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Seria: MATEMATYKA-FIZYKA z. 74 Nr kol. 1285

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ON A CERTAIN FUNCTIONAL EQUATION

Summary. The subject of the paper is the construction of the solution of the equation (1). In the fundamental theorem the general solution is given.

O PEWNYM RÓWNANIU FUNKCYJNYM

Streszczenie. W pracy podaje się wszystkie rozwiązania wektorowego równania funkcyjnego (1) o niewiadomej funkcji wektorowej $g(x)$ przy zadanej funkcji macierzowej $F(x)$ i ciągłości $g(x)$. W twierdzeniu podstawowym podane jest ogólne rozwiązanie.

ÜBER FUNKTIONALGLEICHUNGEN

Zusammenfassung. In die Arbait die functionalgleichung (1) is gelöst. In die Fundamentasatz die allgemeine form der Lösung is gegeben.

1. Introduction

In the present paper we shall give all solutions of the matrix — functional equation

$$g(xy) = F(x)g(x) + g(x) \tag{1}$$

F being given matrix. The matrices F and unknown g are $n \times n$ and $n \times 1$ natriees of real variables x, y respectively. We suppose that F, [2] is mesurable and g is continuous for $x \in \mathbb{R} \setminus \{0\}$.

The general solution of the equation

$$F(x\ y) = F(x)F(y) \tag{2}$$

assuming mesarability of F was given by M. Kuczma and A. Zajtz [2]. To solve the equation (2) we shall introduce the folowing matrices

$$M = \begin{bmatrix} |x|^a & \frac{1}{1!}|x|^a \ln|x| & \frac{1}{2!}|x|^a \ln^2|x| & \cdots & \frac{1}{(p-1)!}|x|^a(\ln|x|)^{p-1} \\ & |x|^a & |x|^a \ln|x| & \cdots & \frac{1}{(p-2)!}|x|^a(\ln|x|)^{p-2} \\ & & \cdots & \cdots & \cdots \\ & & & \cdots & \cdots \\ & & & & |x|^a \end{bmatrix} \tag{3}$$

$$N = \begin{bmatrix} A & A(bin|x|) & \frac{1}{2!}A(bin|x|)^2 & \cdots & \frac{1}{(s-1)!}A(bin|x|)^{s-1} \\ & A & A(bin|x|) & \cdots & \frac{1}{(s-2)!}A(bin|x|)^{s-2} \\ & & \cdots & \cdots & \cdots \\ & & & \cdots & \cdots \\ & & & & A \end{bmatrix} \tag{4}$$

A being the matrix of the form

$$A = \begin{bmatrix} |x|^a \cos(b\ln|x|), & |x|^a \sin(b\ln|x|) \\ |x|^a \sin(b\ln|x|), & |x|^a \cos(b\ln|x|) \end{bmatrix},$$

p, s are the dimension of the matrices M, N respectively, a, b are real parameters. The function F being the solutionn of the equation (2) is of form

$$\widetilde{F}(x) = C^{-1} \begin{bmatrix} B_1(x) & & & \\ & B_2(x) & & \\ & & B_3(x) & \\ & & & \ddots \end{bmatrix} C = C^{-1}F(x)C, \tag{5}$$

where C is non-singular constant matrix and F is generalised diagonal matrix. Every $B_i(x)$ is of form M, N or $(\mathrm{sign}x)M$, or $(\mathrm{sign}x)N$, [2].

2. Some lemmas concerning the equation (1)

By [1] we recall the following.

Lemma 1. *If the $\widetilde{F} = CFC^{-1}$ is the solution of the equation (2), then the solution g of the equation (1) is of form $\widetilde{g} = Cg$. Consequently if C is unite matrix E then $B_j = M$ or $B_i = N$, where $i = 1, 2, \ldots, p$ or $i = 1, 2, \ldots, s$.*

Definition 1. *By (F) we denote such matrices $F(x)$ which can by transformed to the diagonal form with at last two matrices B_i.*

Lemma 2. *If $F \in (F)$, then the function g being the solution of (1), is form*

$$g(x) = \begin{bmatrix} g^{(p_1)}(x) \\ \vdots \\ g^{(p_n)}(x) \end{bmatrix}$$

The dimension of the matrix $g^{(p_j)}(x)$ is equal to dimension of $B_j(x)$. The functional $g^{(p_j)}(x)$ satisfy equation (1) for which $F(x)$ is repleced by $B_j(x)$.

