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ON ANTIDERIVATIVES IN THE GELFAND-SHILOV SPACES $K\{M_P\}'$

Summary. The Gelfand-Shilov spaces $K\{M_p\}'$ of distributions on $\mathbf{R}^d$ are considered for a given sequence $\{M_p\}$ of positive functions on $\mathbf{R}^d$ fulfilling the conditions (M), (N), (P). It is proved that for every $f \in K\{M_p\}'$ and $k \in \mathbf{N}_0^d$ there exists $F \in K\{M_p\}'$ such that $D^k F = f$ in $K\{M_p\}'$.

O PIERWOTNYCH W PRZESTRZENIACH $K\{M_P\}'$ GELFANDA-SZYŁOWA

Streszczenie. Rozważane są przestrzenie $K\{M_p\}'$ dystrybucji Gelfanda-Szyłowa na $\mathbf{R}^d$ dla ustalonego ciągu $\{M_p\}$ funkcji dodatnich na $\mathbf{R}^d$, spełniającego warunki (M), (N), (P). Dowodzi się, że dla dowolnych $f \in K\{M_p\}'$ i $k \in \mathbf{N}_0^d$ istnieje $F \in K\{M_p\}'$, takie że $D^k F = f$ w $K\{M_p\}'$.

О ПЕРВООБРАЗНЫХ В ПРОСТРАНСТВАХ $K\{M_P\}'$ ГЕЛЬФАНДА-ШИЛОВА

Резюме. Рассматриваются пространства $K\{M_p\}'$ обобщенных функций на $\mathbf{R}^d$ Гельфанда-Шилова для данной последовательности $\{M_p\}$ положительных функций на $\mathbf{R}^d$ удовлетворяющей условиям (M), (N), (P). Доказывается, что для любого $f \in K\{M_p\}'$ и $k \in \mathbf{N}_0^d$ существует $F \in K\{M_p\}'$, такое что $D^k F = f$ в $K\{M_p\}'$.

1. Introduction

We shall use mostly the standard notation. By $\mathbf{R}, \mathbf{N}, \mathbf{N}_0$ we denote the sets of all reals, all positive integers and all non-negative integers, respectively; for a fixed $d \in \mathbf{N}$, by $\mathbf{R}^d, \mathbf{N}^d, \mathbf{N}_0^d$ we mean the Cartesian products of d copies of the respective sets. If $a = (\alpha_1, \ldots, \alpha_d), b = (\beta_1, \ldots, \beta_d) \in \mathbf{R}^d, \lambda \in \mathbf{R}$, we denote $a+b := (\alpha_1+\beta_1, \ldots, \alpha_d+\beta_d)$, $\lambda a := (\lambda\alpha_1, \ldots, \lambda\alpha_d)$, $ab := (\alpha_1\beta_1, \ldots, \alpha_d\beta_d)$, $\|a\| := (\alpha_1^2+\ldots+\alpha_d^2)^{-1/2}$, $|a| := |\alpha_1|+\ldots+|\alpha_d|$ (in particular, $|k| = \kappa_1 + \ldots + \kappa_d$ for $k = (\kappa_1, \ldots, \kappa_d) \in \mathbf{N}_0^d$). By e_i we mean the multi-index $(0, \ldots, 0, 1, 0, \ldots, 0) \in \mathbf{N}_0^d$, where 1 appears at ith place.

By *smooth functions* we mean real-valued C^∞-functions on $\mathbf{R}$ or $\mathbf{R}^d$ and by $\mathcal{D}(\mathbf{R})$ and $\mathcal{D}(\mathbf{R}^d)$ the spaces of all smooth functions on $\mathbf{R}$ and $\mathbf{R}^d$, respectively, endowed with the standard topologies. The Gelfand-Shilov spaces $K\{M_p\}'$, defined below, are subspaces of the dual $\mathcal{D}'$ of $\mathcal{D}(\mathbf{R}^d)$, the space of distributions.

