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GROUPS
and
GROUP RINGS

ABSTRACTS

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cn(PSL(2,K))=4 for K=Q, R

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Let G be a group. The smallest integer m satisfying equality $C^m = G$ for each nontrivial conjugacy class C of G is called the covering number and is denoted by $cn(G)$. The problem of investigation of $cn(G)$ is called the *covering problem.*

It is known that

1. If $K = C$, then $cn(PSL(2, K)) = 2$, (see [2]),

2. If K is finite field and $|K| \geq 4$, then $cn(PSL(2, K)) = 3$, (see [5]),

3. If $K = Q$, R, then $cn(PSL(2, K)) \geq 3$, (see [4]).

Using the following theorems

Theorem 1. *If $A \in GL(n, K)$, $A \notin Z$ and also $V = diag(v_1, \ldots, v_n)$, $W = diag(w_1, \ldots, w_n)$, $v_i \neq v_j$, $w_i \neq w_j$ for $i \neq j$, then there exist $X, Y \in GL(n, K)$ such that $A = (X^{-1}VX)(Y^{-1}WY)$ and $det(X)$, $det(Y)$ are arbitrary elements of K^*, (see [1]).*

Theorem 2. *If $A \in GL(n, K)$ ($n = 2, 3$; $K = R, C$), $A \notin Z$, then there exists $X \in SL(n, K)$ such that eigenvalues of $AX^{-1}AX$ are different, (see [2]).*

it has been proved that $cn(PSL(2, R)) \leq 4$, (see [2]).

Now we can prove that if $K = Q, R$, then $PSL(2, K) - Z = C^3$ ($C-$ any nontrivial conjugassy class) and $cn(PSL(2, K)) = 4$.

A similar problem to the covering problem of G by conjugacy classes is the covering problem of G by the sets $K_w = \{g \in G : o(g) = w\}$ of elements of order w.

In the paper [3] it has been proved the following theorem

Theorem 3. *If K is the real field R or $K = GF(p^s)$, $p > 2$, then $O_n(K, f) = K_2K_2$. If K is the complex field C or $K = GF(2^s)$, then $O_n(K, f) \neq K_2K_2$, where $O_n(K, f)$ denotes a group of automorphisms of the vector space $V_n(K)$ which leave invariant a quadratic form f of determinant different from zero.*

It also has been proved that $PGL(2, K) = K_2K_2$ while $PGL(n, K) \neq K_2K_2$ with $n \geq 3$, where $PGL(n, K) = GL(n, K)/Z$.

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IA-automorphisms of free groups

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This talk is about some results which are not new but their interest lies in the fact that the relative problems have drawn again the attention of the algebraists. Let $K = IAut(F)$ be the group of the $IA-$automorphism of the free group F of rank q. Thus K consists of all automorphisms of F which induce the identity on the commutator group.

Two main main questions are examined:First to identify the lower central series of K and second to find the ranks of the lower central factors which are free abelian group.

The last problem is connected to the problem of finding the tame automorphisms of free nilpotent groups,i.e. automorphisms which are induced by automorphisms of the free group.

Lattices of intermediate subgroups

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Let G be any group and G_0 a fixed subgroup of G. On the lattice of all intermediate subgroups $Lat(G_0, G) = \{H | G_0 \leq H \leq G\}$ we introduce the structure of a graph (of normality). Verteces of this graph are all H, and between H_1 and H_2 we have the edge iff H_1 is normal in H_2. The connected components of the graph of normality are called **garlands** of $Lat(G_0, G)$. If G_0 is normal in G then our graph of noramlity is connected and in this case we have only one garland. On the other hand if every H is selfnormalised in G then each garland contains only one subgroup.

In the general case $G_0 \leq G$, we consider two operations (lifting and descent): for any H, $G_0 \leq H \leq G$, we put $H' = N_G(H)$ and $'H = G_0^H$. Iterations of this operations permit to introduce two sequences:

$$H \leq H' \leq H'' \leq ... \tag{1}$$

and

$$H \geq {'H} \geq {''H} \geq ... \tag{2}$$

Garland Γ is called bounded from the top iff for any $H \in \Gamma$ the sequence (1) is finite. Similarly, by means of (2), we introduce the notion of garland which is bounded from the bottom. The garland is called of **a finite type** if it is bounded from the both sides. For any two subgroups H_1 and H_2 in the same garland if from $H_1 \leq H_2$ it follows $H_1' \leq H_2'$ (condition of monotony for lifting) then there exists $1-1$-correspondence between all garlands bounded from the top and all selfnormalized intermediate subgroups.

An intermediate subgroup F is called **full** iff $G_0^F =< G_0^x | x \in F >= F$. All garlands of $Lat(G_0, G)$ which are bounded from the bottom are in bijective correspondence with all full intermediate subgroups. A subgroup G_0 is called **polynormal** iff $G_0^{<x>}$ is full for every $x \in G$ (see [2]). For the polynormal subgroup G_0 in G all garlands of $Lat(G_0, G)$ are in bijective correspondence with all full intermediate subgroups F and each garland coincides with the interval $Lat(F, N_G(F))$.

For example (see [1]), let K be any field (or skew-field) for which $|K| \geq 7$, $G = GL(n, K)$, $G_0 = D(n, K)$ the subgroup of diagonal matrices. Then G_0 is polynormal in G (and even pronormal), the lattice $Lat(G_0, G)$ is finite and is independent from K, all garlands are in bijective correspondence with all topologies on n points, for each full subgroup F the factor-group $N_G(F)/F$ is isomorphic to a section in symmetric group S_n, the subgroup of all monomial matrices over K of degree n is maximal in G.

In the second example we take $G = GL(2, Q)$ (Q is the rational number field) and G_0 a non-split torus which is isomorphic to multiplicative group of a quadratic field $Q(\sqrt{d})$

(see [3]). In this example the subgroup G_0 is not polynormal for every d, the structure of $Lat(G_0, G)$ depends essentially on arithmetical properties of d, all garlands are bounded from the top and have property of monotony for lifting; therefore all garlands are in $1-1$-correspondence with all selfnormalized intermediate subgroups.

If in the quadratic field $Q(\sqrt{(d)})$ we take $1, \sqrt{(d)}$ as the bases then for the nonsplit torus $G_0 = T(d)$ we have the following representation in $G = GL(2, Q)$:

$$T(d) = \{\begin{pmatrix} x & yd \\ y & x \end{pmatrix} | x, y \in Q; (x, y) \neq (0, 0)\}.$$

All garlands of $Lat(T(d), G)$ are of the finite type if and only if $d \equiv 1 (mod 4)$.

