ZESZYTY NAUKOWE POLITECHNIKI ŚLĄSKIEJ 1995
Seria: MATEMATYKA-FIZYKA z. 72 Nr kol. 1279

Brunon SZOCIŃSKI

ON SOME GENERALIZATION OF ELEMENTARY KLEIN SPACE

Summary. The aim of this paper is to formulate definitions and state some properties of the generalized elementary Klein space over arbitrary field. It is show that affine and projective space are the examples of such spaces.

O PEWNYM UOGÓLNIENIU ELEMENTARNEJ PRZESTRZENI KLEINA

Streszczenie. Celem niniejszej pracy jest podanie definicji i pewnych własności uogólnionej przestrzeni elementarnej Kleina nad dowolnym ciałem. Przestrzeń afiniczna i rzutowa są przykładami tego rodzaju przestrzeni.

НЕКОТОРЫЕ ОБОБЩЕНИЕ ЭЛЕМЕНТАРНОГО ПРОСТРАНСТВА КЛЕЙНА

Резюме. Целю настоящей работы является передставление определений и некоторых свойств обобщенного элементарного пространства Клейна над произвольным полем. Аффинное и проективное пространство это примеры этих пространств.

Introduction

M. Kucharzewski in [1] defined, using the concept of abstract object, notions of Klein space, geometric object and Klein geometry. By the abstract object supported by the group G we mean any triplet

$$(X, G, F) \tag{1}$$

consisting of a nonempty set X, an arbitrary group G and an action F of the group G on the set X, i.e. a mapping $F : X \times G \to X$ satisfying the conditions:

$$F\left(F(x,g_1),g_2\right) = F\left(x, g_2 \cdot g_1\right), \qquad F(x,e) = x$$

for all $x \in X$ and g_1, $g_2 \in G$, where e is the unity in G and $g_1 \cdot g_2$ denotes product in G. These conditions are called, respectively, the translation equation and the identity condition. The set X is a fibre of the object (1), where as F is a transformation formula (or transformation law) of this object.

It can be shown (see [1]), that the mapping

$$F_g : X \to X, \qquad F_g(x) := F(x,g)$$

is a bijection of the set X onto itself, and

$$\hat{F} : G \to \mathcal{G}(X), \qquad \hat{F}(g) := F_g$$

is a homomorphism of the group G into the group $\mathcal{G}(X)$ of all transformations of the set X. This homomorphism will be called a representation of the group G in the group of transformations $\mathcal{G}(X)$, or — simply — the representation of the object (1). Abstract object (1) will be called effective iff its representation $\hat{F}$ is a monomorphism. Effective abstract objects are called Klein spaces.

The above definition of KLein space is too general. In [5], using the notion of linear space of translations of the set, the elementary Klein space over arbitrary field was defined. However, the class of such spaces is too narrow: e.g. the projective space is not an elementary Klein space.

The aim of this paper is to formulate definitions (section 1) and state some properties (section 3) of the generalized elementary Klein space over an arbitrary field. In section 2 it is shown that affine, Euclidean, projective and elliptic spaces are the examples of such spaces.

1. Vector structure of a set over the field

The group of transformations $\mathcal{T}(M)$ of a nonempty set M will be called a group of translations of this set iff it acts straightly transitively, i.e. for any $p, q \in M$ there exists one and only one $\tau \in \mathcal{T}(M)$ such that $\tau(p) = q$.

Let K be an arbitrary field, with zero and unity denoted by 0 and 1, respectively. Note that the abelian group of translations $\mathcal{T}(M)$ with outer operation

$$\cdot : K \times \mathcal{T}(M) \to \mathcal{T}(M), \tag{2}$$

satisfying, for all $a, b \in K$ and $\tau_1, \tau_2, \tau \in \mathcal{T}(M)$, conditions:

$$\begin{aligned} a \cdot (\tau_1 \circ \tau_2) &= (a \cdot \tau_1) \circ (a \cdot \tau_2), \\ (a+b) \cdot \tau &= (a \cdot \tau) \circ (b \cdot \tau), \\ (ab) \cdot \tau &= a \cdot (b \cdot \tau), \\ 1 \cdot \tau &= \tau, \end{aligned} \tag{3}$$

is a linear space over K.