By D^k we mean the *differentiation operator* of order $k = (\kappa_1, \ldots, \kappa_d) \in \mathbf{N}_0^d$, i.e. if φ is a smooth function on $\mathbf{R}^d$, then

$$D^k\varphi(x) := \frac{\partial^{\kappa_1+\ldots+\kappa_d}}{\partial\xi_1^{\kappa_1}\ldots\partial\xi_d^{\kappa_d}}\varphi(\xi_1, \ldots, \xi_d)$$

for $x = (\xi_1, \ldots, \xi_d) \in \mathbf{R}^d$. The differentiation operator can be extended in the standard way to the spaces $\mathcal{D}'$ and, in particular, $K\{M_p\}'$.

By H and δ on $\mathbf{R}$ we denote the *Heaviside function* and the *Dirac delta distribution*, respectively, i.e.

$$H(\xi) := \begin{cases} 1 & \text{if } \xi \geq 0 \\ 0 & \text{if } \xi < 0 \end{cases}; \qquad \langle\delta, \varphi\rangle := \varphi(0)$$

for an arbitrary $\varphi \in \mathcal{D}(\mathbf{R})$. In $\mathbf{R}^d$, H and δ are meant as follows:

$$H(x) := H(\xi_1)\cdot\ldots\cdot H(\xi_d); \qquad \delta(x) := \delta(\xi_1)\cdot\ldots\cdot\delta(\xi_d) := \delta(\xi_1)\otimes\ldots\otimes\delta(\xi_d)$$

for $x = (\xi_1, \ldots, \xi_d) \in \mathbf{R}^d$.

By a *unit-sequence* in $\mathbf{R}^d$ we shall mean a sequence $\{\rho_n\}$ of functions of the class $\mathcal{D}(\mathbf{R}^d)$ such that for every compact $K \subset \mathbf{R}^d$ there exists an n_0 so that $\rho_n(x) = 1$ for $x \in K$ and $n \geq n_0$.

Let us recall the definition of the space $K\{M_p\}$ (cf. [1], p. 78). Let $\{M_p\}$ be a non-decreasing sequence of functions $M_p: \mathbf{R}^d \mapsto [1, \infty]$, continuous on the sets $\Omega_p := \{x \in \mathbf{R}^d\colon M_p(x) < \infty\}$, $p \in \mathbf{N}$, which are assumed to be equal, so we put $\Omega_p := \Omega$ for $p \in \mathbf{N}$. We define the space $K\{M_p\}$ as the locally convex space consisting of all smooth functions φ on $\mathbf{R}^d$ for which the expressions

$$\|\varphi\|_p = \max_{|k|\leq p}\sup_{x\in\mathbf{R}^d} M_p(x)|D^k\varphi(x)| \qquad (p \in \mathbf{N}) \tag{1}$$

are finite, endowed with the topology induced by the sequence of norms (1). We adopt here and throughout the whole paper the notation $0 \cdot \infty = 0$; in particular, if $\|\varphi\|_p < \infty$ and $M_p(x_0) = \infty$, then for every multi-index k with $|k| \leq p$ we have $D^k\varphi(x_0) = 0$. Clearly, $K\{M_p\}$ is a Fréchet space. The elements of the dual $K\{M_p\}'$ of $K\{M_p\}$ will be called distributions of Gelfand-Shilov.

Denote by Φ_p ($p \in \mathbf{N}$) the space $K\{M_p\}$ equipped with the topology induced by the single norm $\|\cdot\|$. Let us recall the following characterization of the space $K\{M_p\}'$ (cf. [1], p. 37):

A linear functional f on $K\{M_p\}$ is continuous if and only if there exist a $p \in \mathbf{N}$ and a positive constant K such that

$$|\langle f, \varphi\rangle| \leq K\|\varphi\|_p \qquad (\varphi \in K\{M_p\}) \tag{2}$$

where $\langle f, \varphi\rangle$ denotes the value of f at φ.

If (2) holds, we say that the distribution f has an order $\geq p$. Notice that the space of all distributions of order $\geq p$ is the dual of the space Φ_p and therefore it is a complete normed space. We denote it by Φ_p'.