For any intermediate subgroup H, $T(d) \leq H \leq G$, we introduce the module of transvections

$$A(H) = \{\alpha \in Q | \begin{pmatrix} 1 & 0 \\ \alpha & 1 \end{pmatrix} \in H\}$$

and the ring of coefficients

$$R(H) = \{r \in Q | rA(H) \subseteq A(H)\}.$$

A subring R in Q is called admissible if there exists a subgroup H for which $R(H) = R$. Also a submodule A in Q is called admissible if $A = A(H)$ for any H. For characterization of the d-admissible subrings and submodules we introduce the subring $R(d)$ in Q:

$$R(d) = ring < 1, \frac{2tr(z^2)}{N(z)} | z \in Q(\sqrt{(d)}), z \neq 0 >$$

(here tr and N are trace and norm for the extension $Q(\sqrt{(d)})/Q$). The subring $R(d)$ is generated by $\frac{1}{p}$, p's are odd prime with $(\frac{d}{p}) = 1$, and additionaly by $\frac{1}{2}$, if $d \equiv 1 (mod 8)$. The subring R in Q is d-admissible if and only if $R(d) \subseteq R$. For any garland Γ of $Lat(T(d), G)$ all subgroups H in Γ have the same ring of coefficients R. It follows that the lattice $Lat(T(d), G)$ contains continuum (that is $2^{\aleph_0}$) garlands. Let R be a d-admissible subring. If number 2 is invertible in R or d is even then each nonzero integer ideal in R is d-admissible. If number 2 is not invertible in R and d is odd, then nonzero integer ideal in R is d-admissible if and only if it is even. Now we can get the description of all garlands in $Lat(T(d), G)$ and also to find the structure of each garland [3].

The last results may be extended to nonsplit tori connected with any quadratic extensions of fields of characteristic $\neq 2$.

Another examples of garlands may be found in papers [4] – [8].

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Some problems of group rings of infinite groups

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1. Free subgroups of units in group algebras

Let K be a commutive ring and $t(G)$ the set of torsion elements of G.

The following problem is a very difficult and interesting one:

Problem 1. When does the group of units $U(KG)$ not contain a free group of rank 2?

The first result have been obtained by B. Hartley and P. Pickel:

Theorem *Let G be a solvable-by-finite group and suppose that $U(\mathbb{Z}G)$ does not contain a free group of rank two. Then the following conditions hold:*

1. *$t(G)$ is a subgroup and every subgroup of $t(G)$ is normal in G;*
2. *$t(G)$ is an abelian subgroup or a hamiltonian 2-group.*

We would like to deal with this problem for the group of units $U(KG)$ of a group algebra KG. J. Z. Goncalves (1984) gave necessary and sufficient conditions for this problem in case G is finite, and, moreover, some extensions of this result to infinite groups.

We now define for an arbitary group G the normal subgroup

$$\Lambda(G) = \{g \in G \mid [H : C_H(g)] < \infty\}$$

for every finitely generated subgroup H of G.

Of couse, the torsion part $\Lambda^+(G)$ of $\Lambda(G)$ is a normal subgroup and the factor group $\Lambda(G)/\Lambda^+(G)$ is torsion free and abelian.

Theorem *Let K be a field of characteristic 0 or p and suppose that $U(KG)$ does not contain a free group of rank two. Then one of following conditions hold:*

1. *G is abelian;*
2. *G is a torsion group and K is algebraic over its prime field $GF(p)$;*
3. *K is a field of characteristic 0 and*
   a) *$\Lambda^+(G)$ is an abelian subgroup and each of its subgroups is normal in G;*

b) *the centralizer $C_G(\Lambda^+(G)$ contains all elements of finite order of G;*

c) *for every $a \in \Lambda^+(G)$, which is not central in G, K contains no root of unity of order equal to the order of a;*

4. *K is a field of characteristic p and*

a) *the subgroup $\Lambda^+(G)$ is a semidirect product of the p-Sylow subgroup P of $\Lambda^+(G)$ and an abelian group A;*

b) *if K is not algebraic over its prime field $GF(p)$, then the centraliser $C_G(\Lambda^+(G/p)$ contains all elements of finite order of G/P;*

c) *if $\Lambda^+(G/P)$ is noncentral in G/P and G/P is non-torsion, then the algebraic closure L of $GF(p)$ in K is finite and for all $g \in G/P$ and $a \in AP/P$ there exists a natural number r such that $gag^{-1} = a^{p^r}$. Furthermore, each such r satisfies condition that $[L : GF(p)]$ divides r.*

Theorem *Let K be a field of characteristic 0 or p and G a locally nilpotent group. Then $U(KG)$ does not contain a free group of rank two if and only if one of following conditions hold:*

1. *G is abelian;*

2. *G is a torsion group and K is algebraic over its prime field $GF(p)$;*

3. *K is a field of characteristic 0 and*

a) *the maximal torsion subgroup $t(G)$ is abelian and each of its subgroups is normal in G;*

b) *for every $a \in t(G)$, which is not central in G, K contains no root of unity of order equal to the order of a;*

4. *K is a field of characteristic p and*

a) *the subgroup $t(G)$ is a direct product of the p-Sylow subgroup P of $t(G)$ and an abelian group A;*

b) *if A is noncentral in G then the algebraic closure L of $GF(p)$ in K is finite and for all $g \in G$ and $a \in A$ there exists a natural number r such that $gag^{-1} = a^{p^r}$. Furthermore, each such r satisfies that $[L : GF(p)]$ divides r.*

Problem 2. Suppose that $U(KG)$ does not contain a free group of rank two and $\Lambda^+(G)$ is not a central subgroup of G. If $U(K)$ has an element of infinite order then is it true that $t(G) = \Lambda^+(G)$?

The question has already been answered in the positive in case

1. (Hartley and Pickel) K is a field of characteristic 0 and G is a solvable-by-finite group;

2. (Goncalves) K is a field of characteristic p not algebraic over its prime field $GF(p)$, and if $p = 2$ then the degree of transcendence of K over $GF(2)$ is at least 2; G is a solvable-by-finite group without p-elements.

Problem 3. Let K be a field of characteristic p. When does $U(KG)$ contain a free product of two cyclic subgroups of order p?

For example, let H be a finite group and assume that its commutator subgroup is not a p-group. If G is a direct product of H and an infinite cyclic group then $U(GF(p)G)$ contain a free product of two cyclic subgroups of order p.

Theorem *Let G be a finite nonabelian group which is not Hamiltonian 2 - group. Then*

1. *(Goncalves-Ritter-Sehgal) Every subnormal subgroup of $U(\mathbf{Z}G)$ containing G have a free group of rank two;*

2. *(Hartley) Let N be a subnormal subgroup of $U(\mathbf{Z}G)$. There exists a normal subgroup H of finite index in $U(\mathbf{Z}G)$ such, that either N contains a free group of rank two or $N \cap H$ lies in the centre of $U(\mathbf{Z}G)$.*

Problem 4. Let G be a torsion group which has no hamiltonian subgroups, and N a (sub)normal subgroup of $U(\mathbf{Z}G)$. Then either N contains a free group of rank 2 or is a solvable-by-finite group.

2. The isomorphism problem for group rings of infinite groups

Concerning the isomorphism problem for finite groups deep and interesting results has been obtained recently (Roggenkamp, Scott, Weiss). As these are well-known, we shall examine the group rings of infinite groups.