Proof. It sufficies to proof that if of

$$F(x) = \begin{bmatrix} B_{p_1}(x) & 0 \\ 0 & B_{p_2}(x) \end{bmatrix}$$

and

$$g(x) = \begin{bmatrix} g^{(p_1)}(x) \\ g^{(p_2)}(x) \end{bmatrix}$$

then

$$\begin{bmatrix} g^{(p_1)}(x\,y) \\ g^{(p_2)}(x\,y) \end{bmatrix} = \begin{bmatrix} B_{p_1}(x) & 0 \\ 0 & B_{p_2}(x) \end{bmatrix} \begin{bmatrix} g^{(p_1)}(y) \\ g^{(p_2)}(y) \end{bmatrix} + \begin{bmatrix} g^{(p_1)}(x) \\ g^{(p_2)}(x) \end{bmatrix} =$$

$$\begin{bmatrix} B_{p_1} \cdot g^{(p_1)}(y) \\ B_{p_2} \cdot g^{(p_2)}(y) \end{bmatrix} + \begin{bmatrix} g^{(p_1)}(x) \\ g^{(p_2)}(x) \end{bmatrix}.$$

Consequently

$$g^{(p_1)}(xy) = B_{p_1}(x)g^{(p_1)}(y) + g^{(p_1)}(x) \quad \text{and}$$

$$g^{(p_2)}(xy) = B_{p_2}(x)g^{(p_2)}(y) + g^{(p_2)}(x)$$

By induction we can prove the assertation for arbitrary positive integer p.

Definition 2. *By (G) we denote the set of all solution g of the equation (1).*

Let $\mathbb{R}$denote the set of real numbers.

Lemma 3. *If $g_i(x)$, $i = 1, 2$, belong to (G) and $\lambda_i \in \mathbb{R}$, $i = 1, 2$, then $(\lambda_1 g_1 + \lambda_2 g_2) \in (G)$.*

Proof. Let

$$h(x) \stackrel{def}{=} \lambda_1 g_1(x) + \lambda_2 g_2(x)$$

We fave

$$\begin{aligned} h(x\,y) &= \lambda_1 g_1(x\,y) + \lambda_2 g_2(x\,y) \\ &= \lambda_1[F(x)g_1(y) + g_1(x)] + \lambda_2[F(x)g_2(y) + g_2(x)] \\ &= F(x)[\lambda_1 g_1(y) + \lambda_2 g_2(y)] + [\lambda_1 g_1(x) + \lambda_2 g_2(x)] \\ &= F(x)h(y) + h(x). \end{aligned}$$

In the sequel we shall apply Lemma 3, to the construction of the solution of the equation (1).

3. By we shall consider two cases I and II

I. The matrix b_i is of type (3), and $a \neq 0$ or B_i is of type 4 and $b = 0$. If $b = 0$, then b_i is of type (3). For I we obtain the general solution of (10) applying Lemma 4 paper [1].

Lemma 4. *If there exists $x_0 \in \mathbb{R}$, such that the matrix $F(x_0) - E$ is non-singular, then the general solution of the equation (1) is of the form*

$$g(x) = [F(x) - E]V,$$

V being an arbitrary constant vector.

Proof. By assumption $F(x_0) - E$ is non-singular matrix. By the equation (1) we obtain

$$g(x_0x) = F(x_0)g(x) + g(x_0)$$

and

$$g(xx_0) = F(x)g(x_0) + g(x)$$

and consequently

$$F(x_0)g(x) + g(x_0) = F(x)g(x_0) + g(x)$$
$$[F(x_0) - E]g(x) = [F(x) - E]g(x_0)$$

and

$$g(x) = (f(x_0) - E)^{-1}(F(x) - E)g(x_0).$$

We shall verify that matrices $F(x_0) - E$ and $F(x) - E$, where $F(xy) = F(x)F(y)$, are commutative indeed. We have

$$\begin{aligned}[F(x_0) - E][F(x) - E] &= F(x_0)F(x) - E\,F(x) - F(x_0)E + E^2 = \\ &= F(x_0x) - E\,F(x) - F(x_0)E + E^2\end{aligned}$$

and

$$\begin{aligned}[F(x) - E][F(x_0) - E] &= F(x)F(x_0) - E\,F(x_0) - F(x)E + E^2 = \\ &= F(xx_0) - E\,F(x_0) - F(x)E + E^2\end{aligned}$$

