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THE SYMMETRY OF TRANSFORMATIONS OF THE MAXWELL AND TWO-TERMINAL CAPACITANCE MATRICES

Summary. The paper presents the symmetry property of the transformations of the Maxwell matrix and "two-terminal" matrix describing the capacitance parameters of the planar multiconductor interconnects in a VLSI structures. This symmetry of transformations plays a crucial role in the minimization of errors occurring in the Maxwell matrix which can be computed from measurements, performed by the direct measurement of elements of the "two-terminal" matrix in the multiconductor structures, using active separation.

SYMETRIA PRZEKSZTAŁCEŃ MACIERZY MAXWELLA I POJEMNOŚCIOWEJ MACIERZY DWÓJNIKOWEJ

Streszczenie. W pracy przedstawiono związki pomiędzy przekształceniem macierzy Maxwella i macierzy pojemnościowej dwójnikowej, opisujących pojemnościowe parametry sprzężonych wielolinii połączeń w planarnych strukturach VLSI. Związki te charakteryzują się symetrią. Symetria przekształceń macierzy gra podstawową rolę w minimalizacji błędu wyznaczenia macierzy Maxwella z pomiarów pojemnościowej macierzy dwójnikowej, metodą pomiaru z aktywną separacją.

1. Assumptions

Let us consider the system of multiconductor planar transmission lines of n conductors plus ground conductor with cross-section presented in Fig. 1.

The equivalent "two-terminal" capacitance network for n signal conductors plus ground (reference) conductor is presented in Fig. 2.. These "two-terminal" capacitances T_{ij} form a "two-terminal" matrix $T=[T_{ij}]_{n\times n}$. The ground conductor is denoted as #0.

Let $Q_1, Q_2, \ldots, Q_n$ be the charge per unit length on the conductors with $V_1, V_2, \ldots, V_n$ the corresponding potentials. The vectors Q and V are related by Maxwell matrix $C=[C_{ij}]_{n\times n}$ defined as follows:

$$Q = CV, \tag{1}$$

where:

$Q' = [Q_1, Q_2, \ldots, Q_n]$ - is a vector of PUL charges on the n-conductor transmission line, symbol ' denotes transposition,

$V' = [V_1, V_2, \ldots, V_n]$ - is the vector of line potentials with respect to the reference.

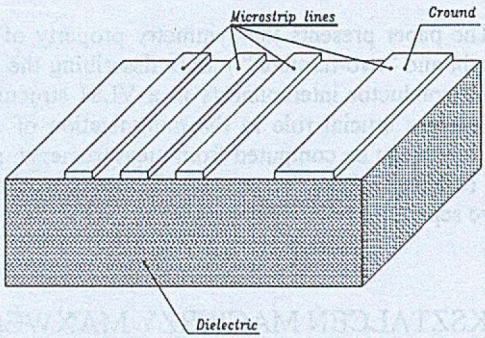

Fig. 1. The cross-section of n-conductor geometry with reference
Rys. 1. Przekrój planarnej wielolinii transmisyjnej z przewodem ziemi

The diagonal elements of the matrix C are called coefficients of capacitance and the off-diagonal elements are called the coefficients of electrostatic induction. The matrix C has the following properties [1]:

$$\begin{aligned} &C_{ij} = C_{ji} && i,j = 1,2,\ldots,n \\ &C_{ii} > 0 && i = 1,2,\ldots,n \\ &C_{ij} < 0 && i \neq j \\ &\sum_{j=1}^{n} C_{ij} > 0 && i = 1,2,\ldots,n. \end{aligned} \tag{2}$$

The goal is determination of Maxwell matrix using the results of measurements of "two-terminal" capacitances for discussed structure.

Analyzing the potential differences between conductors and using "two-terminal" capacitance matrix $T=[T_{ij}]_{n\times n}$, one finds [2]:

$$T_{ij} = -C_{ij} \quad \text{for } i \neq j,$$
$$T_{ii} = \sum_{j=1}^{n} C_{ij}. \tag{3}$$

These "two-terminal" capacitances, T_{ij}, also called circuit capacitances have interpretation presented in Fig. 2.

