

Ludmiła CZECH

AN APPLICATION OF INVOLUTORY L TRANSFORMATION

Summary. In [2] a simple descriptive definition and properties of involutory quadratic plane transformation named L transformation are given. A computer program based on the simple algorithm of L transformation plots conics as straight line images, cubics as images of circle through one of the fundamental points and quartics with three real singular points from a circle not passing through fundamental points. A table presents a visual systematization of quartics with singular point T (isolated, node or cusps), doubly tangent to t_0 at the singular point O (isolated, tacnode or „ramphoid” cusp [1]).

ZASTOSOWANIE INWOLUCYJNEGO PRZEKSZTAŁCENIA L

Streszczenie. W [2] została podana prosta graficzna definicja inwolucyjnego przekształcenia kwadratowego, nazywanego przekształceniem L . Na definicji tej oparte są programy komputerowe kreślące krzywe trzeciego i czwartego rzędu jako obrazy okręgów, a stożkowe jako obrazy prostych w tym przekształceniu. W pracy omówiono własności przekształcenia L . Podana jest klasyfikacja krzywych rzędu czwartego z punktem podwójnym (węzłem, ostrzem, punktem izolowanym) T , po- dwójnie stycznych do prostej t_0 w punkcie O , który jest albo węzłem, albo ostrzem jednostronnie stycznym, albo punktem izolowanym.

ANWENDUNG DER INVOLUTORISCHEN *L* TRANSFORMATION

Zusammenfassung. Einfache geometrische Definition einer involutorischen quadratischen Transformation, die *L* Transformation genannt wurde [2], ist als ein Algorithmus eines kurzen Komputerprogrammes benutzt. Dieses Programm zeichnet die Kurven der dritten und vierten Ordnung als Abbildungen der Kreise und Kegelschnitte als Abbildungen der Geraden in der *L* Transformation. Die Eigenschaften der *L* Transformation sind angegeben. Die erhaltenen affinen Formen von Kurven der vierten Ordnung mit dem doppelten Punkt *T* (isolierter Punkt, Knoten, Sptzpunkt) und doppelttangente in *O* (isolierter Punkt, Knoten, Sptzpunkt) zur Gerade t_0 wurden in einer Tabelle geordnet.

1. Introduction

The theory of curves is connected with transformation. Descriptive geometry methods and constructive geometry enable visualization of curves by means of computer.

The transformation named involutory *E* transformation [2] is a quadratic transformation [3] with one real fundamental point and two imaginary conjugated points on the real straight line (fig. 1a), belonging to the class of Cremona transformation [4]. The *E* transformation had appeared in the study of the relationships between representation of elements of the three dimensional space in the known central projection and the new conical projection [2, 5, 6]. A generalization of construction made it possible to give a descriptive definition of the plane involutory quadratic transformation named *L* transformation¹

2. *L* Transformation

2.1. Geometrical definition of *L* transformation

Involutory quadratic transformation with three real fundamental points from which two are coincident on one straight line may be graphically defined in the following way:

Let two points *O* and *T* be distinguished in the Euclidean plane. From now all the other points will belong to the Euclidean plane complemented with the straight line at infinity (fig. 1b). Let to an arbitrary point \dot{A} of the plane laying neither on the straight line *OT* or on the straight line t_0 through *O* perpendicular to *OT*, a common point \ddot{A} of the straight line *AT* and the straight line through *O* perpendicular to $\dot{A}O$ be assigned.

¹See notices in „References“ at the end of the paper.

The point T corresponds to the whole straight line t_0 through O perpendicular to OT . To every point of the straight line t_0 point T is assigned.

The point O corresponds to the whole straight line OT . To every point of the straight line OT point O is assigned.

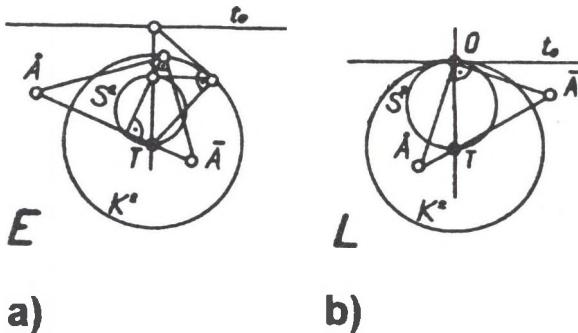


Fig. 1.