Consequently it follows that the matrices $[F(x_0) - E]^{-1}$ and $[F(x_0) - E]$ are commutative and that

$$\begin{aligned}g(x) &= [F(x) - E][F(x_0) - E]^{-1}g(x_0), \\ g(x) &= [F(x) - E]V,\end{aligned}$$

where $V = [F(x_0) - E]^{-1}g(x_0)$. The method of the proof is similar to the method in the paper (1).

Let $x \neq 1$, x_0 be arbitrary real number.

This matrix $F(x_0) - E$ for F of the form (3) by $a \neq 0$ or (4) by $b \neq 0$ the case (1) is non-singular matrix.

Lemma 5. *By assumptions of Lemma 4 the general solution the equation (1) is the form*

$$g(x) = [F(x) - E]V, \quad x \neq 1,$$

where V is an arbitrary constant vector.

II. If B_i is of form (4) and $a = 0$, then

$$M = \begin{bmatrix} 1 & \frac{1}{1!}\ln|x| & \frac{1}{2!}\ln^2|x| & \cdots & \frac{1}{(p-1)!}(\ln|x|)^{p-1} \\ & 1 & \ln|x| & \cdots & \frac{1}{(p-2)!}(\ln|x|)^{p-2} \\ & & & \cdots & \cdots \\ & & & & 1 \end{bmatrix} \tag{6}$$

In this case for the construction of the solution of (1), we shall apply the evident.

Lemma 6. *For every arbitrary constant vector V, the vector function, of the form*

$$[F(X) - E]\,V$$

is the solution of the equation (1).

Putting $v = e_2$, $v = e_3$, ..., $v = e_n$ we obtain the following $(n-1)$ solution

$$[F(X) - E]\,e_2, \ldots [F(X) - E]\,e_n \tag{7}$$

The vector functions (7) are columns of the matrix $F(X) - E$ and evidently are linear independent for $x = 1$.
Let

$$g_k \stackrel{def}{=} [F(X) - E]\,e_{k+1}, k = 1, \ldots, n-1,$$

By (6), (7) ew have

$$g_1 = \begin{bmatrix} \ln|x| \\ 0 \\ \vdots \\ 0 \end{bmatrix}, g_2 = \begin{bmatrix} \frac{1}{2!}\ln^2|x| \\ \ln|x| \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \ldots, g_{n-1} = \begin{bmatrix} \frac{1}{(n-1)!}(\ln|x|)^{n-1} \\ \vdots \\ \ln|x| \\ 0 \end{bmatrix} \tag{8}$$

let

$$g_n(x) \stackrel{def}{=} \begin{bmatrix} \frac{1}{n!}(\ln|x|)^n \\ \vdots \\ \ln|x| \end{bmatrix}$$

By direct calculus we can verify that $g_n(x)$ for $x = 1$ is the solution of the equation (1) and the system

$$g_1(x), ..., g_{n-1}(x), g_n(x) \tag{9}$$

is lineary independent for $x = 1$.

Remark 1. *Since* $g(1) = g(1.1) = F(1)g(1) + g(1) + g(1)$ *thus for arbitrary solution* $g(1) = 0$.

Remark 2. *In sequel we shall suppose that* $p = n$.