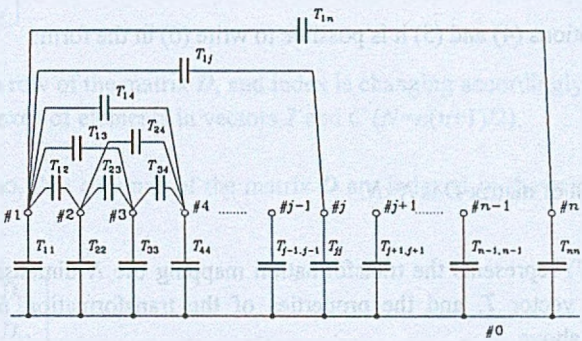

Fig. 2. Capacitance equivalent circuit for n conductors with reference
Rys. 2. Dwójnikowy schemat zastępczy wielolinii transmisyjnej z przewodem ziemi

To simplify the notation we will use the following vector definitions (symmetry of matrix C forces the symmetry of matrix T) – thanks to the symmetry, there are $(n^2 - n)/2 + n = n(n+1)/2$ elements of matrices C and T to consider instead total n^2 elements:

$$C' = [C_{11}, C_{12}, C_{13}, C_{14},\ldots C_{1n}, C_{22}, C_{23}, C_{24},\ldots C_{2n}, C_{33}, C_{34}, C_{35},\ldots C_{3n}, C_{44},\ldots C_{nn}]$$
$$= [C_1, C_2, C_3,\ldots C_N] \tag{4}$$

and

$$T' = [T_{11}, T_{12}, T_{13}, T_{14},\ldots T_{1n}, T_{22}, T_{23}, T_{24},\ldots T_{2n}, T_{33}, T_{34}, T_{35},\ldots T_{3n}, T_{44},\ldots T_{nn}]$$
$$= [T_1, T_2, T_3,\ldots T_N]. \tag{5}$$

Vectors C and T have $N=n(n+1)/2$ entries.

Using relation (3) and symmetry that $T_{ij}=T_{ji}$, one may write:

$$T_{ij} = (\delta_{ij} - 1)C_{ij} + \delta_{ij}\sum_{k=1}^{n} A_{ik} \tag{6}$$

for $i, j = 1, 2, \ldots, n$, and $i \le j$,

where

$$\delta_{ij} = \begin{cases} 1 & \text{for} \quad i = j \\ 0 & \text{for} \quad i \ne j, \end{cases}$$

$$A_{ik} = \begin{cases} C_{ik} & \text{for} \quad i \le k \\ C_{ki} & \text{for} \quad i > k. \end{cases}$$

Using the definitions (4) and (5) it is possible to write (6) in the form:

$$T = DC \tag{7}$$

where the dimension of matrix D is $N \times N$.

The equation (7) represents the transformation mapping the N-dimensional vector C into the N-dimensional vector T, and the properties of the transformation matrix D has been precisely presented above.

2. Theorem

If the transformation matrix D (different from unity matrix – Eqs. (2) and (3)) maps the N-dimensional vector C into the N-dimensional vector T accordingly with relation

$$T = DC,$$

where vectors T, C and matrix D have properties presented in the assumptions, than the inverse matrix $G = D^{-1}$ of the matrix D is equal to itself i.e.:

$$G = D^{-1} = D. \tag{8}$$

3. Proof

Basing on definitions of the vectors T and C, transformation (8) described by Eq. (7), can be rewritten in a form:

$$T = DC = \begin{bmatrix} D_1 \\ D_2 \\ D_3 \\ . \\ . \\ D_r \\ . \\ . \\ D_N \end{bmatrix} C \tag{9}$$

where

D_r - r-th row of the matrix D, and index is changing accordingly to the indexes of elements in vectors T and C ($N=n(n+1)/2$).

We assume also, that columns of the matrix D are indexed in the same manner, than we have:

$$D = \begin{bmatrix} D_1 \\ D_2 \\ D_3 \\ . \\ . \\ . \\ . \\ . \\ . \\ D_r \\ . \\ . \\ D_N \end{bmatrix} = \begin{bmatrix} D_{11} \\ D_{12} \\ D_{13} \\ . \\ D_{1n} \\ D_{22} \\ D_{23} \\ . \\ D_{2n} \\ D_{33} \\ . \\ D_{ij} \\ . \\ D_{nn} \end{bmatrix} = [D_{rs}]_{N \times N}, \tag{10}$$

where

$i, j = 1, 2, 3, \ldots n$, and $i \le j$,

$r, s = 1, 2, 3, \ldots N$.