According to the definition given in [5], the Abelian group of translations $\mathcal{T}(M)$ with outer operation (2) satisfying conditions (3) will be called a linear space of translations of the set M over the field K and denoted by $\mathcal{T}(M, K)$.

The notion of the group of translations and the linear space of translations of the set M can may be generalized as follows:

Definition 1. *The group of transformations $\mathcal{T}_D(M)$ of the set M will be called a group of quasi-translations of this set with the quasi-domain D ($\emptyset \neq D \subset M$) iff it acts straightly transitively on D and for all $\tau \in \mathcal{T}_D(M)$ the condition*

$$\tau\big|_{M \setminus D} = \mathrm{id}_{M \setminus D}$$

holds. The abelian group of quasi-translations $\mathcal{T}_D(M)$ satisfying for all $a, b \in K$, $\tau_1, \tau_2, \tau \in \mathcal{T}_D(M)$ conditions (3) will be called a linear space of quasi-translations of the set M over the field K, and denote by $\mathcal{T}_D(M, K)$.

In particular, if $D = M$, the linear space of quasi-translations is the linear space of translations of M over K.

Definition 2. *Let $\{\mathcal{T}_D(M, K)\}_{D \in \Lambda}$ be a system of linear spaces of quasi-translations of M over K, where Λ is a family of quasi-domains of these spaces, and let $\{\mathcal{A}_p(M)\}_{p \in M}$ be a system of groups of transformations of M, satisfying, for all $p \in M$ and $\alpha \in \mathcal{A}_p(M)$ the equality $\alpha(p) = p$. The pair*

$$\left(\{\mathcal{T}_D(M, K)\}_{D \in \Lambda}, \{\mathcal{A}_p(M)\}_{p \in M}\right) \tag{4}$$

is called a vector structure of the set M over the field K iff the following axioms are true:

V1 *For all $D, D' \in \Lambda$, $D \neq D'$ and each $\tau \in \mathcal{T}_D(M, K)$, $\tau' \in \mathcal{T}_{D'}(M, K)$, $a \in K$ the following conditions hold: $\tau(D') \in \Lambda$ and*

$$\begin{aligned} \tau \circ \mathcal{T}_{D'}(M, K) \circ \tau^{-1} &= \mathcal{T}_{\tau(D')}(M, K), \\ \tau \circ (a \cdot \tau') \circ \tau^{-1} &= a \cdot \left(\tau \circ \tau' \circ \tau^{-1}\right); \end{aligned}$$

V2 *For each $p, q \in M$ there exists a quasi-domain $D \in \Lambda$ such that $p, q \in D$;*

V3 *For all $p \in M$, $D \in \Lambda$ and $\tau \in \mathcal{T}_D(M, K)$*

$$\tau \circ \mathcal{A}_p(M) \circ \tau^{-1} = \mathcal{A}_{\tau(p)}(M);$$

V4 *There exists a point $p \in M$ such that for every two quasi-domains D', D'' of the family*

$$\Lambda_p := \{D \in \Lambda : \ p \in D\}$$

there exists one and only one transformation $\alpha \in \mathcal{A}_p(M)$ satisfying conditions: $\alpha(D') = D''$ and

$$\alpha \circ \mathcal{T}_{D'}(M, K) \circ \alpha^{-1} = \mathcal{T}_{D''}(M, K),$$

where as for each $a \in K$ and $\tau' \in \mathcal{T}_{D'}(M, K)$ the relation

$$\alpha \circ (a \cdot \tau') \circ \alpha^{-1} = a \cdot \left(\alpha \circ \tau' \circ \alpha^{-1}\right)$$

holds.

