) - (r_1 + r_2)(x+y) + \\ &\quad + \mu(x)l(x) + \mu(y)l(y) - \mu(x+y)l(x+y) \end{aligned} \tag{29}$$

in case μ is additive, for $x, y \in P_n$.

Now, we substitute (28)-(29) into (21), applying (24), and separate the variables y and z in the resulting equations. The functions on the left and right hand side after the separation depend neither on y nor on z, so we can denote them as follows:

$$\Phi_2(x,y) + \beta_1\mu(x+y) - \beta_2\mu(y) + r_1(x+y) - r_2(y) =: h(x) \tag{30}$$

in case μ is not additive, and

$$\begin{aligned}\Phi_2(x,y) + \alpha_1\mu(x+y) - \alpha_2\mu(y) + \\ + r_1(x+y) - r_2(y) + \mu(x+y)l(x+y) - \mu(y)l(y) =: k(x)\end{aligned} \tag{31}$$

in case μ is additive, where h, k are functions on P_n. Define

$$h_1(x) := h(x) - (r_1 - r_2)(x); \qquad k_1(x) := k(x) - (r_1 - r_2)(x) - \mu(x)l(x).$$

It follows from μ-homogeneity of Φ_2 and (12) that h_1 and k_1 are also μ-homogeneous. Hence

$$h(x) = (r_1 - r_2)(x) + \beta_4\mu(x),$$
$$k(x) = (r_1 - r_2)(x) + \alpha_4\mu(x)l(x) + \mu(x)l(x),$$

where $\beta_4 := h_1(1)$ and $\alpha_4 := k_1(1)$. Therefore (30) and (31) imply

$$\begin{aligned}\Phi_2(x,y) &= \beta_4\mu(x) + \beta_2\mu(y) - \beta_1\mu(x+y) + \\ &\quad +(r_1 - r_2)(x) + r_2(y) - r_1(x+y)\end{aligned} \tag{32}$$

in case μ is not additive, and

$$\begin{aligned}\Phi_2(x,y) &= \alpha_4\mu(x) + \alpha_2\mu(y) - \alpha_1\mu(x+y) + \\ &\quad +(r_1 - r_2)(x) + r_2(y) - r_1(x+y) + \\ &\quad +\mu(x)l(x) + \mu(y)l(y) - \mu(x+y)l(x+y)\end{aligned} \tag{33}$$

in case μ is additive, for $x, y \in P_n$.

Consider now (22). Interchanging x and y and adding the obtained equations, we get

$$\begin{aligned}2\Phi_3(x+y,z) + \Psi_3(x,y) + \Phi_3(x+z,y) + \\ +\Phi_3(y+z,x) + \Phi_4(x,z) + \Phi_4(y,z) = 0\end{aligned} \tag{34}$$

for $x, y, z \in P_n$, where

$$\Psi_3(x,y) := \Phi_4(x,y) + \Phi_4(y,x), \qquad x, y \in P_n$$

is μ-multiplicative function. Substituting in (34) $z = 1$, replacing in (22) x, y, z by $1, x, y$, respectively, and comparing the obtained equations, we get

$$\Psi_3(x,y) = g_1(x) + g_2(y) - g_3(x+y), \qquad x, y \in P_n, \tag{35}$$

where

$$g_1(x) := \Phi_4(1,x) - \Phi_4(x,1) =: g_2(x); \qquad g_3(x) := 2\Phi_3(x,1) \tag{36}$$

for $x \in P_n$.

Putting $y = z = 1$ in (22), we obtain

$$\Phi_4(x,1) = \Phi_3(x+1,1). \tag{37}$$

Since Ψ_3 is μ-homogeneous, we can apply Proposition 2 to (35) and to find the form of g_3. Substituting it to the second formula in (36), we can find subsequently Φ_3 and Φ_4, due to the formulae (37) and

$$\Phi_i(x,y) = \mu(y)\Phi_i\left(\frac{x}{y},1\right), \qquad i = 3,4.$$

Thus, taking into account (12), we obtain

$$2\Phi_3(x,y) = \mu(y)a\left(\frac{x}{y}\right) + \beta_3\mu(x) + r(x) - r(y), \tag{38}$$

$$-2\Phi_4(x,y) = \mu(y)a\left(\frac{x}{y}+1\right) + \beta_3\mu(x+y) + r(x+y) - r(y) \tag{39}$$

in case μ is not additive, and

$$2\Phi_3(x,y) = \mu(y)a\left(\frac{x}{y}\right) + \alpha\mu(x) + \mu(x)l\left(\frac{x}{y}\right) + r(x) - r(y), \tag{40}$$

$$\begin{aligned} -2\Phi_4(x,y) = {} & \mu(y)a\left(\frac{x}{y}+1\right) + \alpha_3\mu(x+y) + \mu(x+y)l\left(\frac{x}{y}+1\right) + \\ & + r(x+y) - r(y) \end{aligned} \tag{41}$$

in case μ is additive. Substituting the above forms of Φ_3 and Φ_4 into (22) and using the additivity of a, we get

$$\mu(z)a\left(\frac{y}{z}\right) + \mu(y)a\left(\frac{z}{y}\right) - [\mu(y)+\mu(z)]a(1) = 0 \tag{42}$$

in case μ is not additive, and

$$\mu(z)a\left(\frac{y}{z}\right) + \mu(y)a\left(\frac{z}{y}\right) - [\mu(y)+\mu(z)]a(1) + \mu(y)l\left(\frac{z}{y}\right) + \mu(z)l\left(\frac{z}{y}\right) = 0 \tag{43}$$

in case μ is additive, for $y, z \in P_n$.