Evidently, the indexing of elements of matrix D has a form:

$$[D_{rs}]_{N\times N} = \begin{array}{c} \\ 11 \\ 12 \\ 13 \\ 14 \\ \cdot \\ 1n \\ 22 \\ 23 \\ 24 \\ \cdot \\ 2n \\ 33 \\ 34 \\ \cdot \\ 3n \\ 44 \\ \cdot \\ nn \end{array} \begin{array}{c} 11,12,13,14,\ldots 1n,22,23,24,\ldots 2n,33,34,\ldots 3n,44,\ldots nn \\ \left[\begin{array}{c} \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \end{array}\right] \end{array} \tag{11}$$

Using relation (6) and (10) one may write:

$$D_{ij} = \left[\ldots \Delta_{ij,ij} \ldots \delta_{ij,1i} \ldots \delta_{ij,2i} \ldots \delta_{ij,ii} \ldots \delta_{ij,ii+1} \ldots \delta_{ij,ii+2} \ldots \delta_{ij,in} \ldots \right] \tag{12}$$

where

$$\Delta_{ij,ij} = \begin{cases} -1 & \text{for } l \neq j, \text{ and is placed in } ij\text{ - th row and } ij\text{ - th column} \\ & \text{of the matrix } D \text{ (Eq. (11))}, \\ 0 & \text{for } i = j, \text{ and is placed in } ij\text{ - th row and } ij\text{ - th column} \\ & \text{of the matrix } D, \end{cases}$$

$$\delta_{ij,kl} = \begin{cases} 1 & \text{for } i = j, \text{ and is placed in } ij\text{ - th row and } kl\text{ - th column} \\ & \text{of the matrix } D, \\ 0 & \text{for } i \neq j, \text{ and is placed in } ij\text{ - th row and } kl\text{ - th column} \\ & \text{of the matrix } D, \end{cases}$$

dots ... - mean 0's inserted in all remaining columns in considered D_{ij} row.

In (12) all numbers in the indexes ij,kl fulfill the conditions: $i \leq j$ and $k \leq l$. The dependence between the vectors T and C in a non-evident form, can be expressed as follows:

for $i \neq j$

$$T_{ij} = -C_{ij}, \tag{13}$$

for $i = j < n$

$$T_{ij} = T_{ii} = \sum_{k=1}^{n} A_{ik} = \sum_{k=1}^{i} C_{ki} + \sum_{k=i+1}^{n} C_{ik}, \tag{14}$$

for $i = j = n$

$$T_{ij} = T_{ii} = T_{nn} = \sum_{k=1}^{n} A_{ik} = \sum_{k=1}^{n} C_{kn}. \tag{15}$$

Resolving (14) and (15) for i=1, 2, 3, ... n we have:

for $i = 1$

$$T_{11} = \sum_{k=1}^{n} C_{1k},$$

for $i = 2$

$$T_{22} = \sum_{k=1}^{2} C_{k2} + \sum_{k=3}^{n} C_{2k},$$

for $i = 3$

$$T_{33} = \sum_{k=1}^{3} C_{k3} + \sum_{k=4}^{n} C_{3k}, \tag{16}$$

for $i = 4$

$$T_{44} = \sum_{k=1}^{4} C_{k4} + \sum_{k=5}^{n} C_{4k},$$

.

for $i = n$

$$T_{nn} = \sum_{k=1}^{n} C_{kn},$$

Let we consider the product DT of right-side multiplication of the matrix D by the vector T. Basing on the definition of vector T and Eq. (12) we have:

$$D_{11}T = \left[\, \delta_{11,11} \ldots \delta_{11,12} \ldots \delta_{11,13} \ldots \delta_{11,14} \ldots \delta_{11,15} \ldots \delta_{11,1n} \ldots \right] T = \sum_{k=1}^{n} T_{1k},$$

$$D_{22}T = \left[\ldots \delta_{22,12} \ldots \delta_{22,22} \ldots \delta_{22,23} \ldots \delta_{22,24} \ldots \delta_{22,25} \ldots \delta_{22,2n} \ldots \right] T = \sum_{k=1}^{2} T_{k2} + \sum_{k=3}^{n} T_{2k},$$