Let $L(G)$ be the lattice of normal subgroups of G and $L_f(G)$ the lattice of finite normal subgroups of G. Let K be any integral domain of characteristic 0 in which no rational prime is invertible. Let G and H be finite groups and suppose that $KG \cong KH$. Then there exists a lattice isomorphism

$$\delta : L(G) \to L(H).$$

Several authors investigated the properties of δ, which gave lots of information on the isomorphism problem.

At the early 70's it was proved (Bovdi, Bass, Passman) that for an infinite group G, in $\mathbf{Z}G$ torsion units have trace 0. This allowed to transform the results on δ for infinite groups G, which can be found in the books *Group rings* by Bovdi (1974) and *Topics in Group Rings* by Sehgal (1978).

Theorem *Let K be any integral domain of characteristic 0 in which no rational prime is invertible. Let G and H be arbitary groups and suppose that $KG \cong KH$.*

Then there exists a lattice isomorphism

$$\delta : L_f(G) \to L_f(H)$$

with the following properties:

1. *The commutation of any two finite normal subgroup: if G' is commutator subgroup of G and P, L are finite normal subgroups of G, then*

$$\delta(P') = (\delta(P))' \text{ and } \delta((P,L)) = (\delta(P), \delta(L));$$

2. *The isomorphism of normal abelian sections: if P, L are finite normal subgroups of G, $P \subseteq L$ and L/P is abelian, then*

$$L/P \cong \delta(L)/\delta(P).$$

Research on $L(G)$ for infinite G was initiated by Ziegenbalg in 1975. He obtained some results for locally finite groups, which was later extended to arbitrary torsion groups:

Theorem *(Bovdi 1976) Let K be any integral domain of characteristic 0 in which no rational prime is invertible. Let G and H be torsion groups and suppose that $KG \cong KH$. Then there exists a lattice isomorphism*

$$\delta : L(G) \to L(H)$$

with following properties:

1. *The commutation of any two normal subgroups: if P, L are normal subgroups of G, then*

$$\delta(P') = (\delta(P))' \text{ and } \delta((P,L)) = (\delta(P), \delta(L));$$

2. *The isomorphism of normal abelian sections: if P, L are normal subgroups of G, $P \subseteq L$ and L/P is abelian, then*

$$L/P \cong \delta(L)/\delta(P).$$

We do not know any result about δ for non-torsion groups G.

It is very interesting that if $KG \cong KH$, then

1. if G is a locally finite group then H is also is a locally finite group;

2. if G is a p-group then H is also a p-group.

If G is torsion, then is it true that H is also torsion?

Theorem *Let* $\mathbf{Z}G$) *an integral group ring and* $\mathbf{Z}G \cong \mathbf{Z}H$. *Then*

1. *(Magnus and P. Cohn) If* G *is an abelian group then* G *and* H *are isomorphic;*

2. *(Whitcomb) If* G *and* H *are torsion groups, then*

$$G/G'' \cong H/H'';$$

3. *(Furukawa) If* G *and* H *are torsion groups, then*

$$G/D_4(G) \cong H/D_4(H);$$

4. *(Röhl) If* G *is the circle group of a nilpotent ring then* G *and* H *are isomorphic;*

5. *5. (Strojnowski) If* G *is a u.p. group then* G *and* H *are isomorphic.*

Hales and Passi proved that a nilpotent group G of class 2 is the circle group of a nilpotent ring of index 3 if

1. the centre $\zeta(G)$ is 2-divisible and extraction of square roots is unique in $\zeta(G)$;

2. $G/\zeta(G)$ is a direct product of cyclic groups;

3. G/G' is divisible or torsion;

4. G/G' is torsion free and completely decomposable.

They also showed that not every nilpotent group of class 2 is a circle group. Therefore there exists a nilpotent group of class 2 for which the isomorphism problem is open.

Let $V(\mathbf{Z}G)$ be the group of units with augmentation 1 of $\mathbf{Z}G$ and I an ideal of $\mathbf{Z}G$ contained in the augmentation ideal. Then the normal subgroup

$$I^+ = \{x \in V(\mathbf{Z}G) \mid x - 1 \in I\}$$

is called a *congruence subgroup* of $V(\mathbf{Z}\mathbf{G})$.

Theorem *(Sandling 1974 and later reproved by Bovdi 1977) Let K be a commutative ring and G the circle group of a radical algebra R over K. Then there exists a congruence subgroup I^+ such that*

$$V(KG) = G \ltimes I^+.$$

Problems

Let K be a commutative domain of characteristic 0. Let G be the circle group of a radical algebra R over K and suppose that the order of any torsion element of G has no inverse in K. The following problems are open:

1. **The isomorphism problem for circle groups.**

2. **Is the congruence subgroup I^+ torsion free?**

We remark that the congruence subgroup I^+ is torsion free provided

a. (Passman-P. Smith 1981) R is a finite ring;

b. (Bovdi 1984) R is a locally nilpotent ring.

c. (Furukawa 1987) R is residually nilpotent.

3. The Zassenhaus problem for group rings of infinite groups

If $u \in U(KG)$ and $u \notin G \times U(K)$ then u is called a nontrivial unit of KG.

Let K be a ring of characteristic 0. There is one obvious way to obtain a nontrivial torsion unit of $U(\mathbf{Z}G)$: We conjugate a torsion element $g \in G$ by the unit $a \in U(KG)$ of the overring KG such that $a^{-1}ga \in U(\mathbf{Z}\mathbf{G})$.

Zassenhaus conjected that this is the only way to obtain a torsion unit for a finite group G.

The Zassenhaus Conjecture has been proved for several classes of groups. The strongest result belongs to Weiss:

Theorem *If G is a finite nilpotent group and H is a finite subgroup in $V(\mathbf{Z}G)$ then there exists a unit $w \in U(\mathbf{Q}G)$ such that $w^{-1}Hw \subseteq G$.*

Roggenkamp and Scott constructed a metabelian group G for which the Zassenhaus Conjecture is not valid.

A necessary condition for the existence of conjugation $w^{-1}uw = \pm g$ can be expressed in terms of the Stallings-Bass functions $t_C(x)$. They are defined for each conjugacy classes C of G by the formula

$$t_C(\sum_{g\in G} c_g g) = \sum_{g\in C} c_g.$$

The Stallings-Bass function $t_C(x)$ is additive and has the following trace property: $t_C(xy) = t_C(yx)$ for all $x, y \in KG$.

Clearly, the number $t_C(u) = t_C(w^{-1}uw) = \pm t_C(g)$ is nonzero only if g belongs to C. For finite groups this necessary condition is also sufficient.

A group G is said to satisfy the *unique trace property* if for each torsion unit $u \in \mathbf{Z}G$ there exists a unique conjugacy class C such that $t_C(u) \neq 0$.

For an infinite group G little is known about the unique trace property.