Now, let

$$\mathcal{T}_\Lambda(M, K) := \bigcup_{D \in \Lambda} \mathcal{T}_D(M, K) \tag{5}$$

and consider, for $\tau \in \mathcal{T}_\Lambda(M, K)$, a function

$$L^*_\tau : \mathcal{T}_\Lambda(M, K) \to \mathcal{T}_\Lambda(M, K), \qquad L^*_\tau(\hat{\tau}) = \tau \circ \hat{\tau} \circ \tau^{-1}. \tag{6}$$

As the immediate consequence of axioms **V1**–**V4** we get the following two corollaries:

Corollary 1. *For each $\tau \in \mathcal{T}_\Lambda(M, K)$ mapping (6) is a bijection. Moreover, for any $D \in \Lambda$, its restriction $L^*_\tau\big|_{\mathcal{T}_D(M,K)}$ is a linear isomorphism of the linear space $\mathcal{T}_D(M, K)$ onto $\mathcal{T}_{\tau(D)}(M, K)$.*

Corollary 2. *Conditions stated in axiom* **V4** *are satisfied at any point $p \in M$.*

Using these corollaries and axioms **V1**–**V4**, we will prove

Corollary 3. *All linear spaces of the system $\{\mathcal{T}_D(M, K)\}_{D \in \Lambda}$ are isomorphic.*

Proof. Let us consider arbitrary two linear spaces $\mathcal{T}_{D'}(M, K)$ and $\mathcal{T}_{D''}(M, K)$, $D', D'' \in \Lambda$ and two points $p \in D'$, $q \in D''$. According to **V2** there exists $\tau \in \mathcal{T}_\Lambda(M, K)$ such that $\tau(p) = q$. By Corollary 2, spaces $\mathcal{T}_{D'}(M, K)$ and $\mathcal{T}_{\tau(D')}(M, K)$ are isomorphic and $q \in \tau(D')$. In virtue of axiom **V4** and Corollary 2 we infer that spaces $\mathcal{T}_{\tau(D')}(M, K)$ and $\mathcal{T}_{D''}(M, K)$ are isomorphic, which ends the proof. ∎

Corollary 3 implies that all linear spaces of the system $\{\mathcal{T}_D(M, K)\}_{D \in \Lambda}$ are of the same dimension.

Definition 3. *The common dimension of all linear spaces $\mathcal{T}_D(M,K)$, $D \in \Lambda$ will be called the dimension of vector structure (4).*

Now, let us consider a Klein space

$$(M, G, f) \tag{7}$$

and vector structure of the fibre of this space.

Definition 4. *Vector structure (4) of the fibre of Klein space (7) will be called compatible with this space iff the following two compatibility conditions are satisfied:*

(*i*) *For each $g \in G$, $D \in \Lambda$, $\tau \in \mathcal{T}_D(M,K)$ and $a \in K$*

$$\begin{aligned} f_g \circ \mathcal{T}_D(M,K) \circ f_g^{-1} &= \mathcal{T}_{f_g(D)}(M,K), \\ f_g \circ (a \cdot \tau) \circ f_g^{-1} &= a \cdot \left(f_g \circ \tau \circ f_g^{-1}\right); \end{aligned}$$

(*ii*) *For any $g \in G$ and $p \in M$*

$$f_g \circ \mathcal{A}_p(M) \circ f_g^{-1} = \mathcal{A}_{f_g(p)}(M).$$

Definition 5. *Klein space (7) will be called n-dimensional generalized elementary Klein space iff there exists an n-dimensional vector structure (4) of the fibre M over K, compatible with this space.*

2. Examples of the generalized elementary Klein space

2.1. Elementary Klein space

Let $\mathcal{T}(M,K)$ be a linear space of translations of the set M over the field K and let for each $p \in M$ $\mathcal{A}_p(M)$ be a trivial group $\{\mathrm{id}_M\}$. It is easily seen that the pair

$$\left(\{\mathcal{T}(M,K)\}, \{\mathcal{A}_p(M)\}_{p\in M}\right), \qquad \mathcal{A}_p(M) = \{\mathrm{id}_M\} \quad \text{for } p \in M \tag{8}$$

satisfies axioms **V1–V4**. Hence it is a vector structure of M over K. The pair (8) is called an elementary vector structure.

Klein space (7) is called an n-dimensional elementary Klein space over K (cf. [5]) if there exists n-dimensional elementary vector structure (8) compatible with this space.