For every multi-index $k \in \mathbf{N}^d$ the operation $\varphi \mapsto D^k\varphi$ in $K\{M_p\}$ is linear and continuous and so for each distribution $f \in K\{M_p\}'$, the functional $D^k f$ defined by

$$\langle D^k f, \varphi\rangle = (-1)^{|k|}\langle f, D^k\varphi\rangle$$

is a distribution of Gelfand-Shilov, called the (distributional) derivative of order k of f. The functional f is then said to be an antiderivative of order k of $F = D^k f$.

To obtain some important properties of the considered spaces $K\{M_p\}$ and $K\{M_p\}'$ I.M. Gelfand and G.I. Shilov assumed the following three conditions on the sequence $\{M_p\}$ (see [1], p. 92, 111):

(M) *for each $p \in \mathbf{N}$ there exists a constant $C_p > 0$ such that*

$$M_p(\xi_1, \ldots, \xi_i', \ldots, \xi_d) \leq C_p M_p(\xi_1, \ldots, \xi_i'', \ldots, \xi_d)$$

whenever $|\xi'| \leq |\xi_i''|$ and $\xi_i' \xi_i'' \geq 0$;

(N) *for each $p \in \mathbf{N}$ there exists a $p' \in \mathbf{N}$, $p' > p$ such that*

$$m_{p,p'} := M_p M_{p'}^{-1} \in L^1(\mathbf{R}^d);$$

(P) *for each $p \in \mathbf{N}$ there exists a $p' \in \mathbf{N}$, $p' > p$ such that for every $\varepsilon > 0$ there is a $T > 0$ for which $m_{p,p'} < \varepsilon$ whenever $\|x\| > T$ or $M_p(x) > T$.*

In particular, if the sequence $\{M_p\}$ satisfies conditions (M), (N), (P), then every distribution $f \in K\{M_p\}'$ can be represented in the form:

$$f = \sum_{|k| \leq p} D^k(M_p f_k), \tag{3}$$

where f_k are bounded measurable functions.

A. Kamiński proved in [2] that if the sequence $\{M_p\}$ satisfies an additional condition (N′), then the representation (3) can be simplified, namely the distributions of the space $K\{M_p\}'$ may be represented then in the form

$$f = D^k(M_p f_k), \tag{4}$$

where $k \in \mathbf{N}_0^d$ and f_k is a bounded measurable function. It follows directly from Kamiński's result that

Theorem. *If the sequence $\{M_p\}$ satisfies the conditions* (M), (N), (P), (N′), *then for every distribution $f \in K\{M_p\}'$ and a multi-index $k \in \mathbf{N}_0^d$ there exists an antiderivative F in $K\{M_p\}'$ of order k of f and the distribution F is again of the form* (4).

In [4], it was proved that distributions of the space $K\{M_p\}'$ need not be represented in the form (4) unless condition (N′) is assumed.

The aim of this note is to show that the Theorem holds true, even if the sequence $\{M_p\}$ satisfies only conditions (M), (N), (P). An analogous property holds also for the convergence in $K\{M_p\}'$ (cf. Theorem 2).

2. Auxiliary results

We shall need two lemmas.

Lemma 1. *If F is a locally integrable function on $\mathbf{R}$ vanishing at infinity and for every unit-sequence $\{\rho_n\}$ the limit*

$$A = \lim_{n\to\infty} \int_a^\infty \rho_n F \tag{5}$$

exists for some $a \in \mathbf{R}^1$, then F is improperly integrable on $[a,\infty)$ and the limit, which does not depend on the choice of a unit-sequence, is equal to the improper integral $\int_a^\infty F$. An analogous property holds for intervals of the form $(-\infty,a)$ and $(-\infty,\infty)$.