Theorem *The Zassenhaus Conjecture is valid if*

1. *(Lichtman-Sehgal) G is a free product of finite cyclic groups;*
2. *(F. Levin-Sehgal) G is an infinite dihedral group;*
3. *(Bovdi-Marciniak-Sehgal) G is a direct product of a group with the unique trace property and an abelian torsion free group.*

Sehgal and Zalesskii constructed an infinite group for which the Zassenhaus Conjecture is not valid.

Bovdi-Marciniak-Sehgal described some classes of groups with the unique trace property (for example groups residually by locally nilpotent groups).

Defining relations for Hurwitz groups

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In this talk I will describe some recent joint work with E.F. Robertson (St Andrews) and M.D.E. Conder (Auckland). Much of the material is in a joint paper 'Defining relations for Hurwitz groups' which will appear in the Glasgow Mathematical Journal.

The $(2,3,7)$-triangle group is the group with presentation

$$\Delta(2,3,7) = \langle x, y \mid x^2 = y^3 = (xy)^7 = 1 \rangle$$

or, alternatively,

$$\Delta(2,3,7) = \langle A, B \mid (AB)^2 = (A^{-1}B)^3 = B^7 = 1 \rangle.$$

A *Hurwitz group* is any finite non-trivial quotient of $\Delta(2,3,7)$. John Leech, who sadly died on 28 September 1992, was interested in the situation when a fourth relation $w(x,y) = 1$ (or $w(A,B) = 1$) in the generators x and y is added. In this talk, as in the paper, I will describe some results on what is known for various fourth relations $w(A,B) = 1$. In particular answers will be given to some questions raised by John Leech over twenty years ago.

When $w = A^k$ (or $w = [x,y]^k$), a complete picture now exists, the most recent results showing that the group with presentation

$$\langle A, B \mid (AB)^2 = (A^{-1}B)^3 = B^7 = A^k = 1$$

is infinite when $k \geq 9$ [$k = 10$ and $k \geq 12$ being proved by Holt and Plesken (and also, independently, by Howie and Thomas) and the case $k = 11$ being proved by Edjvet].

We describe how we may then assume that the fourth relator w

is of the form $(A^rB^s)^k$ where $s \in \{2,4,6\}$, $k \geq 2$ and $r \geq 2$ if $s = 2$ or 4 and $r \geq 4$ if $s = 6$. Examples are given for various relators w of the form $(A^rB^s)^k$.

Some of Leech's questions were relatively easily solved, some were deeper and some are still open. For example, an easily determined answer is that when $w = (A^4B^4)^3$, the presentation defines an extension by $SL(2,8)$ of an extra-special 2-group of order 2^{15}. More difficult is the case $w = (A^2B^4)^6$ where the group is an extension of C_2^6 by $PSL(2,7) \times PSL(2,13)$ of order 11,741,184. When $w = (A^3B^2)^5$ the question is still open.

Semigroup identities in varieties of groups

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Let F be a free group generated by set $X = \{x_1, x_2, \ldots\}$ and Φ be a free semigroup generated by the same set X. Let a group $G = F/N$ be a qoutient group of F. An identity $w_1 = w_2$, for $w_1, w_2 \in F$, is called a semigroup identity if $w_1, w_2 \in \Phi$.

By S we denote a cancellative semigroup with a nontrivial identity $u = v$. It follows from Ore's theorem that S is embeddable in a group of fractions $G = S^{-1}S = SS^{-1}$. We call S a base semigroup of G.

Since in any group $xy = yx$, implies $xy^{-1} = y^{-1}x$, it follows that if S is commutative, G then is also commutative. It is known from [3] that if S is nilpotent of class n, then G is also nilpotent of class n. Does the same hold in the case of an arbitrary semigroup identity? This question was posed by G. M. Bergman in 1981 [1], and another formulation of it can be found in a work of Shevrin and Volkov [4].

Definition: A semigroup identity $u = v$ is called **transferable** when if the semigroup S satisfies $u = v$ then the group of fractions G also satisfies $u = v$.

Since every periodic cancellative semigroup coincides with its group of fractions, the identity $x^{n+1} = x$ (giving periodicity) is transferable. Moreover, since for a semigroup S with this identity one has $S = G$, obviously each other identity of S is the identity of G. So the identity $x^{n+1} = x$ has a property of being transferring.

Definition: The semigroup identity $u = v$ we call **transferring** if any other identity $w_1 = w_2$ which is satisfied in S is necessary satisfied in G under condition that S satisfies $u = v$.

Proposition: The set of transferring semigroup identities forms a modular lattice. This lattice is not closed to infinite sums.

Although for a semigroup S with the identity $x^n y^n = y^n x^n$ one has usually $S \neq G$, it is shown in [2] by J. Krempa and O. Macedońska that the identity $x^n y^n = y^n x^n$ is both transferable and transferring.

Proposition: The identity $(xy)^n(yx)^n$ is transferring.

Proposition: A variety with semigroup identities has a transferring identity if and only if any group in the variety having a relatively free base semigroup is a relatively free group.

Question: Does there exist identities which are transferable but not transferring and are transferring but not transferable?

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Gradings, derivations and automorphisms of some algebras which are nearly associative.

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We examine a class of algebras which includes Lie algebras, Lie color algebras, right alternative algebras, left alternative algebras, antiassociative algebras and associative algebras. We call this class of algebras (α, β, γ)-algebras and we examine gradings of these algebras by groups with finite support.

We generalize various results on associative algebras and finite - dimensional Lie algebras.

Two of our main results are

Theorem A. Let A be a $G-$graded left (α, β, γ)-algebra and $V = \oplus_{g \in G} V_g$ a $G-$graded left $A-$module with finite support, where G is a torsion free abelian group. If A_0 acts nilpotently on V, then A also acts nilpotently on V.

Theorem B. Let A be a $G-$graded $(\alpha, \beta, \gamma)-$algebra with finite support, where $G = T \times \mathbf{Z}_m$ and T is a torsion free abelian group. If the identity component $A_{(0,0)}$ acts nilpotently on A on both sides, then A is solvable.

These results can be used to examine the invariants of automorphisms and derivations. In particular we have

Corollary 1. Let $L = \oplus_{g \in G} L_g$ be a Lie color algebra over a field K of characteristic 0 and let D be a finite - dimensional nilpotent Lie algebra of homogeneous derivations of L which are algebraic as $K-$linear transformations of L. If the subalgebra of constants $L^D = \{x \in L | \delta(x) = 0, \forall \delta \in D\}$ is zero then L is nilpotent.

Corollary 2. Let $L = \oplus_{g \in G} L_g$ be a Lie color algebra over a field K and let σ be a homogeneous automorphism of L which is algebraic as a $K-$linear transformation of L. If

either $L^\sigma = 0$ or L^σ acts nilpotently on L where σ satisfies a separable polynomial, then L is solvable. Furthermore, if $L^\sigma = 0$, where the order of σ is prime, then L is nilpotent.

These results generalize old works of G. Higman ([H]), N. Jacobson ([J]), V. A. Kreknin and I. A. Kostrikin ([KK, Kr]) and D. Winter ([W]).

This is joint work with Jeffrey Bergen (DePaul University).