Vector space, affine space, Euclidean and pseudo-Euclidean space are the examples of elementary Klein spaces (cf. [2], pp. 14–17).

2.2. Projective space

Consider an n-dimensional projective Klein space (cf. [3], pp. 32):

$$(P^n, GP(n,K), f). \tag{9}$$

Let Λ denotes the family of all subsets of the fibre P^n of the form $D = P^n \setminus H$, where H is an $(n-1)$-dimensional projective hyperplane, whereas $\mathcal{T}_D(P^n)$ will denote the group of projective translations. Such a group acts straightly transitively on the set $P^n \setminus H$, and its elements τ satisfy the condition $\tau\big|_H = \mathrm{id}_H$. Stability subgroup (cf. [3], pp. 23) of hyperplane H is isomorphic with affine group (cf. [3], pp. 36), whence for each group $\mathcal{T}_D(P^n)$, $D \in \Lambda$ we can define outer operation

$$\cdot : K \times \mathcal{T}_D(P^n) \to \mathcal{T}_D(P^n)$$

in such a way that $\mathcal{T}_D(P^n)$ becomes a linear space $\mathcal{T}_D(P^n, K)$ of quasi-translations of the set P^n.

The duality principle implies that for each $p \in P^n$ there exists a subgroup $\mathcal{A}_p(P^n)$ of stability group in p, acting straightly transitively on the set of $(n-1)$-dimensional hyperplanes not containing p. It can be easily shown that a pair obtained in such a way

$$\left(\{\mathcal{T}_D(P^n, K)\}_{D\in\Lambda}, \{\mathcal{A}_p(P^n)\}_{p\in P^n}\right) \tag{10}$$

satisfies axioms **V1–V4**. Therefore (10) is a vector structure of the set P^n.

It is easy to verify that this structure is compatible with projective Klein space (9). Hence, projective Klein space is a generalized elementary Klein Space.

It is obvious elliptic space (cf. [3], p. 41) is a generalized elementary space as well.

3. Tangent bundle

In [2] and [4] some techniques of construction of geometric objects are shown. Using one of them we can define a factor object. Let

$$(X, G, F) \tag{11}$$

be a geometric object of Klein space (7) and r a congruence of the fibre of this object, i.e. an equivalence relation satisfying condition

$$x_1\, r\, x_2 \Longrightarrow F(x_1, g)\, r F(x_2, g)$$

for all x_1, $x_2 \in X$ and $g \in G$. Then the mapping

$$F^r : \left(X/_r\right) \times G \to X/_r, \qquad F^r([x], g) = [F(x, g)]$$

is an action ofthe group G on factor set $X/_r$. Thus, we can define a new geometric object

$$\left(X/_r, G, F^r\right)$$

of Klein space (7), called a factor object of the object (11) by congruence r.

Now, consider n-dimensional generalized elementary Klein space (7) over K and a set

$$\mathcal{T}M := \{(p,\tau) : \; p \in M, \; \tau \in \mathcal{T}_D(M,K), \; D \in \Lambda_p\}.$$

It is easily seen that the function $F : \mathcal{T}M \times G \to \mathcal{T}M$, defined by the formula

$$F\left((p,\tau),g\right) = \left(f(p,g), f_g \circ \tau \circ f_g^{-1}\right),$$

is an action of G on $\mathcal{T}M$. Thus the triplet

$$(\mathcal{T}M, G, F) \tag{12}$$

is a geometric object of space (7). We will define a certain relation in the fibre of this object.

Definition 6. *We say that (p,τ), $(q,\tau_1) \in \mathcal{T}M$ are in relation r iff $p = q$ and there exists $\alpha \in \mathcal{A}_p(M)$ such that $\alpha \circ \tau \circ \alpha^{-1} = \tau_1$.*

It is easy to note that r is a congruence in the fibre of object (12). Let us denote

$$TM := \mathcal{T}M/_r$$

and define factor object

$$(TM, G, F^r) \tag{13}$$

of object (12) with respect to congruence r.