Proof. Without loss of generality we can assume that $F = 0$ on $(-\infty,a]$. Let $I_n = [a_n,b_n]$ be a sequence of intervals such that $a_n \to \infty$, $b_n \to \infty$. Denote by χ_n the characteristic functions of I_n and by ρ an arbitrary smooth function of bounded support with $\int \rho = 1$. Suppose that $\operatorname{supp} \rho \subset [-c,c]$ for some $c > 0$. It is easy to see that (ρ_n), where $\rho_n = \chi_n * \rho$ for $n \in \mathbf{N}$, is a unit-sequence and

$$A_n := \operatorname{supp}\,(\rho_n - \chi_n) \subset [a_n - c, a_n + c] \cup [b_n - c, b_n + c]$$

for $n \in \mathbf{N}$. Thus, by the assumption,

$$\begin{aligned}
&\left|\int_{-\infty}^{\infty} \chi_n F - \int_{-\infty}^{\infty} \rho_n F\right| \\
&\leq \int_{A_n} \left[\int_{-c}^{c} |\chi_n(x) - \chi_n(x-t)||\rho(t)|\,dt\right] |F(x)|\,dx \\
&\leq 2\int_{-c}^{c} |\rho(t)|dt \cdot \int_{A_n} |F(x)|\,dx \\
&\leq 4c \operatorname{sup\,ess}_{x\in A_n} |F(x)| \int_{-\infty}^{\infty} |\rho(t)|\,dt
\end{aligned}$$

for $n \in \mathbf{N}$. Hence

$$\lim_{n\to\infty} \int_a^{b_n} F = \lim_{n\to\infty} \int_{-\infty}^{\infty} \chi_n F = A + \lim_{n\to\infty} \left(\int_{-\infty}^{\infty} \chi_n F - \int_{-\infty}^{\infty} \rho_n F\right) = A.$$

Notice that if (ϱ_n') and (ϱ_n'') are two unit-sequences, then the interlaced sequence (ϱ_n) defined by

$$\varrho_{2n-1} := \varrho_n'; \qquad \varrho_{2n} := \varrho_n'' \qquad\qquad \text{for } n \in \mathbf{N}$$

is also a unit-sequence. Hence the limit (5) does not depend on the choice of a unit-sequence and since the sequence $\{I_n\}$ of intervals is arbitrary, we conclude that f is improperly integrable and $\int_a^\infty F = A$. □

In the next two lemmas the following notation will be used. Let

$$S := \{s = (\sigma_1, \ldots, \sigma_d) : \ \sigma_i = -1 \text{ or } 1 \ \text{ for } i = 1, \ldots, d\}.$$

Given an $s = (\sigma_1, \ldots, \sigma_d) \in S$ (fixed temporarily), $y = (\eta_1, \ldots, \eta_d) \in \mathbf{R}^d$ and $i \in \{1, \ldots, d\}$, denote

$$I_y^i := I_{s,y}^i := \begin{cases} (-\infty, \eta_i) & \text{if } \sigma_i = -1 \\ (\eta_i, \infty) & \text{if } \sigma_i = 1. \end{cases}$$

Now, let $J_y := J_{s,y} := I_y^1 \times \ldots \times I_y^d$ and

$$Q_s := \{y = (\eta_1, \ldots, \eta_d) \in \mathbf{R}^d : \ \sigma_i \eta_i \geq 0 \text{ for } \ i = 1, \ldots, d\}.$$

Given $p \in \mathbf{N}$ and $y \in Q_s$, consider the following function of variable $x \in \mathbf{R}^d$:

$$f_y(x) := f_{s,y}(x) := M_p(y)\chi(y)H(s(x-y)),$$

where χ denotes the characteristic function of the set Ω.

Lemma 2. *Given $p \in \mathbf{N}$ let $p' \in \mathbf{N}$ be such an index greater than p for which condition (N) holds. The distribution f_y for $y \in Q_s$ is of order $\geq p'$ and $D^{(0,1,\ldots,1)} f_y$ is of order $\geq p' + d - 1$. Moreover, if $y_n \in Q_s$ for $n \in \mathbf{N}$, then*

$$D^{(0,1,\ldots,1)} f_{y_n} \to 0 \ \ \textit{in} \ \ \Phi_{p'}', \textit{ whenever } |y_n| \to \infty \tag{6}$$

and

$$D^{(0,1,\dots,1)} f_{y_n} \to D^{(0,1,\dots,1)} f_{y_0} \quad in \quad \Phi'_{p'} \ whenever \ y_n \to y_0 \in Q_s \tag{7}$$

as $n \to \infty$.