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Varieties of groups with no semigroup identity

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The question we consider is: which groups satisfy nontrivial semigroups identities.

From Zorn's lemma it follows that each variety of groups which has no semigroup identities contain the minimal subvariety with this property.

In his article Shevrin calls such varieties " nearly satisfying " a semigroup identities.

The known example of such varieties $\mathfrak{A}_p \cdot \mathfrak{A}$. The first proof of the fact that in the variety of all metabelian groups there is no semigroup identity was given by Malcev.

We give hare a sketch of our own (easier) proof of the same fact.

Let F be a free group on two generators.

Theorem:

There is no a semigroup identity in the group F/F''.

Sketch of the proof:

Let's assume that in F/F'' there is a semigroup identity. We may find then a binary one:

$$x^{k_1}y^{l_1}...x^{k_s}y^{k_s} = x^{m_1}y^{n_1}...x^{m_t}y^{n_t}, \qquad (1)$$

where

$$\sum_{i=1}^{s} k_i = \sum_{i=1}^{t} m_i \quad \text{and} \quad \sum_{i=1}^{s} l_i = \sum_{i=1}^{t} n_i.$$

We shall prove that it is the trivial identity. If apply s times the commutator equality $ab = ba[a,b]$ to both side of (1), we get:

$$x^{\sum_{i=1}^{s} k_i} y^{\sum_{i=1}^{s} l_i} [y^{\sum_{i=1}^{s} l_i}, x^{\sum_{i=2}^{s} k_i}][x^{\sum_{i=2}^{s} k_i}, y^{\sum_{i=2}^{s} l_i}][y^{\sum_{i=2}^{s} l_i}, x^{\sum_{i=3}^{s} k_i}][x^{\sum_{i=3}^{s} k_i}, y^{\sum_{i=3}^{s} l_i}]\ldots$$

$$\ldots[x^{l_{s-1}+l_s}, x^{k_s}][x^{k_s}, y^{l_s}] = x^{\sum_{i=1}^{t} m_i} y^{\sum_{i=1}^{t} n_i} [y^{\sum_{i=1}^{t} n_i}, x^{\sum_{i=2}^{t} m_i}]\ldots[x^{m_t}, y^{n_t}] \qquad (2)$$

We show that modulo F'' the left-hand and right-hand sides are equal so the identity is trivial in F/F''.

Symmetric words in groups

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An n-ary word w is called $n-$ *symmetricword* in a group G (or simply *symmetric word* if there is no ambiguity) if

$$(*)\ w(g_1, \ldots, g_n) = w(g_{\sigma(1)}, \ldots, g_{\sigma(n)})$$

for all $g_1, \ldots, g_n$ from G and all permutations σ from symmetric group S_n.

Symmetric words in a given group G are in a close connection with fixed points of the automorphisms permuting generators in the corresponding relatively free group, with symmetric operations in universal algebra and symmetric identities.

The $n-$symmetric words in G build a group $S^{(n)}(G)$ introduced by E.Płonka in [P1]. We consider with G a sequence $\{S^{(n)}(G)\}_{n\geq 1}$ of its groups of $n-$symmetric words.

We concern here with the following general

Problem. *For a given group G describe a sequence* $\{S^{(n)}(G)\}_{n\geq 1}$.

In our talk we give a survey of results on symmetric words in nilpotent, soluble and finite groups, on generation of groups of $n-$symmetric words and pose some open questions.

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On rings which are sums of two subrings

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More than 30 years ago Kegel, motivated by some results on groups, proved that a ring which is a sum of two nilpotent subrings is nilpotent itself. This result inspired further fruitful studies of rings which are sums of two subrigs.

The aim of the talk is to present some problems and new results in the area. It seems to be interesting to ask whether they have their counterparts in the theory of groups.

A problem of Sehgal

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Let R be a commutative noetherian ring with identity, and let I be an ideal with $I \neq R$. The Krull intersection theorem states that if $x \in \bigcap_{n=1}^{\infty} I^n$ then there exists a t in I such that $x = xt$.

The object of this paper is to generalize this result to group rings. (see [2], problem 38).

Let R be an integral domain, G a group and RG its group ring and let $A(RG)$ denote the *augmentation ideal* of RG. We shall say tha t the *intersection theorem* holds for $A(RG)$ if there exists an element $a \in A(RG)$ such that $\bigcap_{i=1}^{\infty} A^i(RG))(1-a) = 0$.

Sufficient conditions for the intersection theorem to hold for certain RG are given in [1]. In [1] neccessary and sufficient conditions are given in the cases when G is finitely genereted with a nontrivial torsion element and $R = Z$ is the ring of integers, or if G is finitely genereted and $R = Z_p$ is the ring of p-adic integers.

In this paper we give neccessary and sufficient conditions for the intersection theorem to hold for an arbitrary group ring over a commutative integral domain.

Let $D_n(RG)$ be the n^{th} dimension subgroup of the group G, and let
$D_\Omega(RG) = \bigcap_{n=1}^{\infty} D_n(RG)$,

$W_p(G) = \bigcap_{n=1}^{\infty} G^{p^n}\gamma_n(G)$, where $\gamma_n(G)$ is the n^{th} term of the lower central series of G.

We use the following notations for group classes: $\overline{\mathcal{N}}_p$ — *nilpotent p-groups of finite exponent*, $\overline{\mathcal{N}}_\Omega = \bigcup_{p\in\Omega} \overline{\mathcal{N}}_p$, where Ω is a subset of primes.

THEOREM. *Let R be a commutative integral domain. The intersection theorem holds for $A(RG)$ if and only if $D_\omega(RG)$ is the largest finite subgroup of order invertible in R and at least one of the following conditions holds:*

$G/D_\omega(RG)$ is a residually torsion free nilpotent group;
there exists a subset Ω of the primes such that $G/D_\omega(RG)$ is discriminated by the class of groups $\overline{\mathcal{N}}_\Omega$, $\bigcap_{p\in\Omega} J_p(R) = 0$ and for arbitrary subset Λ of Ω the ideal $\bigcap_{p\in\Lambda} J_p(R) = 0$ or $G/D_\omega(RG)$ is discriminated by the class of groups $\overline{\mathcal{N}}_{\Omega\setminus\Lambda}$;
the set of the elements of finite order of G forms a finite normal subgroup $T(G)$ and for every prime divisor p of $|T(G)|$ which is non-invertible in R, the group $W_p(G/D_\omega(RG))$ is finite with no p–elements and $J_p(R) = 0$.

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Wreath Products in the Unit Group of Modular Group Algebras of 2-groups of Maximal Class

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Let G is a finite p-group, K is a field of characteristic p and $U(KG)$ denotes the group of normalized units of KG, namely $U(KG) = 1 + \Delta(G)$, where $\Delta(G)$ is the augmentation ideal of KG.

In [1] A.Shalev stated the question whether the wreath product of a group of order p and the commutator subgroup of G is always involved in $U(KG)$.

In [2] he gave a positive answer to this question for the case of odd p and a cyclic commutator subgroup of G.