Definition 7. *Geometric object (13) of generalized elementary Klein space (7) will be called an abstract tangent bundle and its fibre a tangent bundle of this space. Abstraction classes $[(p,\tau)]$ will be called a tangent vectors to space (7) at the point p. The set of all tangent vectors to the space at a fixed point p will be denoted by T_pM and called tangent space to space (7) at the point p.*

Axiom **V4**, Corollary 2 and Definition 6 imply that for each tangent vector $[(p,\tau)]$ and $D' \in \Lambda_p$ there exists exactly one quasi-translation $\tau' \in \mathcal{T}_{D'}(M,K)$ such that $(p.\tau') \in [(p,\tau)]$. Moreover, if

$$v_p = [(p,\tau)], \quad w_p = [(p,\hat{\tau})], \qquad \text{where } \tau, \hat{\tau} \in \mathcal{T}_D(M,K),$$

and

$$(p, \tau') \in v_p, \quad (p, \hat{\tau}') \in w_p, \qquad \text{where } \tau', \hat{\tau}' \in \mathcal{T}_{D'}(M, K),$$

then

$$(p, \tau' \circ \hat{\tau}') \in [(p, \tau \circ \hat{\tau})] \quad \text{and } (p, a \cdot \tau') \in [(p, a \cdot \tau)] \quad \text{for each } a \in K.$$

Thus, in each space T_pM we can define the addition of two vectors

$$[(p, \tau)] + [(p, \hat{\tau})] := [(p, \tau \circ \hat{\tau})], \qquad \text{where } \tau, \hat{\tau} \in \mathcal{T}_D(M, K), \tag{14}$$

and the multiplication of a vector by an element of the field K

$$a \cdot [(p, \tau)] := [(p, a \cdot \tau)]. \tag{15}$$

Hence, tangent spaces T_pM with operations (14) and (15) form n-dimensional linear spaces over K.

Now, we will prove that for each $p \in M$, $v_p, w_p \in T_pM$, $a \in K$ and $g \in G$ the following equalities hold:

$$F^r(v_p + w_p, g) = F^r(v_p, g) + F^r(w_p, g), \tag{16}$$

$$F^r(av_p, g) = aF^r(v_p, g). \tag{17}$$

Indeed, for each $\tau, \hat{\tau} \in \mathcal{T}_D(M, K)$, where $D \in \Lambda_P$, we get

$$F^r\left([(p,\tau)] + [(p,\hat{\tau})], g\right) = F^r\left([(p, \tau \circ \hat{\tau})], g\right) =$$

$$= [F((p, \tau \circ \hat{\tau}), g)] = \left[\left(f(p,g), f_g \circ \tau \circ \hat{\tau} \circ f_g^{-1}\right)\right] =$$

$$= \left[\left(f(p,g), f_g \circ \tau \circ f_g^{-1} \circ f_g \circ \hat{\tau} \circ f_g^{-1}\right)\right] =$$

$$\left[\left(f(p,g), f_g \circ \tau \circ f_g^{-1}\right)\right] + \left[\left(f(p,g), f_g \circ \hat{\tau} \circ f_g^{-1}\right)\right] =$$

$$= [F((p,\tau), g)] + [F((p,\hat{\tau}), g)] =$$

$$= F^r\left([(p,\tau)], g\right) + F^r\left([(p,\hat{\tau})], g\right).$$

Thus, equality (16) is proven. Similarly, one can prove (17).

Results obtained above can be expressed the following theorem.

Theorem 1. *Abstract tangent bundle (13) of n-dimensional generalized elementary Klein space (7) has the following properties:*

(a)

$$TM = \bigcup_{p \in M} T_pM$$

(b) *Tangent spaces T_pM with operations (14) and (15) form n-dimensional linear spaces over the field K.*

(c) *For each $g \in G$ and $p \in M$ the mapping $F_g^\tau\big|_{T_pM}$ is a linear isomorphism of the tangent space T_pM onto the tangent space $T_{f(p,g)}M$, i.e. for each $p \in M$, v_p, $w_p \in T_pM$, $a \in K$ and $g \in G$ the relations (16) and (17) hold.*

For elementary Klein space (7) abstract tangent bundle (13) is an object equivalent to the object

$$(M \times \mathcal{T}(M,K), G, F), \qquad F((p,\tau),g) = \left(f(p,g), f_g \circ \tau \circ f_g^{-1}\right),$$

which is a product (cf. [3], p. 17) of space (7) and the object

$$(\mathcal{T}(M,K), G, F_0), \qquad F_0(\tau,g) = f_g \circ \tau \circ f_g^{-1}. \tag{18}$$

For elementary Klein space object (18) instead abstract tangent bundle is usually considered. It is called an abstract free vector (cf. [5]).