Now, we replace y by $2y$ in (42) and (43) and compare the equations so obtained with the original equations (42) and (43), respectively. We get

$$\left[2 - \frac{\mu(2)}{2}\right] \mu(y)a\left(\frac{z}{y}\right) - \mu(z)a(1) - [2-\mu(2)]\,\mu(y)a(1) = 0 \tag{44}$$

in case μ is not additive, and

$$\mu(y)a\left(\frac{z}{y}\right) + \mu(z)l\left(\frac{z}{y}\right) = 2l(2)\mu(y) + [a(1) - l(2)]\,\mu(z) \tag{45}$$

in case μ is additive.

Consider the case μ is non-additive and $\mu \not\equiv 1$. Obviously, $\mu \not\equiv 1$, but now the multiplicity of μ implies $\mu(1) = 1$. Hence, putting $y = z = 1$ in (44), we get $[1 - \mu(2)/2]\,a(1) = 0$. The supposition $a(1) \neq 0$ would imply $\mu(2) = 2$ and, substituting the last value to (44) with $y = 1$, we would obtain $\mu(z) = a(z)/a(1)$, i.e. the contradiction with the assumption that μ is non-additive. Therefore $a(1) = 0$ and (42) with $y = 1$ yields the equation

$$a(z) + \mu(z)a\left(\frac{1}{z}\right) = 0, \qquad z \in P_n. \tag{46}$$

The general solution of (46) is given by one of the following forms:

(a_1) $a(x) = d(\xi_k) - \xi_k$, $\mu(x) = \xi_k^2$ for some k,

(b_1) $a(x) = \alpha\,\mathrm{Im}\,[\varphi(\xi_k)]$, $\mu(x) = |\varphi(\xi_k)|^2$ for some k,

(c_1) $a(x) = \alpha(\xi_j - \xi_k)$, $\mu(x) = \xi_j\xi_k$ for some j, k, $j \neq k$,

(d_1) $a(x) = 0$

for $x = (\xi_1, \ldots, \xi_n), y = (\eta_1, \ldots, \eta_n) \in P_n$, which is proved in [4] (see also [9]). We substitute (a_1) – (d_1) into (28), (32), (38) and (39), taking into account (13). Hence, by (23), we get formulae (III) in Theorem (after renaming the constants).

In case $\mu \equiv 1$, equation (42) for $y = 1$ yields $a(z) = \frac{4}{3}a(1)$ which, by virtue of the additivity of a, gives $a \equiv 0$. Now, substitutting (28), (32), (38), (39), with r's given by (13), into (23), we arrive at the formulae (II) with appropriate l's and α's.

Finally, suppose that μ is additive. Substituting (45) into (40), (41) and using (13) again, we see that $\Phi_3(x, y)$ and $\Phi_4(x, y)$ are linear combinations of $\mu(x)$ and $\mu(y)$. This and equations (29), (33), due to (13) and (23), lead to formulae (I) in Theorem, after the appropriate change of the constants.

To complete the proof of Theorem it remains to notice that F's given by formulae (I) - (III) satisfy (5) - (6).

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Streszczenie

Spośród rozmaitych charakteryzacji miar informacji (w tym entropii Shannona) za pomocą równań funkcyjnych szczególnie interesujący jest opis podany w pracy [5] ze względu na najogólniejszą postać rozpatrywanego tam równania, tzw. uogólnionego równania informacji typu multyplikatywnego:

$$f_1(s) + \mu(1-s)f_2\left(\frac{t}{1-s}\right) = f_3(t) + \mu(1-t)f_4\left(\frac{s}{1-t}\right), \qquad s,t \in D_n,$$

rozważanego na obszarze otwartym

$$D_n := \{(s,t) : \ s,t,s+t \in I_n\},$$

gdzie f_i są szukanymi funkcjami rzeczywistymi na zbiorze $I_n := (0,1)^n$ dla $i = 1,2,3,4$, a μ jest ustaloną funkcją multyplikatywną na I_n, tzn. taką, że $\mu(st) = \mu(s)\mu(t)$ dla $s, t \in I_n$.

Można dowieść, że uogólnione równanie informacji typu multyplikatywnego jest równoważne następującemu uogólnionemu równaniu entropii:

$$F_1(x+y,z) + F_2(x,y) = F_3(x+z,y) + F_4(x,z), \qquad (x,y,z) \in \bar{D}_n,$$

gdzie

$$\bar{D}_n := \{(x,y,z)\colon\ x,y,z,x+y+z \in I_n\},$$

a $F_i\colon\ D_n \to \mathbf{R}^1$ $(i = 1,2,3,4)$ są poszukiwanymi funkcjami μ-jednorodnymi, tzn.

$$F_i(tx,ty) = \mu(t)F_i(x,y), \qquad t \in I_n,\ (x,y) \in D_n.$$

W prezentowanej pracy znajduje się ogólne rozwiązania uogólnionego równania entropii metodą bezpośrednią, bez korzystania z wyników dotyczących równania informacji.