Consequently arbitrary solution of the equation (1) is linear combination of the functions (9) i.e.

$$g(x) = \sum_{i=0}^{n} \lambda_i(x) g_i(x), x \neq 1 \tag{10}$$

and a priori the coefficients λ_i can be the functions of x. Substituting (6) and (10) to (1) and putting $\alpha(x) = \ln|x|$, we obtain

$$\begin{bmatrix} \lambda_1(xy)\alpha(xy) + \lambda_2(xy)\frac{1}{2!}\alpha^2(xy) + \ldots + \lambda_n(xy)\frac{1}{n!}\alpha^n(xy) \\ \lambda_2(xy)\alpha(xy) + \lambda_3(xy)\frac{1}{3!}\alpha^3(xy) + \ldots + \lambda_n(xy)\frac{1}{(n-1)!}\alpha^{n-1}(xy) \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots \\ (xy)\alpha(xy) \end{bmatrix} =$$

$$= \begin{bmatrix} 1 & \alpha(y) & \ldots & \frac{1}{(n-1)!}\alpha^{n-1}(y) \\ & 1 & \ldots & \frac{1}{(n-1)!}\alpha^{n-2}(y) \\ & & & \ldots\ldots \\ & & & 1 \end{bmatrix} \begin{bmatrix} \lambda_1(x)\alpha(x) + \ldots + \lambda_n(x)\frac{1}{n!}\alpha^n(x) \\ \lambda_2(x)\alpha(x) + \ldots + \lambda_n(x)\frac{1}{(n-1)!}\alpha^{n-1}(x) \\ \ldots\ldots\ldots\ldots\ldots \\ \lambda_n(x)\alpha(x) \end{bmatrix} +$$

$$+ \begin{bmatrix} \lambda_1(y)\alpha(y) + \ldots + \lambda_n(y)\frac{1}{n!}\alpha^n(y) \\ \lambda_2(y)\alpha(y) + \ldots + \lambda_n(y)\frac{1}{(n-1)!}\alpha^{n-1}(y) \\ \ldots\ldots\ldots\ldots\ldots \\ \lambda_n(y)\alpha(y) \end{bmatrix}$$

After the convenient transformations, by the formula $\alpha(xy) = \alpha(x) + \alpha(y)$ we obtain the following system of scalar for $\lambda_i(x)$, $i = 1, 2, \ldots, n$.

$$\begin{gathered} [\lambda_1(xy) - \lambda_1(x)]\,\alpha(x) + [\lambda_1(xy) - \lambda_1(y)]\,\alpha(y) + \\ + \sum_{k=2}^{n} \frac{1}{k!}\,[\lambda_k(xy) - \lambda_k(x)]\,\alpha^k(xy) + \sum_{k=2}^{n} [\lambda_k(x) - \lambda_k(y)]\,\alpha^k(xy) = 0 \\ [\lambda_2(xy) - \lambda_2(x)]\,\alpha(x) + [\lambda_2(xy) - \lambda_2(y)]\,\alpha(y) + \\ + \sum_{k=2}^{n-1} \frac{1}{k!}\,[\lambda_{k+1}(xy) - \lambda_{k+1}(x)]\,\alpha^k(xy) + \sum_{k=2}^{n-1} \frac{1}{k!}\,[\lambda_{k+1}(x) - \lambda_{k+1}(y)]\,\alpha^k(y) = 0 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{gathered} \tag{11}$$

$$[\lambda_{n-1}(xy)-\lambda_{n-1}(x)]\,\alpha(x)+[\lambda_{n-1}(xy)-\lambda_{n-1}(y)]\,\alpha(y)+$$
$$+\frac{1}{(n-1)!}\,[\lambda_n(xy)-\lambda_n(x)]\,\alpha^{n-1}(xy)+\frac{1}{(n-1)!}\,[\lambda_n(x)-\lambda_n(y)]\,\alpha(y)=0$$
$$[\lambda_n(xy)-\lambda_n(x)]\,\alpha(x)+[\lambda_n(xy)-\lambda_n(y)]\,\alpha(y)=0.$$

Putting in the equation (11) $\lambda_n(xy)=\lambda(xy)$, we obtain

$$[\lambda(xy)-\lambda(x)]\,\alpha(x)+[\lambda(xy)-\lambda(y)]\,\alpha(y)=0 \tag{12}$$

or equivalent equation

$$\lambda(xy)\,[\lambda(x)+\alpha(y)]=\lambda(x)\alpha(x)+\lambda(y)\alpha(y). \tag{13}$$