We consider 2-groups of maximal class, namely, the dihedral, semidihedral and generalized quaternion groups, which we denote by D_n, S_n and Q_n respectively.

We give positive answer for D_n and S_n for every n.

Theorem 1: Let K is a field of characteristic two, G is a dihedral or semidihedral 2-group. Then the wreath product C_2 *wr* G' of a cyclic group of order two and the commutator subgroup of G is involved in $U(KG)$.

The theorem was proved by constructing of the subgroup of $U(KG)$ which homomorfic image is isomorphic to the wreath product.

For the generalized quaternion group we show by direct calculation, then such wreath product is involved in $U(KG)$ when $|G| \leq 32$. The proof for general case is now being in consideration. Clearly, in general case we can involve in $U(KG)$ wreath product of C_2 and a cyclic group of order $\frac{1}{2}|G'|$.

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Rings with periodic unit groups

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Let A be a torsion free abelian group and let G be the group of its automorphisms. Hallett and Hirsch proved that, if G is finite then G has to be a subdirect product of some copies of 6 explicitly distinguished small groups.

Our aim here is to extend this result to the case when G is periodic. We proceed in the context of rings and their groups of units. In particular integral group rings with periodic group of units will be described.

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On group algebras with metabelian unit groups

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Let G be a finite group and $\mathbf{F}$ a field of positive characteristic p. Although the problem of its solvability has formerly been resolved, few facts are known of the derived length of the group of units $U(\mathbf{FG})$ of the group algebra $\mathbf{FG}$. In [1] Shalev showed that $U(\mathbf{FG})$ is metabelian if and only if G is abelian provided $p > 3$; moreover, if and only if G is abelian or nilpotent with a commutator subgroup of order 3 provided $p = 3$. Our aim is to treat the case $p = 2$.

Theorem Let G be a finite group and $\mathbf{F}$ a field of characteristic 2. The group of units $U(\mathbf{FG})$ is metabelian if and only if one of the following conditions holds:

(i) G is abelian;

(ii) G is nilpotent of class 2 and has an elementary abelian commutator subgroup of order 2 or 4;

(iii) $\mathbf{F} = \mathbf{F_2}$, the field of two elements, and G is an extension of an elementary abelian 3-group H by the group $\langle b\rangle$ of order 2 with $b^{-1}ab = a^{-1}$ for every $a \in H$.

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On group and semigroup identities

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Let $\mathcal{F}$ and F denote respectively the free n-generator semigroup and the free n-generator group, both generated by n elements $x, y, z, \ldots, t$.

Let G be a group generated by $g_1, \ldots, g_n$, and S be a semigroup generated by the same $g_1, \ldots, g_n$. The group G is called the group of fractions for S if $G = SS^{-1} = S^{-1}S$. The necessary and sufficient condition for this situation is the Ore condition satisfied in S, that is: for each $a, b \in S$ the intersections $aS \cap bS$ and $Sa \cap Sb$ are not empty.

Elements of S in G can satisfy an identity *i.* which is not necessary a semigroup identity *s.i.* The question, we consider, concerns possibilities for S and G to satisfy an identity or even a semigroup identity.

The possibilities can be described by a table

	s.i.	*i.*
G	$\pm$	$\pm$
S	$\pm$	$\pm$

We prove that not more then fife different possibile tables exist, give some examples and formulate problems. These fife tables are:

I.

	s.i.	*i.*
G	+	+
S	+	+

II.

	s.i.	*i.*
G	−	−
S	−	−

III.

	s.i.	*i.*
G	−	+
S	−	+

IV.

	s.i.	*i.*
G	−	−
S	+	+

V.

	s.i.	*i.*
G	−	+
S	+	+

We note first that if G satisfies a *s.i.* then we have the case I.

Example 1 Each group with a semigroup identity provides an example with the table I. It is enough to take S nilpotent [3], or with the identity $x^n y^n = y^n x^n$ [2], which imply the same semigroup identities in G.

Example 2 If S satisfies no identity, then we have the case II. To have an example we denote by $\gamma_k(F)$ the k-th member of the low central series in F, then define $G = \Pi^\times F/\gamma_k(F)$, the infinite direct product of free nilpotent groups of classes $k = 1, 2, 3, ..$ and $S = \Pi^\times S_k$, where S_k is the image of $\mathcal{F}$ in $F/\gamma_k(F)$. Since by [3] $F/\gamma_k(F)$ has a *s.i.*, it is the group of fractions for S_k, and hence $G = SS^{-1} = S^{-1}S$. However S has no identities because $\cap\gamma_k(F) = 1$ [4].

Question 1 Does there exist a finitely generated example for the table II?

We shall show that a situation with the table

	s.i.	*i.*
G	−	−
S	−	+

is impossible. It follows from the

Theorem If S satisfies an identity $w = 1$ then S satisfies a semigroup identity.

Example 3 To have the situation as in the table III, we define

$$G = \Pi^\times F/(F''\gamma_k(F)).$$

Here G satisfies a metabelian identity. Similarly to the Example 2, G is locally nilpotent and hence $G = SS^{-1} = S^{-1}S$. If S has a semigroup identity $u = v$, say, then $uv^{-1} \in \cap F''\gamma_k(F)$. By [4] this intersection is equal to F''. It follows from Malcev [3] that $uv^{-1} \equiv 1$. So S has no semigroup identity and hence G also does not have one.

Question 2 Does there exist a finitely generated example for the table III?

Question 3 Does there exist an example for tables IV and V? This is a well known question of G.M.Bergman [1], posed in 1981.

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On distributive group rings

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Let R be a ring, G a group and RG the group ring of G over R. We are going to survey some known results (for example from [1], [2], [3]) concerning necessary and sufficient conditions on G and R under which RG has a distributive lattice of right ideals.

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Semigroups of matrices

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The aim of this talk is to present a general approach to the study of linear semigroups, that is, subsemigroups S of the full monoid $M_n(K)$ of n×n matrices over a field K (and more generally, over a division ring K). First, we discuss the general structure theorem for such semigroups, obtained in [1]. It describes S in terms of so called *uniform components*, which are semigroups that can be viewed as orders in completely 0-simple semigroups $\mathcal{M}(G, X, Y; P)$ over certain linear groups G. This allows to reduce several problems on S to the study of cancellative semigroups $S \cap D$, where D are maximal subgroups of $M_n(K)$, the study of the linear groups $S_D \subseteq D$ they generate and of the sandwich matrices of the uniform components of S.
In order to apply the structure theorem, exploiting the powerful results on linear groups, one needs to be able to transfer properties from the nonempty intersections $S \cap D$ to the groups S_D. In this direction, we present the 'generalised Tits alternative', recently proved in [2], which asserts that for a finitely generated semigroup $S \subseteq M_n(K)$ the following conditions are equivalent:
1) S satisfies an identity, 2) S does not have a free noncommutative subsemigroup, 3) each nonempty $S \cap D$ is generates an almost nilpotent group. Consequences for the general philosophy of studying linear semigroups via the structure theorem are discussed. Finally, examples of combinatorial and structural applications are given. They are concerned with the growth problem for semigroups of matrices and certain problems in the theory of graded rings (extending known results on group-graded rings and semigroup rings).