Proof. According to (M) and (N), we have $m_{p,p'} \in L^1(\mathbf{R}^d)$ and

$$|\langle f_y, \psi\rangle| \le (C_p)^d \int_{J_y} M_p(x)|\psi(x)|dx \le K_y \|\psi\|_{p'}, \tag{8}$$

where

$$K_y = (C_p)^d \int_{J_y} m_{p,p'}(x)\, dx, \tag{9}$$

for some constant $C_p > 0$ and arbitrary $\psi \in K\{M_p\}$. This means that f_y is a distribution in $K\{M_p\}'$ of order $\ge p'$ and, consequently, $D^{(0,1,\dots,1)} f_y$ is of order $\ge p' + d - 1$.

For every $\psi \in K\{M_p\}$, we have

$$|\langle D^{(0,1,\dots,1)} f_y, \psi\rangle| = M_p(y)\chi(y) \left| \int_{I_1^y} \psi(y_\tau)\, d\tau \right|, \tag{10}$$

where the integral on the right hand side is improper. In fact, given arbitrary unit-sequences $\{\rho_{i,n}\}$ in $\mathcal{D}(\mathbf{R}^1)$ for $i = 1, \dots, d$, the sequence (ρ_n), defined by

$$\rho_n := \rho_{1,n} \cdot \ldots \cdot \rho_{d,n}, \qquad n \in \mathbf{N},$$

is also a unit-sequence in $\mathcal{D}(\mathbf{R}^d)$ and

$$\rho_{1,n}(\xi_1) = \rho_n(\xi_1, \eta_2, \dots, \eta_d)$$

for every $(\eta_2, \dots, \eta_d) \in \mathbf{R}^{d-1}$ and sufficiently large $n \in \mathbf{N}$. Hence, in view of (N), $\rho_n\psi \to \psi$ in $K\{M_p\}$ as $n \to \infty$ and

$$|\langle D^{(0,1,\dots,1)} f_y, \psi\rangle| = \lim_{n\to\infty} |\langle D^{(0,1,\dots,1)} f_y, \rho_n\psi\rangle| =$$

$$= \lim_{n\to\infty} |\langle\, M_p(y)\chi(y)H(\sigma_1(\xi_1 - \eta_1))\delta(\bar{s}(\bar{x} - \bar{y}),\ \rho_n(x)\psi(x)\,\rangle|$$

$$= M_p(y)\chi(y) \lim_{n\to\infty} \left| \int_{I_1^y} \rho_{1n}(\tau)\psi(y_\tau)\, d\tau \right|,$$

where $\bar{s} := (\sigma_2, \dots, \sigma_d), \bar{x} := (\xi_2, \dots, \xi_d), \bar{y} := (\eta_2, \dots, \eta_d)$. Since the sequence $\{\rho_{1,n}\}$ was chosen arbitrarily, we conclude, by virtue of Lemma 1, that the improper integral in (10) exists and equality (10) holds.

Notice that if $|y_n| \to \infty$, then $f_{y_n} \to 0$ in Φ'_p as $n \to \infty$, in view of (8), (9) and (N), i.e. (6) is proved.

Similarly, due to (M) and (N), for every $\psi \in K\{M_p\}$ and $y_0 \in Q_s$ we have

$$|\langle f_y - f_{y_0}, \psi\rangle|$$

$$= \int_{J_y \cup J_{y_0}} [M_p(y)H(s(x-y)) - M_p(y_0)H(s(x-y_0))]\, \psi(x)\, dx$$

$$\leq (C_p)^d \int_{I_{y,y_0}} M_p(x)|\psi(x)|\, dx + |M_p(y) - M_p(y_0)| \int_{J_{y,y_0}} |\psi(x)|\, dx \leq D_{y,y_0} \|\psi\|_{p'}$$

with

$$D_{y,y_0} := (C_p)^d \int_{I_{y,y_0}} m_{p,p'}(x)\, dx + |M_p(y) - M_p(y_0)| \int_{J_{y,y_0}} |\chi(x)|\, m_{p,p'}(x)\, dx,$$

where $I_{y,y_0} := J_y \triangle J_{y_0}$ and $J_{y,y_0} := J_y \cap J_{y_0}$. Hence $y_n \to y_0$ implies $f_{y_n} \to f_{y_0}$ in Φ_p' as $n \to \infty$ and so (7) holds true. □