Lemma 7. *If $\lambda(x)$ satisfies (12) or (13) and there exists $\lim\lambda(x)$ if $x\to 0$ or $x\to\infty$, then $\lambda(x)$ is constant for $x=0,1$.*

Proof. Let x, y be arbitrary number different from 0 and 1. Let us suppose that $\lim\limits_{\zeta\to 0}\lambda(\zeta)=\sigma$.
At first we shall verity that

$$\lambda(x^p)=\lambda(x)=\lambda\left(\frac{1}{x}\right), \qquad p \text{ is positive integer.}$$

Let $xy=1$; $y=\frac{1}{x}$, $x\neq 0$; then

$$\alpha(y)=\alpha\left(\frac{1}{x}\right)=-\alpha(x).$$

By (13), we obtain

$$\lambda(x)\alpha(x)-\lambda\left(\frac{1}{x}\right)\alpha(x)=0.$$

Since $\alpha(x)=0$ for $x\neq 1$ thus

$$\lambda(x)=\lambda\left(\frac{1}{x}\right). \tag{14}$$

By induction with respect to p we shall verify that

$$\lambda(x^p)=\lambda(x).$$

Putting $y=x$ in (13) we obtain

$$\lambda(x^2)\,[\alpha(x)+\alpha(x)]=\lambda(x)\alpha(x)+\lambda(x)\alpha(x),$$

$$\lambda(x^2)2\alpha(x)=2\lambda(x)\alpha(x), \qquad \lambda(x^2)=\lambda(x).$$

If $\lambda(x^{p-1})=\lambda(x)$, then $\lambda(x^p)=\lambda(x)$.

If $y = x^{p-1}$, in (13) then we get

$$\lambda(x^p)\left[\alpha(x)+\alpha(x^{p-1})\right] = \lambda(x)\alpha(x)+\lambda(x^{p-1})\alpha(x^{p-1}), \tag{15}$$

Consequently

$$\lambda(x^p)\left[\alpha(x)+\alpha(x^{p-1})\right] = \lambda(x)\left[\alpha(x)+\alpha(x^{p-1})\right]$$

and

$$\alpha(x)+\alpha(x^{p-1}) = p \qquad \alpha(x) \neq 0.$$

Thus

$$\lambda(x^p) = \lambda(x). \tag{16}$$

Finally, by (14), (16), we obtain

$$\lambda(x^p) = \lambda(x) = \lambda\left(\frac{1}{x^p}\right). \tag{17}$$

Let $|x| < 1$, $|y| < 1$. By (17) we get

$$\lambda(x) = \lambda(x^n), \qquad \lambda(y) = \lambda(y^n) \tag{18}$$

for ever positive integer n.
If $n \to \infty$, then $x^n \to 0$, $y^n \to 0$ for ever $x, y \in (-1,1)$. Since $\lim_{\zeta\to 0}\lambda(\zeta) = \sigma$, σ is finite number and by (13) we obtain

$$\begin{aligned}
\lambda(x) &= \lim_{n\to\infty}\lambda(x^n) = \sigma \\
\lambda(y) &= \lim_{n\to\infty}\lambda(y^n) = \sigma \\
\lambda(x) &= \lambda(y).
\end{aligned}$$

Consequently $\lambda(x) =$ constant.
If $|x| > 1$ or $|y| > 1$ we replace (18) by

$$\lambda(x) = \lambda\left(\frac{1}{x^n}\right) \qquad \text{or} \qquad \lambda(y) = \lambda\left(\frac{1}{x^n}\right).$$

If $n \to \infty$, then $\lambda(x) = \lambda(y)$.
Similarly we obtain the same result if there exists $\lim \lambda(\zeta)$.