$$D_{33}T = \left[\ldots \delta_{33,13} \ldots \delta_{33,23} \ldots \delta_{33,33} \ldots \delta_{33,34} \ldots \delta_{33,35} \ldots \delta_{33,3n} \ldots \right] T = \sum_{k=1}^{3} T_{k3} + \sum_{k=4}^{n} T_{3k},$$

$$D_{44}T = \left[\ldots \delta_{44,14} \ldots \delta_{44,24} \ldots \delta_{44,34} \ldots \delta_{44,44} \ldots \delta_{44,45} \ldots \delta_{44,4n} \ldots \right] T = \sum_{k=1}^{4} T_{k4} + \sum_{k=5}^{n} T_{4k}, \tag{17}$$

.

$$D_{nn}T = \left[\ldots \delta_{nn,1n} \ldots \delta_{nn,2n} \ldots \delta_{nn,3n} \ldots \delta_{nn,4n} \ldots \delta_{nn,5n} \ldots \delta_{nn,nn} \right] T = \sum_{k=1}^{n} T_{kn}.$$

For all remaining i,j we have (Eq. (13)):

$$D_{ij}T = -T_{ij}\,. \tag{18}$$

Equations (17) and (18) in a matrix notation have a form:

$$DT = \begin{bmatrix} \sum_{k=1}^{n} T_{1k} \\ -T_{12} \\ -T_{13} \\ -T_{14} \\ . \\ -T_{1n} \\ \sum_{k=1}^{2} T_{k2} + \sum_{k=3}^{n} T_{2k} \\ -T_{23} \\ -T_{24} \\ . \\ -T_{2n} \\ \sum_{k=1}^{3} T_{k3} + \sum_{k=4}^{n} T_{3k} \\ -T_{34} \\ . \\ -T_{3n} \\ \sum_{k=1}^{4} T_{k4} + \sum_{k=5}^{n} T_{4k} \\ . \\ \sum_{k=1}^{n} T_{kn} \end{bmatrix} \tag{19}$$

The sums of T_{ij} in Eq. (19) can be expressed by C_{ij} from Eqs. (13), (14), (15), and (16):

$$\sum_{k=1}^{n} T_{1k} = T_{11} + \sum_{k=2}^{n} T_{1k} = \sum_{k=1}^{1} C_{k1} + \sum_{k=2}^{n} C_{1k} + \sum_{k=2}^{n} (-C_{1k}) = C_{11}\,,$$

$$\sum_{k=1}^{2} T_{k2} + \sum_{k=3}^{n} T_{2k} = T_{12} + T_{22} + \sum_{k=3}^{n} T_{2k} = -C_{12} + \sum_{k=1}^{2} C_{k2} + \sum_{k=3}^{n} C_{2k} + \sum_{k=3}^{n} (-C_{2k}) = C_{22}\,,$$

$$\sum_{k=1}^{3} T_{k3} + \sum_{k=4}^{n} T_{3k} = \sum_{k=1}^{2} T_{k3} + T_{33} + \sum_{k=4}^{n} T_{3k} = \sum_{k=1}^{2} (-C_{k3}) + \sum_{k=1}^{3} C_{k3} + \sum_{k=4}^{n} C_{3k} + \sum_{k=4}^{n} (-C_{3k}) = C_{33},$$

$$\sum_{k=1}^{4} T_{k4} + \sum_{k=5}^{n} T_{4k} = \sum_{k=1}^{3} T_{k4} + T_{44} + \sum_{k=5}^{n} T_{4k} = \sum_{k=1}^{3} (-C_{k4}) + \sum_{k=1}^{4} C_{k4} + \sum_{k=5}^{n} C_{4k} + \sum_{k=5}^{n} (-C_{4k}) = C_{44},$$

$$\sum_{k=1}^{n} T_{kn} = \sum_{k=1}^{n-1} T_{kn} + T_{nn} = \sum_{k=1}^{n-1} (-C_{kn}) + \sum_{k=1}^{n} C_{kn} = C_{nn}.$$

As a result we have:

$$DT = C. \tag{20}$$

Matrix D is non-singular, through mutual subtraction of the rows and column we find, that determinant of the matrix D equals to 1 or –1, depending of value n.

