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HOMOGENEOUS FUNCTIONS AND DIFFERENTIAL EQUATIONS

Summary. In this paper a partial differential equation for so-called μ-homogeneous functions is considered. Also the result about the general form of solutions of this equation is presented. Moreover, the problem of local and global μ-homogenity of solutions of this equation is investigated.

FUNKCJE JEDNORODNE I RÓWNANIA RÓŻNICZKOWE

Streszczenie. Niech E, E_1 będą przestrzeniami liniowymi nad $\mathbb{R}$ i niech $f : E \to E_1$ i $\mu : \mathbb{R} \to \mathbb{R}$ będą danymi funkcjami. Jeżeli

$$F(tx) = \mu(t)\ f(x) \qquad \text{dla } x \in E,\ t \in \mathbb{R},$$

to f nazywa się μ-jednorodną funkcją.
W pracy zajmiemy się pewnymi własnościami jednorodnych funkcji. W szczególności bada się równanie różniczkowe cząstkowe związane z tymi funkcjami i przedstawia postać jego rozwiązania. Ponadto, powstaje problem tak zwanej lokalnej i globalnej jednorodności rozwiązania tego równania, którym zajmujemy się w tej pracy.

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ОДНОРОДНЫЕ ФУНКЦИИ И ДИФФЕРЕНЦИАЛЬНЫЕ УРАВНЕНИЯ

Резюме. В настоящей работе рассматриваются μ-однородные функции и изучаются их некоторые свойства. Исследуется дифференциальное уравнение связанное с этими функциями и найдено представление для его решения. С этой проблемой связанна проблема локальной и глобальной однородности решений этого уравнения. В работе также представлены результаты связанные с этой проблемой.

1. We shall introduce the following definition.

Definition 1. *Let E, E_1 be linear spaces over $\mathbb{R}$ (the set of real numbers) and $f : E \to E_1$ and $\mu : \mathbb{R} \to \mathbb{R}$ two given functions. If*

$$F(tx) = \mu(t)\ f(x) \qquad \textit{for } x \in E,\ t \in \mathbb{R}, \tag{1}$$

then f is said to be an μ-homogeneous function.

Now we state three simple lemmas.

Lemma 1. *If f is a nontrivial μ-homogeneous function, then μ is a multiplicative function, i. e.*

$$\mu(tv) = \mu(t)\mu(v) \qquad \textit{for } t, v \in \mathbb{R} \tag{2}$$

Proof. We have for $t, v \in \mathbb{R}$ and $x \in E$,

$$F(tvx) = \mu(tv)f(x) = \mu(t)\mu(v)f(x)$$

and hence we get (2).∎

Denote by $\mathbb{R}_+ := (0, \infty)$, $\mathbb{R}_0 := \mathbb{R} \setminus \{0\}$.

Lemma 2. ([1]) *If $\mu : \mathbb{R}_+ \to \mathbb{R}$ is measurable, then has one of following forms:*

$$\mu = 0; \quad \mu = 1; \quad \mu(x) = x^c, \qquad x \in \mathbb{R}_+,$$

where $c \in \mathbb{R}$.

Lemma 3. *If $f : \mathbb{R}_0 \times \mathbb{R}^n \to \mathbb{R}$ is an μ-homogeneous function, then there exists a function $\omega : \mathbb{R}^n \to \mathbb{R}$ such that*

$$f(x, y) = \mu(x)\omega(\frac{y}{x}), \qquad \textit{for } x, y \in \mathbb{R}_0 \times \mathbb{R}^n. \tag{3}$$

Proof. Let $x, y \in \mathbb{R}_0 \times \mathbb{R}^n$, then we get

$$f(x,y) = f(x, \frac{y}{x}x) = \mu(x) f(1, \frac{y}{x}) = \mu(x)\omega(\frac{y}{x}),$$

where

$$\omega(y) := f(1,y) \qquad \text{for } y \in \mathbb{R}^n. \blacksquare$$

Theorem 1. *Let $\mu(t) = t^s$, $t \in \mathbb{R}_+$, where $s \in \mathbb{R}$ is arbiturarily fixed and let $z : \mathbb{R}_+ \times \mathbb{R}^m \to \mathbb{R}$ be an s-homogeneous function with all first partial derivatives in $\mathbb{R}_+ \times \mathbb{R}^m$. Then z satisfies the following partial differential equations*

$$\frac{\partial z}{\partial x} + \frac{y_1}{x}\frac{\partial z}{\partial y_1} + \cdots + \frac{y_m}{x}\frac{\partial z}{\partial y_m} = s\frac{z}{x} \qquad \textit{for } (x,y) \in \mathbb{R}_+ \times \mathbb{R}^m. \tag{4}$$