In the Lemma 7 the assumptio $\lim_{\zeta\to 0}\lambda(\zeta) = \sigma < \infty$ or $\lim_{\zeta\to\infty}\lambda(\zeta) = \sigma_1$ can be replaced by the condition $\lambda(x) \in C(\mathbb{R}\setminus\{0\})$. Indeed. The following lemma holds

Lemma 8. *If $g(x)$ defined by formula (10) is continuous vector function for $x \in \mathbb{R}\setminus\{0\}$, then in formula (10) sre the functions $\lambda_i(x) \in C(\mathbb{R}\setminus\{0\})$.*

Proof. Let

$$g(x) = \begin{bmatrix} g^1(x) \\ \vdots \\ g^n(x) \end{bmatrix}.$$

By (18), (12) the equality (10) take the form

$$\begin{bmatrix} g^1(x) \\ \vdots \\ g^n(x) \end{bmatrix} = \lambda_1(x) \begin{bmatrix} \ln|x| \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \lambda_2(x) \begin{bmatrix} \frac{1}{2!}\ln^2|x| \\ \ln|x| \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \ldots + \lambda_n(x) \begin{bmatrix} \frac{1}{n!}(\ln|x|)^n \\ \vdots \\ \ln|x| \end{bmatrix}.$$

Consequently we obtain

$$\begin{aligned} g^1(x) &= \lambda_1(x)\ln|x| + \frac{1}{2!}\lambda_2(x)\ln^2|x| + \ldots + \frac{1}{n!}\lambda_n(x)(\ln|x|)^n, \\ g^2(x) &= \lambda_2(x)\ln|x| + \ldots + \frac{1}{(n-1)!}\lambda_n(x)(\ln|x|)^{n-1}, \\ \ldots & \ldots \ldots \\ g^{n-1}(x) &= \lambda_{n-1}(x)\ln|x| + \frac{1}{2!}\lambda_n(x)\ln^2|x|, \qquad (19) \\ g^n(x) &= \lambda_n(x)\ln|x|. \qquad (20) \end{aligned}$$

By continuity of $g^1(x)$, $i = 1, \ldots, n$, and by (20) that $\lambda_n(x)$ is also continious for $x = 0$. By (20) and (19) follows that $\lambda_{n-1}(x)$ is continious for $x = 0$. Similarly we can verify that the functions $\lambda_{n-2}(x), \ldots, \lambda_1(x)$ are continious for $x = 0$.

Lemma 9. *If the function $\lambda(x)$ satisfying (12) is continious for $x = 0$ and $x = 1$, then $\lambda(x) =$ constant.*

Proof. Let us suppose that $x > 0$ and p is arbitrary positive integer for $y = x^{\frac{1}{p}}$. There exist $y > 0$ such that $x = y^p$ and by (17) for $y = x^{\frac{1}{p}}$ we have

$$\lambda(y^p) = \lambda(y) \qquad \text{or } \lambda(x) = \lambda\left(x^{\frac{1}{p}}\right).$$

Consequently for ever $x > 0$ and ever p being positive integer we obtain

$$\lambda(x) = \lambda\left(x^{\frac{1}{p}}\right). \qquad (21)$$

Let p, q denote arbitrary positive integers. By (17), (21) we get

$$\lambda(x) = \lambda\left(x^{\frac{1}{q}}\right) = \lambda\left(x^{\frac{p}{q}}\right). \qquad (22)$$

By yhe identity

$$\lambda\left(\frac{1}{x}\right) = \lambda\left(\frac{1}{x^p}\right) = \lambda\left(x^{-p}\right) = \lambda(x)$$

and by (22) for rational $\frac{p}{q} < 0$ holds. Consequantly if r ia an arbitrary rational number, and $x > 0$, then

$$\lambda(x) = \lambda(x^r) \tag{23}$$

For $x < 0$, we have

$$\lambda(x) = \lambda(x^2) = \lambda\left[(-x)^2\right] = \lambda(-x).$$

Consequantly the function λ is even function and (23) for $x < 0$ holds also if x^r is defined. Let x, y be arbitrary real numbers x, $y \neq 1$. Since λ is even function it sufficies consider the case if $x > 0$, $y > 0$.
Then, we have

$$y = x^{\log_x y} \tag{24}$$

Let

$$\mu \stackrel{\text{def}}{=} \log_x y$$

and let r_n denotes an arbitrary sequence of rational numbers for which

$$\lim_{n\to\infty} r_n = \mu. \tag{25}$$

By (23) we obtain

$$\lambda(y) = \lambda\left(x^{r_n}\right). \tag{26}$$

By (25) we have

$$\lim_{n\to\infty} x^{r_n} = x^{\mu} = y.$$

By continuity of the function $\lambda(x)$ and by (26), we obtain

$$\lambda(y) = \lim_{n\to\infty} \lambda\left(x^{r_n}\right) = \lambda\left(\lim_{n\to\infty} x^{r_n}\right) = \lambda(y).$$

Consequently we get the assertion of Lemma 8.