From (7) and (20) results:

$$DT = D^{-1}T = GT,$$

where

$G = D^{-1}$ – inverse matrix of D,

and finally we have

$$G = [G_{rs}]_{N \times N} = D, \tag{21}$$

because all elements of vector T are positive.

This completes the proof of the theorem.

4. Application

The absolute error ΔC_r in calculation of the r-th coordinate C_r of the vector C from measurements of coordinates of the vector T (Fig. 2) has the form [3, 4]:

$$\Delta C_r = \sum_{s=1}^{N} |G_{rs}||\Delta T_s| \tag{22}$$

where

$|\Delta T_s|$ - absolute error at measurement of the s-th element of the vector T,

G_{rs} - element of the matrix G (Eq. (21)),

and r=1, 2, ... , N.

The relative error for coordinates of vector C has the form:

$$\delta_{C_r} = \frac{\Delta C_r}{|C_r|}. \tag{23}$$

From (20) and (21) one finds:

$$C_r = G_r T \tag{24}$$

where

G_r - r-th row of the matrix G.

Using definition

$$\delta_{T_s} = \frac{|\Delta T_s|}{T_s} = \delta, \tag{25}$$

where symbol δ denotes relative error of measurements of the all elements of "two-terminal" matrix T (usual case), from (22), (23), (24), and (25) we have

$$\delta_{C_r} = \frac{\delta}{|G_r T|} \sum_{s=1}^{N} |G_{rs}| T_s. \tag{26}$$

From (12) we can observe, that row D_r (Eq. (10)) of the matrix D has either one column equals to -1 with all remaining columns equal 0, or few columns with all elements equal 1 with all remaining columns equal 0. Consequently basing on the theorem we can write for vector T (Eq. (5)):

$$\frac{\sum_{s=1}^{N} |D_{rs}| T_s}{|D_r T|} = \frac{\sum_{s=1}^{N} |G_{rs}| T_s}{|G_r T|} = 1, \tag{27}$$

for all $r = 1, 2, 3, \ldots N$.

From (26) and (27) we finally get an important dependency:

$$\delta_{C_r} = \delta_{C_{ij}} = \delta,$$

for all $r = 1, 2, \ldots, N$ $(i, j = 1, 2, \ldots, n,$ and $i \leq j)$,

what means, that accuracy of determining the Maxwell matrix is equal to the accuracy of measurement of the elements in the "two-terminal" matrix, when the method of measurements with active separation [5, 3] is used, without additional error propagation.

5. Conclusion

The symmetry of transformations of the Maxwell matrix and the "two-terminal" capacitance matrix plays a crucial role in the minimization of errors occurring in the computed Maxwell matrix from measurements performed by the direct measurement of elements of the "two-terminal" matrix using active separation of the network capacitors, except the measured element of "two-terminal" matrix. The symmetry of transformations assures the same accuracy in computing the elements of Maxwell matrix as the accuracy of direct measurement of capacitance.

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Omówienie

W pracy sformułowano twierdzenie i przedstawiono dowód o symetrii przekształcenia macierzy Maxwella i pojemnościowej macierzy dwójnikowej, opisujących parametry pojemnościowe wielolinii transmisyjnych, realizujących połączenia w planarnych strukturach VLSI. Przekształcenie macierzy jest symetryczne, co wyraża się tym, że macierz przekształcenia macierzy dwójnikowej w macierz Maxwella jest równa jej macierzy odwrotnej. Konsekwencją tej właściwości przekształcenia jest to, że obliczenie macierzy Maxwella na podstawie zrealizowanych pomiarów pojemności, występujących w macierzy dwójnikowej, jest obarczone tylko błędem równym błędowi pomiaru pojemności reprezentujących elementy macierzy dwójnikowej. Zastosowana metoda aktywnej separacji [5] przy pomiarach bezpośrednich pojemności, występujących w macierzy dwójnikowej łącznie z przedstawionymi właściwościami przekształcenia, pozwala znaleźć elementy macierzy Maxwella z minimalnym błędem równym błędowi pomiaru elementów macierzy dwójnikowej [3], bez propagacji błędu pomiarowego, czego nie można uniknąć przy pomiarach pośrednich [4], a co w konsekwencji wymaga ekstremalnie dokładnych pomiarów i bardzo kosztownych stanowisk pomiarowych niezbędnych przy pomiarach pośrednich.