2.2. Properties of L transformation

- 1° Every straight line through T is transformed into the same straight line; where every its point corresponds to another point of this line in such a way that the two points form a pair in the elliptic involution defined on that line (see 7° and 8°).
- 2° Every straight line through O is transformed to the perpendicular straight line through O . The lines assigned to each other form a pair in the elliptic involution in the pencil (O).
- 3° There is a bijection between the set of straight lines belonging to the plane of transformation without pencils (O) and (T) and the homaloidal net of conics [4] with three real fundamental points, two of which coinciding at one point O on line t_0 ; the third is point T .
- 4° The straight line at infinity of the plane of L transformation is transformed into the circle S^2 of diameter OT and reciprocally: to an arbitrary point of circle S^2 the point at infinity is assigned.
- 5° The circle K^2 with center T and radius OT is fixed; L transformation for points of this circle is a central symmetry with center T (it changes the ends of diameters in pairs).

- 6° The straight line t_0 cuts the plane of transformation into two half-planes. The half-plane without point T is transformed into disk S^2 and reciprocally: disk S^2 is transformed into the half-plane.
- 7° For any straight line of the fixed pencil (T) elliptical involution is determined by means of two pairs of points: 1) point T and the common point of the straight line of pencil (T) and line t_0 , 2) ends of an arbitrary diameter of the circle K^2 . In these two pairs one can be replaced by the following pair of points: the point at infinity and the center of the involution i.e. a common point of the line of pencil (T) and the circle S^2 .
- 8° The circle S^2 (see 4°) is the locus of the centers of elliptic involution (see 1°) on the straight lines of the fixed pencil (T) (Theorem 6.3 in [2]).
- 9° The common point of the transformed line and line t_0 connected with point T gives a tangent at the singular point T of the image of the line in L transformation.
- 10° Disk K^2 and its complements of the L transformation plane are interchanged.

3. Creation of curves

3.1. Conics

- 1° A secant of the circle S^2 is transformed in L transformation into a **hyperbola** and vice versa: hyperbola belonging to the homoloidal net (T, O, t_0) i.e. hyperbola through T tangent in O to the straight line $t_0 \perp AT$, is transformed into a secant of the circle S^2 .
- 2° Any tangent to the circle S^2 is transformed into a **parabola** and reciprocally: a parabola through T , tangent at O to t_0 (i.e. a parabola belonging to the homaloidal net (T, O, t_0)) is transformed into a straight line which is tangent to circle S^2 .
- 3° The straight line without any real common points with circle S^2 is transformed into an **ellipse** and reciprocally: ellipse through T , tangent in O to t_0 (i.e. a ellipse belonging to the homaloidal net (T, O, t_0)) is transformed into a straight line without common points with the circle S^2 .

VISUAL SYSTEMATIZATION OF QUARTIC CURVES

WITH THREE DOUBLE POINTS

TWO OF WHICH ARE COINCIDENT IN O ON ONE TANGENT t_0

UNSYMMETRICAL FORMS with "isolated lemnisc" O on t_0		SYMMETRICAL FORMS with isolated singular point T		UNSYMMETRICAL FORMS with node or cusp in singular point T with tacnode or "taenphoid" cusp in point O on t_0 (two singular points coinciding on t_0)			
							<img alt="Quartic curve

3.2. Cubic curves

It is known [4] that quadratic transformation increases the degree of curve twice. If the curve contains one fundamental point, its image by L transformation breaks up into a fixed line corresponding to the fundamental point and a residual curve of a lower degree.

Thus the result of L transformation of conic (or circle) containing points T and O or conic (or circle) tangent to t_0 at point O is another conic (or circle) through T and O or conic (or circle) tangent to t_0 in point O .

The image by L transformation of conic (or circle) containing one fundamental point consist of a cubic and straight line through the two other fundamental points.

The image by L transformation of a circle through O which is not tangent to t_0 consists of the straight line OT and a **cubic** with

- a) a node at O if the circle cuts line OT ,
- b) a cups at O if the circle is tangent to line OT .

Straight line $t_0 \perp OT$ is always tangent or is one of the tangents to the cubic at point O .

The image by L transformation of a circle through T consists of the straight line t_0 and a **cubic** with

- a) the isolated singular point T (see row 3 column 6 in the table) if t_0 has no common points with the circle,
- b) a cups at point T if t_0 is tangent to the circle,
- c) a node at point T if t_0 cuts the circle.

3.3. Quartic curves

Circles not containing fundamental points of L transformation are transformed on quartic with singular point T doubly tangent to t_0 at the double point O . A big variety of affine curve forms is systematized in the table. Six symmetrical forms are separated in column 6. The remaining 51 unsymmetrical forms are divided into vertical columns, where as criterions the existence of two isolated singular points coinciding at O on t_0 (columns 1–3), the existence of singular point at T (column 3–5) or the existence of cups or node with double tangent t_0 at O and cups or node in T (columns 7–10).