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Quotients of finite exponent for hyperbolic groups

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The following statement answers to Gromov's question.

Theorem. *Let G be any hyperbolic group that is not cyclic-by-finite. Then there exists an integer $n = n(G)$ such that G possesses an infinite quotient of exponent n.*

Finitely presented groups and semigroups

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We consider generalising Nielsen-Reidemeister-Schreier theory for finitely presented groups to finitely presented semigroups. Since a subsemigroup of a free semigroup is not necessarily free, the semigroup results must look different.

Theorem (Campbell, Robertson, Ruškuc, Thomas). *Let F be a free semigroup. Then*

(a) *a proper ideal of F is never free;*

(b) *a proper right ideal R of F, where R is finitely generated as a semigroup, is always finitely presented and never free;*

(c) *F contains infinitely generated free right ideals.*

Next we ask whether a subsemigroup T of finite index of a finitely presented semigroup S is necessarily finitely presented. But then we have to specify what is meant by "index". We follow Jura and define the index of T in S to be $|S - T| + 1$, the justification for this being that if T is an ideal $|S - T| + 1$ is the order of the Rees quotient S/T. Thus, T has finite index in S if and only if $S - T$ is a finite set.

Theorem (Campbell, Robertson, Ruškuc, Thomas). *A right ideal of finite index of a finitely presented semigroup is finitely presented (+an explicit presentation).*

One may also try to find presentations for some subgroups of a given semigroup. The most obvious first candidates are the group of units and the maximal subgroups of a minimal ideal.

Theorem (Zhang). *Let S be a finitely presented special monoid, i.e. a monoid defined by a finite number of relations of the form $w = 1$. The group of units of S is finitely presented and has a solvable word problem if and only if S has a solvable word problem.*

Theorem (Campbell, Robertson, Ruškuc, Thomas). *Let S be a finitely presented semigroup, and let I be a minimal ideal of S which is a completely simple semigroup. If I has finitely many minimal left ideals or finitely many minimal right ideals then I is finitely presented and any maximal subgroup of I is finitely presented (+ an explicit presentation).*

Let us suppose that the semigroup S is defined by a presentation Π,
that G is the group defined by the same presentation, and that S possesses a minimal ideal I which is a completely simple semigroup. The computer evidence, obtained by using an implementation of the Todd-Coxeter enumeration algorithm for semigroups, suggested that in many cases I is a union of groups isomorphic to G. A necessary and sufficient condition for this to happen is

Theorem (Campbell, Robertson, Ruškuc, Thomas). *The ideal I is a union of groups isomorphic to G if and only if the idempotents of I are closed. In particular, this is the case if I has a unique minimal left or unique minimal right ideal.*

Question. Is a subsemigroup of finite index in a finitely presented semigroup necessarily finitely presented?

On some undecidable problems in varieties of groups

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It will be a report about results of my former students Yu.G.Kleiman and M.I.Anokhin. The results are a part of Anokhin's PhD thesis. In 1979 Kleiman constructed an example of finitely based variety with undecidable word problem in free groups. Consequently the problem of implication is not decidable for group identities. One of the most interesting statement of these works is the following

Theorem. *There exists a soluble finitely based variety in which the free group of rank 2 has decidable word problem but it has undecidable problem of implication of identities.*

On torsion-free groups with trivial center

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Let Γ be a Bieberbach group which means that Γ is a to rsion free group and it has a free abelian normal subgroup A of finite index. If A is maximal it is called the translation subgroup, and then Γ/A is called the point group of Γ. We will show a series of examples of Bieberbach groups with trivial center and with the translation group A of minimal rank. Such groups are of interest to differential geometers, since they arise as fundamental groups of primitive, compact flat Riemannian manifolds. All these groups are torsion free but not right orderable. Some of them does not satisfy the unique product condition. Hence the group rings over such groups are good for testing the Kaplansky's conjecture about units in group rings.

Direct limits of finite symmetric groups with diagonal embedding

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Subgroups of infinite symmetric groups which are direct limit of finite symmetric groups with diagonal embedding [1, ch 6] are investigated.

We prove, that such subgroups form a complete upper semilattice with natural supplement to the lattice. This lattice is isomorphic to the lattice of subgroups for additive group of rational numbers, containing group of integer numbers.

Every subgroup is simple or contains the simple subgroup of index 2. Conjugacy classes are described. We obtain the characterization of stabilizers of partition on integer numbers. Such a characterization gives a simple criterion of isomorphism for subgroups from the semilattice.

We can identify minimal elements if this semilattice as finitary birational automorphism group over locally finite fields. We have the description of Sylov subgroups for this groups.

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On the adjoint group of a radical ring

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An associative ring R is radical if the set of all its elements forms a group under the operation $r \circ s = r + s + rs$ (r and s are of R) which is called the adjoint group of R and is denoted by $R_\circ$.

The basic problem which will be considered refers to the study of relations between the structures of a subgroup G of $R_\circ$ and the subring $[G]$ of R generated by G.

Our purpose is to present the recent results and indicate the various approaches to the problem.

In particular, we will show that if R is a nil ring and G is a subgroup of finite Pruefer rank in the adjoint group of R, then the subgroup G and the subring $[G]$ of R are locally nilpotent.

As a corollary we obtain that if G is a group in which every finitely generated subgroup has a finite Pruefer rank and the augmentation ideal of the group ring KG over a commutative ring K with unity is a nil ideal, then the group G is locally nilpotent.

Unitary subgroup of the Sylow p-subgroup of the group of normalized units in a commutative group ring

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Let G be an abelian group, K a ring of prime characteristic p and let V(KG) denote the group of normalized units of the group ring KG. An element $u = \sum_{g\in G} \alpha_g g \in V(KG)$ is called unitary if u^{-1} coincides with the element $u^* = \sum_{g\in G} \alpha_g g^{-1}$. The set of all unitary elements of the group $V(KG)$ forms the unitary subgroup $V_*(KG)$ of $V(KG)$.

S. P. Novikov had raised the problem of determining the invariants of the group $V_*(KG)$ when G has a p-power order and K is a finite field of characteristic p. This was solved by A. Bovdi and the author in [1]. Here we give the invariants of the unitary subgroup of the Sylow p-subgroup of $V(KG)$ whenever G is an arbitrary abelian group and K is a commutative ring of prime characteristic p without nilpotent elements.

Let G^p denote the subgroup $\{g^p \mid g \in G\}$ and ω an arbitrary ordinal. The subgroup G^{p^ω} of the group G is defined in following way: $G^{p^0} = G$, for non-limited ordinals (that is if $\omega = \nu + 1$): $G^{p^\omega} = (G^{p^\nu})^p$, and if ω is a limited ordinal, then $G^{p^\omega} = \bigcap_{\nu<\omega} G^{p^\nu}$. The subring K^{p^ω} of the ring K is defined similarly. The ring K is called p-divisible if $K^p = K$.