Proof. We have

$$z(t\overline{x}, t\overline{y}_1, \ldots, t\overline{y}_m) = t^s z(x, \overline{y}_1, \ldots, \overline{y}_m) \qquad \text{for } t > 0,\ (x,y) \in \mathbb{R}_+ \times \mathbb{R}^m.$$

Differentiating both sides with respect to t, we get

$$\overline{x}\frac{\partial z}{\partial x} + \overline{y}_1\frac{\partial z}{\partial y_1} + \ldots + \overline{y}_m\frac{\partial z}{\partial y_m} = st^{s-1}z,$$

and, multiplying by t, we obtain

$$t\overline{x}\frac{\partial z}{\partial x} + t\overline{y}_1\frac{\partial z}{\partial y_1} + \ldots + t\overline{y}_m\frac{\partial z}{\partial y_m} = sz(t\overline{x}, t\overline{y}_1, \ldots, t\overline{y}_m).$$

Now the substitution

$$t\overline{x} = x, \quad t\overline{y}_1 = y_1, \quad \ldots, t\overline{y}_m = y_m,$$

leads to the following equation

$$x\frac{\partial z}{\partial x} + y_1\frac{\partial z}{\partial y_1} + \ldots + y_m\frac{\partial z}{\partial y_m} = sz,$$

and consequently to the equation (4). $\blacksquare$

2. Denote $D := \{(x,y) : x \in \mathbb{R}_+,\ y \in \mathbb{R}^m\}$. We can prove the following result.

Theorem 2. *Let $z : D \to \mathbb{R}$ be a solution of class C^2 of the equation*

$$\frac{\partial z}{\partial x} = -\frac{y_1}{x}\frac{\partial z}{\partial y_1} - \ldots - \frac{y_m}{x}\frac{\partial z}{\partial y_m} + s\frac{z}{x}, \qquad (x,y) \in D, \tag{5}$$

where $s \in \mathbb{R}$ is arbitrarily fixed. Then for every point $(x_0, y^0) \in D$ there exist a neighbourhood V of (x_0, y^0) and a function $\omega = \omega(y)$, $y \in \mathbb{R}^m$ of class C^2 such that

$$z(x,y) = \left(\frac{x}{x_0}\right)^s \omega\left[\frac{x_0 y_1}{x}, \ldots, \frac{x_0 y_m}{x}\right] \qquad \textit{for } (x,y) \in V. \tag{6}$$

Proof. Consider the system of characteristic equations for the equation (5) (see [2], p. 263)

$$\begin{aligned}\frac{dy^r}{dx} &= \frac{y^r}{x},\\ \frac{dz}{dx} &= s\frac{z}{x},\\ \frac{dq^r}{dx} &= (s-1)\frac{q^r}{x}, \qquad r=1,\dots,m.\end{aligned}$$

The only solution of that system going through the point

$$(x_0,\xi,\eta,\lambda) \qquad (\xi=(\xi_1,\dots,\xi_m), \quad \lambda=(\lambda_1,\dots,\lambda_m))$$

i. e. satisfying the conditions

$$\begin{aligned}\overline{y}^r(x_0;x_0,\xi,\eta,\lambda) &= \xi_r,\\ \overline{z}(x_0;x_0,\xi,\eta,\lambda) &= \eta,\\ \overline{q}^r(x_0;x_0,\xi,\eta,\lambda) &= \lambda_r, \qquad r=1,\dots,m,\end{aligned}$$

has the form

$$\begin{aligned}\overline{y}^r &= \frac{\xi_r}{x_0}x,\\ \overline{z} &= \eta x_0^{-s}x^s,\\ \overline{q}^r &= \lambda_r x_0^{1-s}x^{s-1}, \qquad r=1,\dots,m.\end{aligned}$$

Put

$$\omega(y) := z(x_0,y) \qquad \text{for } y\in\mathbb{R}^m.$$

Define for $h=(h_1,\dots,h_m)$

$$\begin{aligned}\overline{\overline{y}}^r(x,h) &:= \overline{y}^r\left(x;x_0,h,\omega(h),\frac{\partial\omega}{\partial y_1}(h),\dots,\frac{\partial\omega}{\partial y_m}(h)\right)=\frac{h_r}{x_0}x,\\ \overline{\overline{z}}(x,h) &:= \overline{z}\left(x;x_0,h,\omega(h),\frac{\partial\omega}{\partial y_1}(h),\dots,\frac{\partial\omega}{\partial y_m}(h)\right)=(x_0)^{-s}\omega(h)x^s.\end{aligned}$$

Now we consider the system of equations

$$\overline{\overline{y}}(x,\gamma)=y,$$

where

$$\gamma=(\gamma_1,\dots,\gamma_m), \qquad y=(y_1,\dots,y_m), \qquad \overline{\overline{y}}=\left(\overline{\overline{y}}^1,\dots,\overline{\overline{y}}^m\right).$$