It is easily seen that for each $\tau \in \mathcal{T}_\Lambda(M,K)$ the mapping

$$L_\tau : TM \to TM, \qquad L_\tau([(p,\hat{\tau})]) = \left[\left(\tau(p), \tau \circ \hat{\tau} \circ \tau^{-1}\right)\right] \tag{19}$$

is well defined and is a bijection.

Definition 8. *Mapping (19) will be called a parallel transposition of tangent bundle TM. If $w_{\tau(p)} = L_\tau(v_p)$, we say that tangent vector $w_{\tau(p)}$ is defined parallel transposition (19) of tangent vector v_p from the point p to the point $\tau(p)$.*

Axiom **V2** implies that each vector v_p can be parallely transposed from the point p to an arbitrary point q.

It is easy to note that for each $\tau \in \mathcal{T}_\Lambda(M,K)$, $p \in M$, $v_p, w_p \in T_pM$ and $a \in K$ the equalities:

$$\begin{aligned} L_\tau(v_p + w_p) &= L_\tau(v_p) + L_\tau(w_p), \\ L_\tau(av_p) &= aL_\tau(v_p) \end{aligned}$$

remain true. Hence, the following theorem is also true:

Theorem 2. *Each parallel transposition (19) is a bijection and for each $p \in M$ its restriction $L_\tau\big|_{T_pM}$ is a linear isomorphism of tangent space T_pM onto tangent space $T_{\tau(p)}M$.*

Using the notation of tangent bundle we can define k-dimensional hyperplane in n-dimensional generalized elementary Klein space ($1 \leqslant k \leqslant n-1$).

References

[1] M. Kucharzewski, *Pojęcie przestrzeni Kleina i jej geometrii,* Zeszyty Naukowe Pol. Śl. Mat.-Fiz. **42** (1983), 65-73.

[2] M. Kucharzewski, *Własności przestrzeni Kleina I,* Skrypt Pol. Śl. 1209, Gliwice 1985.

[3] M. Kucharzewski, *Własności przestrzeni Kleina II,* Skrypt Pol. Śl. 1295, Gliwice 1986.

[4] B. Szociński, *Basic concepts of Klein geometries,* Zeszyty Naukowe Pol. Śl. Mat.-Fiz. **62** (1990).

[5] B. Szociński, *Elementary Klein space,* Demonstratio Math. **25** (1992), 785-793.

Recenzent: Prof. dr hab. Włodzimierz Waliszewski

Wpłynęło do redakcji 03.01.1992 r.

Streszczenie

Celem niniejszej pracy jest podanie definicji i pewnych własności uogólnionej przestrzeni elementarnej Kleina nad dowolnym ciałem.

Wprowadzona przez Kucharzewskiego w pracy [1] definicja przestrzeni Kleina jest zbyt ogólna. Na podstawie tej definicji w pracy [5], wykorzystując pojęcie przestrzeni liniowej translacji zbioru nad ciałem, określono pojęcie elementarnej przestrzeni Kleina nad dowolnym ciałem. Klasa tych przestrzeni jest jednak zbyt wąska, np. przestrzeń rzutowa nie jest elementarną przestrzenią Kleina.

W niniejszej pracy podano definicję (rozdział 1) i pewne własności (rozdział 3) uogólnionej przestrzeni elementarnej Kleina nad dowolnym ciałem. W rozdziale 2 pokazano, że przykładami tego rodzaju przestrzeni są takie przestrzenie, jak: afiniczna, euklidesowa, rzutowa i eliptyczna.