The partition into rows in the table depends generally on the amount of points at infinity belonging to the curve, on the situation with respect to the circle S^2 and on the intuitive resemblance of the symmetrical forms.

4. Conclusion

The use of computer programs considerably enlarges cognitive possibilities of synthetical geometry and enables an easy and prompt checking of the way of thinking by means of computer drawing. Simple computer programs (written in Pascal) which accomplish the L transformation gave an easy computer visualization of curves: conic, cubics and quartic with three real singular point from which two are coinciding at one point on a line tangent to the quartics.

References

- [1] R. J. Walker, *Algebraic curves*, Springer, New York, Haidelberg, Berlin 1978.
- [2] L. Czech, *Involucyjne przekształcenia kwadratowe w ujęciu geometrii konstrukcyjnej* (*Involutory quadratic transformation formulated in constructional geometry*), Wyd. Politech. Krak., Monografia **133**, Kraków 1992.
- [3] K. Doeblemann, *Geometrische Transformationen*, Berlin, Leipzig 1930.
- [4] H. Hudson, *Cremona transformations*, London 1927.
- [5] L. Czech, *Rzut stożkowy (Conical projection). Wybrane zagadnienia geometrii wykreślnej (Chosen problems of descriptive geometry)*. Wyd. Politech. Krak., Monografia **64**, Kraków 1988, 15-28.
- [6] L. Czech, *Example of a closed projection base system*. Proceedings of the Third International Conference on Engineering Graphics and Descriptive Geometry, Vienna 1988, vol. I, 85-91.

Remarks

L transformation was presented in author's lecture: „Über eine ebene Transformation und ihre Anwendungen” on Österreichisch-jugoslawische Geometrietagung, Saggauberg, 20-25 Mai 1990.

Systematization of quartic with three real singular points from which two are coinciding in the paper: L. Czech, *Use of involutory L transformation for plotting of computer generated curves*, Proceedings of the Fifth International Conference on Engineering Computer Graphics and Descriptive Geometry, August 17-21 1992. Royal Melbourne Institute of Technology, vol. I. p. 260-263.

Added in proof

Already after submitting the paper, the autor has learned from the book K. Fladt *Analytische Geometric spezieller ebenen Kurven*, Akademische Verlagsgesellschaft, Frankfurt 1962 (p. 214-217) that two of the quartic given in the table (in column 6 rows 4 and 8) were described by means of analytical equations by Jerabek in 1885 and these two curves are called now Jerabek's curves.

Recenzent: Dr hab. Eugeniusz Korczak

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Streszczenie

Podano wykreślną definicję i własności przekształcenia kwadratowego nazwanego przekształceniem L [2]. Przekształcenie L jest inwolucyjnym przekształceniem kwadratowym [3, 4] o trzech rzeczywistych punktach fundamentalnych, z których dwa jednoczą się (na prostej prostopadłej do prostej przechodzącej przez punkty fundamentalne).

Program komputerowy, napisany w języku Pascal, oparty na prostym algorytmie przekształcenia L , kreśli stożkowe jako obrazy prostych, krzywe rzędu trzeciego jako obrazy okręgów przechodzących przez jeden punkt fundamentalny, które są rozpadającymi się krzywymi rzędu czwartego na tą krzywą i prostą, wreszcie krzywe rzędu czwartego jako obrazy okręgów nie przechodzących przez punkty fundamentalne przekształcenia.

Wykorzystując ten krótki program jako narzędzie umożliwiające obok szybkiego sprawdzania rysunkiem syntetycznego sposobu myślenia, wizualizację krzywych, przebadano wszystkie (jak się wydaje) możliwe afinczne postacie krzywych rzędu czwartego z punktem podwójnym T : izolowanym, ostrzem, węzłem, podwójnie styczne do prostej t_0 w punkcie O , który jest albo izolowanym punktem podwójnej styczności z prostą t_0 , albo węzłem podwójnie stycznym do prostej t_0 , albo ostrzem jednostronnie stycznym² do prostej t_0 . Z siedmiu form symetrycznych, z których jedna jest rozpadającą się, i z 51 form nie posiadających osi symetrii ułożono tablice, przyjmując jako kryterium podziału na kolumny rodzaj punktów osobliwych krzywej, a ilość punktów niewłaściwych i intuicyjne podobieństwo do odpowiedniej formy symetrycznej jako kryterium podziału na rzędy tej tablicy.

²W polskiej nomenklaturze nie znaleziono odpowiednika „rampoid” cups [1].