Lemma 3. *For each $p \in \mathbf{N}$ and multi-index k there exists an integer $q > p$ such that $\|D^k \varphi_n\|_q \to 0$ implies $\|\varphi_n\|_p \to 0$ for every sequence $\{\varphi_n\}$ in $K\{M_p\}$. In particular, if $\varphi_n \in K\{M_p\}$ for $n \in \mathbf{N}$ and $D^k \varphi_n \to 0$ in $K\{M_p\}$, then $\varphi_n \to 0$ in $K\{M_p\}$.*

Proof. First we shall show the assertion for $k = e_1$. Fix $p \in \mathbf{N}$ and let $p' \in \mathbf{N}$ be as in Lemma 2. Define $q := p + p' + d - 1$ and assume that $\|D^{e_1}\varphi_n\|_q \to 0$ as $n \to \infty$. We shall prove that $\|\varphi_n\|_p \to 0$ as $n \to \infty$, which means that

$$M_p(y)|D^l \varphi_n(y)| \to 0 \qquad \text{uniformly on } \mathbf{R}^d \tag{11}$$

for every multi-index l such that $|l| \leq p$. Since $q > p$ and the sequence $\{M_p\}$ is nondecreasing, it is clear that it is enough to show (11) for every l of the form $l = (0, \lambda_2, \ldots, \lambda_d)$ with $|l| \leq p$.

Fix $l = (0, \lambda_2, \ldots, \lambda_d)$ with $|l| \leq p$. Since, in view of (N), $D^l \varphi_n(y) \to 0$ as $|y| \to \infty$ for every fixed $n \in \mathbf{N}$, we can write

$$D^l \varphi_n(y) = \int_{-\infty}^{\eta_1} D^{\tilde{l}} \varphi_n(y_\tau)\, d\tau = -\int_{\eta_1}^{\infty} D^{\tilde{l}} \varphi_n(y_\tau)\, d\tau,$$

where $\tilde{l} = e_1 + l = (1, \lambda_2, \ldots, \lambda_d)$, $y = (\eta_1, \ldots, \eta_d)$, $y_\tau = (\tau, \eta_2, \ldots, \eta_d)$ and the integrals are improper. By (10),

$$M_p(y)|D^l \varphi_n(y)| = M_p(y)\chi(y) \left| \int_{-\infty}^{\eta_1} D^{\tilde{l}} \varphi_n(y_\tau) d\tau \right| = |\langle D^{(0,1,\ldots,1)} f_y, D^{\tilde{l}} \varphi_n \rangle|.$$

In view of (P), the functions $M_p(y)|D^l \varphi_n(y)|$ $(n \in \mathbf{N})$ are continuous and vanish at infinity. Therefore

$$\sup_{y \in Q_s} M_p(y)|D^l \varphi_n(y)| = M_p(y_n)|D^l \varphi_n(y_n)| = |\langle D^{(0,1,\ldots,1)} f_{y_n}, D^{\tilde{l}} \varphi_n \rangle|$$

for some $y_n \in Q_s$ $(n \in \mathbf{N})$.