Let $G[p]$ denote the subgroup $\{g \in G \mid g^p = 1\}$ of G. The factorgroup $G^{p^\omega}[p]/G^{p^{\omega+1}}[p]$ can be considered as a vector space over the field of p elements. The cardinality of a basis of this vector space is called the ω-th Ulm–Kaplansky invariant $f_\omega(G)$ of the group G concerning to p.

Theorem. *Let ω be an arbitrary ordinal, K a commutative ring of prime characteristic p without nilpotent elements, P the maximal divisible subgroup of the Sylow p-subgroup S of an abelian group G, $G_\omega = G^{p^\omega}$, $S_\omega = S^{p^\omega}$ and $K_\omega = K^{p^\omega}$. Let, further on, $V_p = V_p(KG)$ denote the Sylow p-subgroup of the group of normalized units $V(KG)$ in the group ring KG and $W = W(KG)$ the unitary subgroup of*

$V_p(KG)$. In case $P \neq 1$ we assume that the ring K is p-divisible, and if $p = 2$ then K without zero divisors.

1) If $G_\omega = G_{\omega+1}$ or $S_\omega = 1$ then $f_\omega(W_p) = f_\omega(V_p) = 0$.

2) If $G_\omega \neq G_{\omega+1}$, $S_\omega \neq 1$ and at least one of the ordinals $|K_\omega|$ and $|G_\omega|$ is infinite then

$$f_\omega(W_p) = \begin{cases} f_\omega(V_p) = \max\{|G_\omega|, |K_\omega|\}, & \text{when } p > 2, \\ \max\{|G|, |K|\}, & \text{when } p = 2 \text{ and } \omega = 0, \\ f_\omega(V_2) = \max\{|G_\omega|, |K_\omega|\}, & \text{when } p = 2, \omega > 0 \text{ and } G_{\omega+1} \neq 1, \\ f_\omega(G), & \text{when } p = 2, \omega > 0 \text{ and } G_{\omega+1} = 1. \end{cases}$$

3) If $S_\omega \neq 1$, the ring K_ω and the group G_ω are finite then

$$f_\omega(W_p) = \begin{cases} \frac{1}{2}[G_\omega : S_\omega](|S_\omega| - 2|S_{\omega+1}| + |S_{\omega+2}|)\log_p |K_\omega|, & \text{when } p > 2, \\ t_0 - 2t_1 + t_2 - f_1(S) + (|S[2]| - 1)\log_2 |K|, & \text{when } p = 2 \text{ and } \omega = 0, \\ t_\omega - 2t_{\omega+1} + t_{\omega+2} + f_\omega(S) - f_{\omega+1}(S), & \text{when } p = 2 \text{ and } \omega > 0, \end{cases}$$

where $t_\omega = \frac{1}{2}([G_\omega : S_\omega](|S_\omega| - 1) - |S_\omega[2]| + 1)\log_2 |K_\omega|$.

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On the fixed points series for permutations of generators

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Let F be a two-generated free group and let σ be the automorphism which permutes generators. If the subgroup $N \subseteq F$ is σ-invariant, normal then we define $S(N) = gpn(s; s^{-1}s^\sigma \in N)$. In another words, $S(N)$ is the normal closure of a preimage of the subgroup of fixed points for $\bar\sigma$ in F/N (where $\bar\sigma$ is the automorphism of F/N, induced by σ). We will be considered the ascending fixed points series: $N \trianglelefteq S(N) \trianglelefteq S^2(N) \trianglelefteq S^3(N) \trianglelefteq \ldots$. We describe the behaviour of such a series in some cases, we give some examples and we discuss some questions.

Outer automorphisms group may become inner in the integral group ring

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For a group G and a commutative ring R the group ring RG is the ring with the free R-module on the set G as additive structure and multiplication induced by the group structure of G. It has the universal property that every group homomorphism of G to the unit group of an R-algebra S extends uniquely to an R-algebra homomorphism from RG to S.

In [2] the question is raised by Jackowski and Marciniak if there is a finite group G and a non inner automorphism of G that extends to an inner automorphism of the integral group ring $\mathbb{Z}G$.

This question is closely related to the isomorphism problem for integral group rings: Does $\mathbb{Z}G \simeq \mathbb{Z}H$ imply $G \simeq H$? The isomorphism problem is open in general, however, a conjecture of Zassenhaus states that if the isomorphism of the rings is an isomorphism as augmented algebras, then the group H is conjugate in the rational group ring $\mathbb{Q}G$ to H. Since the augmentation condition can always be assumed the Zassenhaus conjecture implies a positive answer to the isomorphism problem. Roggenkamp and Scott proved a much stronger statement for p-groups in [5]. In general the Zassenhaus conjecture is false, Roggenkamp and Scott gave a metabelian counterexample in [6].

The methods developed for the counterexample to the Zassenhaus conjecture are strong enough to construct a finite, solvable group G and a group automorphism α such that for each semilocal Dedekind domain R the automorphism α extended to RG is inner [8]. Standard techniques (cf. [3]) involving class field theory can be applied to give a global Dedekind domain S of finite degree over $\mathbb{Z}$ such that α is inner in SG. Up to now it is not known how to construct a counterexample over $\mathbb{Z}$.

The principal method used in the proof is a version of Clifford theory as described in [7, Chapter X]. To apply the Clifford theory it is necessary to examine the structure of the group very explicitely. We need for example that the inertia groups are normal, that the action of the automorphism on the normal subgroup to apply Clifford theory is trivial and that the automorphism varies each group element by an element of the inertia group.

The group G is given by successive extensions of a 2-group having a non inner class sum preserving automorphism of order 2 (cf. [2]) to a group of order $p^2 \cdot q^2 \cdot 128$ with two different odd primes p and q. The 2-Sylow subgroup has a normal complement and the automorphism acts as identity on the 2-Sylow and the p-Sylow subgroup and as inversion on the q-Sylow subgroup. In the whole the group G is an extension of a combination of the group mentioned in [2, 4.9 ff].

The fact that the automorphism is non inner is easily proven and purely group theoretical. The statement that the automorphism becomes inner in the group ring is proved a prime at a time using a result of Fröhlich [1, Theorem 6]. Using the Noether Deuring theorem we may assume that the ground ring R contains enough roots of unity. Locally, fixing a prime $\wp$ of R, Clifford theory can be applied to the various $\wp$-blocks and the automorphism is realized to be inner in $R_{\wp}G$. Fröhlich's result then allows us to see that the automorphism is inner in RG.

Jan Krempa proved (cf. [4]) that the square of a non inner group automorphism that becomes inner in $\mathbb{Z}G$ has to be inner in G, a property that is shared by our automorphism. Jackowski and Marciniak proved in [2] that the group G has to have a non normal Sylow 2-group, a requirement that is satisfied by our G. Therefore there is hope to be able to prove that a slight modification of or even the group itself constructed in [8] can lead to an example for an outer group automorphism of a finite group G that becomes inner in $\mathbb{Z}G$.

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