4. The properties of the system (11)

In the sequel we shall suppose that the functions λ_i, $i = 1, \dots, n$, are continuous for $x = 0$ or that there exist $\lim_{\zeta\to 0} \lambda_i(\zeta)$ or $\lim_{\zeta\to\infty} \lambda_i(\zeta)$, $i = 1, \dots, n$. The function $\lambda_n(x)$ satisfies the equation (12) being the last equation of the system (11). By Lemmas 7, 8 follows that $\lambda_n(x) =$ constant and consequently

$$\lambda_n(xy) - \lambda_n(x) = 0, \qquad \lambda_n(xy) = \lambda_n(y) = 0 \tag{27}$$

By (27) applied to last equation of the system (11), we obtain

$$\left[\lambda_{n-1}(xy) - \lambda_n(x)\right]\alpha(x) + \left[\lambda_{n-1}(xy) - \lambda_{n-1}(y)\right]\alpha(y) = 0.$$

Consequently $\lambda_{n-1}(x)$ satisfies the equation (12). Similarly as for λ_{n-1} we can conclute that λ_{n-1} = constant. Since λ_n and λ_{n-1} are constant, thus by the equation (12) we have λ_{n-2} = constant. Finally we obtain $\lambda_{n-1}, \ldots, \lambda_1$ are constants.

Remark 3. *By the Lemma 3 the general solution of (1) is a linear combination with constants coefficiens of the particular solutions (9).*

5. Fundamental theorem

By foregoing lemmas we obtain the following result.

Theorem. *If $F(x)$ is mesurable for $x \in \mathbb{R}$, and the unknown vector function $g(x)$ is continuous, then the general solution of (1) in wich F a quasi-diagonal form, is the vector function $g(x)$ of form*

$$g(x) = \begin{bmatrix} g^1(x) \\ \vdots \\ g^{(p_n)}(x) \end{bmatrix},$$

where $g^{(p_n)}(x)$ is vector function for which p_j being the number of the coordinates equal the dimensional of the block. Moreover the matrices $B_j(x)$ satisfy the equation (1) in which $F(x)$ is replaced by $B_j(x)$. The solution g is of form depending of $B_j(x)$ and

1^o $g^{(p_n)}(x) = [B_j(x) - E]\, V$, *if $a \neq 0$ or $b \neq 0$, V being arbitrary constant vector or*

2^o $g^{(p_n)}(x) = \sum_{i=1}^{p_j} \lambda_i g_i(x)$, *if $a = b = 0$ and λ_i are arbitrary real numbers, where*

$$g_1 = \begin{bmatrix} \ln|x| \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \ldots, g_{p_j} = \begin{bmatrix} \frac{1}{(p_j)!}(\ln|x|)^{p_j} \\ \frac{1}{(p_j-1)!}(\ln|x|)^{p_j-1} \\ \vdots \\ \ln|x| \end{bmatrix}.$$

Remark 4. *The application of the equation (1) to the theory of geometric objects will be given in the other paper.*

References

[1] M. Kucharzewski, M. Kuczma, *On a system of functional equations occurring the theory of geometric objects.*

[2] M. Kuczma, A. Zajtz, *On the form of real solutions of the matrix functional equation $\phi(x)\phi(y) = \phi(xy)$ for non-singular matrices,* Publ. Math. Debrecen **13** (1966), 257-262.

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Wpłynęło do redakcji 30.03.1989 r.

Streszczenie

Tematem pracy jest konstrukcja rozwiązania równania wektorowego $g(xy) = F(x)g(y) + g(x)$, gdzie g jest nieznaną funkcją wektorową, a F jest opisana w pracy [2]. Macierz F jest nieosobliwa. Zakłada się, że funkcja g jest ciągła. Metodę konstrukcji podano w pracy [1].