Note that if $|m| \leq p' + d - 1$ for some multi-index m, then the multi-index $m' := l + m$ satisfies the inequality $|m'| \leq p + p' + d - 1$. Therefore

$$\|D^{\tilde{l}}\varphi_n\|_{p'+d-1} = \max_{|m|\leq p'+d-1} \sup_{y\in\mathbf{R}^d} M_{p'+d-1}(y)\,|D^{\tilde{l}+m}\varphi_n(y)|$$
$$\leq \max_{|m'|\leq q} \sup_{y\in\mathbf{R}^d} M_q(y)|D^{e_1+m'}\varphi_n(y)| = \|D^{e_1}\varphi_n\|_q.$$

Hence, by the assumption, $\|D^{\tilde{l}}\varphi_n\|_{p'+d-1} \to 0$ as $n \to \infty$. Since the bilinear form $\langle f, \varphi\rangle$ is separately continuous on $\Phi_p' \times \Phi_p$, it is continuous on $\Phi_p' \times \Phi_p$ (cf. e.g. [3], Th. 2.17). Thus, in view of Lemma 2, every subsequence of the sequence $\{a_n\}$ of numbers

$$a_n := |\langle D^{(0,1,\ldots,1)} f_{y_n}, D^{\tilde{l}}\varphi_n\rangle|$$

contains a subsequence converging to 0. Therefore $a_n \to 0$, i.e.,

$$M_p(y)|D^l\varphi_n(y)| \to 0 \quad \text{uniformly in } Q_s.$$

To derive (11) it remains to observe that $\mathbf{R}^d = \bigcup_{s\in S} Q_s$ and the set S is finite.

Analogously, one can prove the assertion for $k = e_i$ $(i = 2, \ldots, d)$. By the induction principle, the assertion of the lemma follows for any $k \in \mathbf{N}^d$.

Finally notice that $\varphi_n \to 0$ in $K\{M_p\}$ if and only if $\|\varphi_n\|_p \to 0$ for every $p \in \mathbf{N}$. Hence $D^k\varphi_n \to 0$ in $K\{M_p\}$ implies $\varphi_n \to 0$ in $K\{M_p\}$ for every sequence $\{\varphi_n\}$ in $K\{M_p\}$. □

Corollary. *For each $p \in \mathbf{N}$ and multi-index k there exist an integer $q > p$ and a constant $B_p > 0$ such that $\|\varphi\|_p \leq B_p\|D^k\varphi\|_q$ for every $\varphi \in K\{M_p\}$.*

Proof. Suppose the contrary. Thus there is a $p \in \mathbf{N}$ such that for each $n \in \mathbf{N}$ there exists a function $\varphi_n \in K\{M_p\}$ for which

$$n\|D^k\varphi_n\|_{p+n} < \|\varphi_n\|_p \qquad (n \in \mathbf{N}).$$

Hence $\{\psi_n\}$ with $\psi_n := \varphi_n/\|\varphi_n\|_p$ is a sequence of functions of $K\{M_p\}$ such that $\|\psi_n\|_p = 1$ for $n \in \mathbf{N}$ and $\|D^k\psi_n\|_q \to 0$ for every $q > p$. But this contradicts Lemma 3. □

3. Main Results

Applying the results of the preceding section, we are going to prove now the main theorems of the paper.

Theorem 1. *For each $f \in K\{M_p\}'$ and every multi-index $k \in \mathbf{N}_0^d$ there exists a distribution $F \in K\{M_p\}'$ such that $D^k F = f$.*

Proof. Define a linear functional F on the subspace

$$K^k\{M_p\} := \{D^k\varphi\colon \varphi \in K\{M_p\}\}$$

as follows:

$$\langle F, D^k\varphi\rangle := (-1)^{|k|}\langle f, \varphi\rangle$$

for $\varphi \in K\{M_p\}$. In view of Lemma 3, the functional F is continuous with respect to the locally convex topology of $K\{M_p\}$. Therefore, by the Hahn-Banach theorem, there exists a continuous linear extension $\tilde{F}$ of F to the whole space $K\{M_p\}$. It is easy to check that $D^k\tilde{F} = f$ which completes the proof. □

Theorem 2. *For each multi-index $k \in \mathbf{N}_0^d$ and distributions $F, f, f_n \in K\{M_p\}'$ such that $f_n \to f$ in $K\{M_p\}'$ as $n \to \infty$ and $D^k F = f$ there exists a sequence $F_n \in K\{M_p\}'$ for which $D^k F_n = f_n$ $(n \in \mathbf{N})$ and $F_n \to F$ as $n \to \infty$.*