Then we have the equality

$$\left(\frac{\gamma_1}{x_0}x,\dots,\frac{\gamma_m}{x_0}x\right)=(y_1,\dots,y_m)$$

and hence

$$\gamma_r = \frac{x_0 y_r}{x}, \qquad r = 1, \dots, m.$$

Define

$$u(x,y) := \bar{\bar{z}}(x, \gamma(x,y)) = (x_0)^{-s}\omega\left[\frac{x_0 y_1}{x}, \dots \frac{x_0 y_m}{x}\right] x^s \tag{7}$$

for $(x,y) \in D$.

It is well known that u is a solution of the equation (5) in D and u is of class C^2 in D. We also have

$$z(x_0, y) = \omega(y) = u(x_0, y). \tag{8}$$

In view of the fact that the right side of the equation (5) satisfies locally the Lipschitz condition, then for every $(x_0, y^0) \in D$ there exists a neighbourhood V of (x_0, y^0) such that the equation (5) has exactly one solution of class C^2 satisfying the initial condition (8). Therefore

$$z(x,y) = u(x,y) \qquad \text{for } (x,y) \in V$$

and we have got the formula (6). This completes the proof.■

Corollary 1. *Solutions z of class C^2 of the equation (5) are locally s-homogeneous, i. e. for every point $(x_0, y^0) \in D$ there exists a neighbourhood V of this point such that*

$$z(tx, ty) = t^s z(x,y)$$

for all $(x,y) \in V$ and all $t > 0$ satisfying the condition $(tx, ty) \in V$.

During the Polish-Austrian Seminar on Functional Equations and Iteration Theory held 26-31 October 93 in Cieszyn, I have stated the problem if the solution of the equation (5), which is locally s-homogeneous, is also a globally s-homogeneous function.

This problem has been solved by Professor Karol Baron from Silesian University in Katowice. He has proved the following

Lemma 4. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a continuous function satisfying the condition: for every $x \in \mathbb{R}^n$ there exists an $\delta > 0$ such that*

$$f(tx) = t^s f(x) \tag{9}$$

for $t \in (1-\delta, 1+\delta)$. Then for every $x \in \mathbb{R}^n$

$$A := \{t > 0 : \ f(tx) = t^s f(x)\} = \mathbb{R}_+. \tag{10}$$

Proof. Let $x \in \mathbb{R}^n$ and

$$A := \{t > 0 : \ f(tx) = t^s f(x)\}.$$

We shall prove that A is a closed and open set in $\mathbb{R}_+$. Take $t_0 \in A$. Then for $y = t_0 x$ there exists an $\delta > 0$ such that

$$f(ty) = t^s f(y) \qquad \text{for all} \ \ t \in (1-\delta, 1+\delta).$$

Consequently

$$f(ty) = t^s f(y) = t^s f(t_o x) = (tt_0)^s f(x)$$

for $t \in (1-\delta, 1+\delta)$, which means that A is an open set. Obviously, since f is continuous, A is also an closed set. This proves that $A = \mathbb{R}_+$.∎

Remark. During the mentioned Polish-Austrian Seminar Professor Maciej Sablik has proved the same result without the assumption of continuity of the function f.

Therefore we can establish the following

Corollary 2. *Solutions z of class C^2 of the equation (5) are globally s-homogeneous on D.*

I would like to express my thanks to Prof. K. Baron and Prof. M. Sablik for their fruitful contribution to the problem considered in this paper.

References

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Streszczenie

Niech E, E_1 będą przestrzeniami liniowymi nad ciałem liczb rzeczywistych $\mathbb{R}$ i niech $f : E \to E_1$ i $\mu : \mathbb{R} \to \mathbb{R}$ będą danymi funkcjami. Jeżeli spełniony jest warunek

$$f(tx) = \mu(t)\, f(x) \qquad \text{dla } x \in E,\ t \in \mathbb{R},$$

to f nazywa się μ-jednorodną funkcją.

W pracy zajmujemy się pewnymi własnościami jednorodnych funkcji. W szczególności takie funkcje spełniają równanie różniczkowe cząstkowe pierwszego rzędu (4). Korzystając z metody charakterystyk przedstawia się ogólną postać jego rozwiązania.

Jednocześnie pojawia się problem tak zwanej lokalnej (w pewnym otoczeniu) i globalnej jednorodności rozwiązania tego równania. W przedstawionej pracy pokazuje się równoważność (przy pewnych założeniach regularnościowych) tych dwóch pojęć.

Wydaje się interesujące badanie zagadnienia lokalnej i globalnej jednorodności przy innych, słabszych, założeniach.