Proof. For each $n \in \mathbf{N}$ define a linear functional H_n on $K^k\{M_p\}$, defined in the proof of the previous theorem, as follows:

$$\langle H_n, D^k\varphi\rangle := (-1)^{|k|}\langle f_n - f, \varphi\rangle.$$

By the assumption, $f_n - f \to 0$ in $K\{M_p\}'$, which means that there is a sequence (ε_n) of positive numbers such that $\varepsilon_n \to 0$ as $n \to \infty$ and

$$|\langle f_n - f, \varphi\rangle| \le \varepsilon_n \|\varphi\|_p$$

for $n \in \mathbf{N}$. In view of Corollary,

$$|\langle H_n, D^k\varphi| \le \varepsilon_n B_p \|D^k\varphi\|_q$$

for $n \in \mathbf{N}$. Hence, by the Hahn-Banach theorem, for each $n \in \mathbf{N}$ there exist continuous linear extensions $\tilde{H}_n$ of H_n to the whole space $K\{M_p\}$ which satisfy the inequality

$$|\langle \tilde{H}_n, \varphi\rangle| \le \varepsilon_n B_p \|\varphi\|_q$$

for $n \in \mathbf{N}$. Thus $\tilde{H}_n \to 0$ in $K\{M_p\}'$ as $n \to \infty$ and $D^k\tilde{H}_n = f_n - f = f_n - D^k F$. Consequently, the sequence $F_n := \tilde{H}_n + F$ fulfils the assertion of the theorem. □

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Streszczenie

Rozważane są przestrzenie $K\{M_p\}'$ dystrybucji na $\mathbf{R}^d$, wprowadzone przez I.M. Gelfanda i G.I. Szyłowa, definiowane dla ustalonego ciągu $\{M_p\}$ funkcji dodatnich na $\mathbf{R}^d$, przy czym zakłada się jedynie, że ciąg $\{M_p\}$ spełnia warunki (M), (N), (P) Gelfanda-Szyłowa.

W pracy rozważane jest zagadnienie istnienia, dla dowolnej dystrybucji $f \in K\{M_p\}'$ i dowolnego wielowskaźnika $k \in \mathbf{N}^d$, pierwotnej dystrybucji f rzędu k w $K\{M_p\}'$, tzn. takiej dystrybucji $F \in K\{M_p\}'$, że $D^kF = f$ w $K\{M_p\}'$.

W przypadku, gdy ciąg $\{M_p\}$ oprócz (M), (N), (P) spełnia dodatkowy warunek (N'), istnienie pierwotnej wynika z twierdzenia A. Kamińskiego o reprezentacji dystrybucji w przestrzeniach $K\{M_p\}'$. Celem pracy jest udowodnienie istnienia pierwotnej dowolnego rzędu dowolnej dystrybucji w $K\{M_p\}'$ bez zakładania dodatkowego warunku (N'). Jest to o tyle istotne, że – jak zostało to pokazane w pracy [4] – istnieją przestrzenie $K\{M_p\}'$ Gelfanda-Szyłowa z $\{M_p\}$ spełniającym (M), (N), (P) i nie spełniającym (N') oraz dystrybucje w tych przestrzeniach, które nie dadzą się reprezentować w postaci zagwarantowanej przez twierdzenie Kamińskiego.

Oczywiście, dla danej dystrybucji $f \in K\{M_p\}'$ i wielowskaźnika $k \in \mathbf{N}^d$ istnieje wiele pierwotnych F rzędu k. Dowodzi się, że dla ustalonego $k \in \mathbf{N}^d$ pierwotne rzędu k można wybrać w sposób ciągły, a dokładniej, że:

Dla danego wielowskaźnika $k \in \mathbf{N}^d$ oraz dystrybucji $F, f, f_n \in K\{M_p\}'$, takich że $f_n \to f$ w $K\{M_p\}'$ i $D^kF = f$ istnieje ciąg (F_n) w $K\{M_p\}'$, dla którego $D^kF_n = f_n$ $(n \in \mathbf{N})$ oraz $F_n \to F$ w $K\{M_p\}'$.