ACTA PHYSICA POLONICA

VOL. IX, FASC. 2—4

KRAKÓW 1947—1948

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First reprint by PWN — Polish Scientific Publishers 1969

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WYDANE Z ZASIŁKU WYDZIAŁU NAUKI MINISTERSTWA OŚWIATY

J. K. LUBAŃSKI

Obituary notice by L. ROSENFELD, Manchester.

The circle of Lubański's friends was not large. During the wartime which was so difficult for him he lived very retired and he was rather shy and taciturn. But those who were in closer contact with him have been able to discover his sensitive and refined personality. After the liberation a striking change came over him, that made us only then understand how much he had suffered during the war. He surprised us by an unknown alacrity and optimism. His new task in Delft gave him great satisfaction and he fulfilled it with an enthusiasm and energy that authorised the greatest expectations for his scientific and personal future. This made the shock the more violent for his friends when the news reached them of his unexpected death on the 8-th December 1946, after only a very short illness.

Jozeph Kazimir Lubański was born in 1914 in Rumania from Polish parents. He spent his youth in Russia; only in 1926 did he come to Poland where he studied Physics at the Universities of Wilno and Kraków. In Kraków he worked under the direction of the very original theoretician Mathisson (who died in England during the war); his first paper in 1937 is based on Mathisson's theories. Until the autumn of 1938 Lubański was assistant at the Institute of Theoretical Physics at Wilno. In December 1938 he came to Leiden with a stipend from the Polish Government to work under the direction of Professor Kramers. Since that time he lived in Holland, where he was greatly helped during the war by the Lorentz Fund. As a Polish citizen he was forced already in 1940 to leave the coastal region and after a short stay in the country-side he settled in Utrecht, where he stayed until his appointment at Delft in October 1945.

In Leiden he worked with Kramers and Belinfante on the Theory of Particles with arbitrary spin. His investigations on this subject are set-out in three papers, published in the Dutch journal Physica. These papers witness his profound knowledge of the abstract theories of modern algebra and his mastery in applying them to fundamental problems of theoretical physics. The same qualities characterize his further publications which contained the results of the work he did in Utrecht.

He was for me a true and inspiring collaborator. His answers were always considered and to the point. His remarks revealed refined understanding of the problem under discussion. Our conversations were not limited to physics, and I must always admire his many-sided interests and the originality of his judgement. He was a great lover of French literature.

Thanks to his allround knowledge of mathematics and physics he could adapt himself without difficulty to his new function as assistant to Professor J. M. B u r g e r s at the Laboratory for Aero and Hydrodynamics at Delft. To this fascinating domain of theoretical and experimental research he devoted his whole energy. In November 1946 he had been promoted Chief Assistant.

After a long separation he had just renewed contact with his family in Poland and he was considering the possibility of offering his talent to the service of his country.

For us who had learned to know and love him his untimely death is a very heavy loss indeed. He continues to live in our memory not only as a distinguished physicist, but also as a modest and good man.

Papers published by late J. K. Lubański

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KONSTANTY ZAKRZEWSKI

1876—1948

Obituary notice by Tadeusz PIECH, Physical Laboratory of the Mining Academy, Kraków.

Professor Konstanty Zakrzewski was born in Warsaw, on the 14-th of January 1876, where he also received his primary education. His secondary studies at the State School Nr. 5 were interrupted in 1893 by his being arrested for belonging to secret youth organizations. After having spent a short time in the Warsaw prison of Pawiak and in the fortress of St. Petersburg, Zakrzewski was set free with a so-called «wolf-ticket«, depriving him of the right of studying in any educational establishment of the Russian Empire; he was sentenced to a forced sojourn in the country, in the province of Kielce. Wishing with all his heart to continue his studies, he crosses illegally the frontier of Galicia (Austria) and finishes, in 1895, his secondary studies in Lwów. In the same year he enlists as undergraduate at the Faculty of Philosophy of the Kraków University, choosing physics as his speciality. His devotion to his task drew the attention of prof. Witkowski, a prominent pedagogue, who entrusted him with the functions of demonstrator. His studies at the Kraków University terminated by a doctors degree obtained for his study «On the electromotive force generated by the motion of a liquid in a silver-plated tube». Zakrzewski continues his work abroad, at the Göttingen University with Prof. Voigt and Riecke — later on at Leyden with Prof. Kamerlingh-Onnes. He stays at Leyden for two years, during which he is also assistant at the laboratory. On his return home, in 1904, he is appointed assistant at the Institute of Physics of the Kraków University. In 1908 he becomes lecturer and in 1911 professor extraordinary of experimental physics. He leaves Kraków for a couple of years in

1913 and becómes professor of theoretical physics at the University of Lwów. After the sudden death of Prof. Smoluchowski, in 1917, the chair of experimental physics being vacant, the Senate of the Kraków University nominates Zakrzewski for this post and entrusts him with the directorship of the Institute of Physics, which had already, thanks to the works of such remarkable individualites as Prof. Wróblewski, Witkowski and Smoluchowski, a considerable reputation in the international scientific world.

Owing to his research work, chiefly on optics, Zakrzewski got to be known as one of the most prominent Polish experimental physicists. At that period the electron theory of metals was flourishing, and it was most important for proving its assumptions to measure experimentally the dispersion of the refractive index and the extinction coefficient of metals. Zakrzewski undertook that difficult work, carried out simultaneously in different physical laboratories, with his habitual conscientiousness. He sacrificed much of his time in getting familiar with these difficult measurements and constructed at the same time a half-shadow elliptical analyser, known in scientific literature as the analyser of Zakrzewski. This instrument, which serves to measure the elliptical elements of light reflected from metallic mirrors, is based on the half-shadow principle and is therefore more precise than all other similar instruments formerly used, set on the maximum of darkness of the field of vision. Zakrzewski's analyser has the advantage of not needing a calibration required by the others. In his following research work Prof. Zakrzewski measures the dispersion of optical constants of platinum, graphite, nickel and zinc. The rich experimental material accumulated by that time enables him to take a critical attitude towards the basic principles of the electron theory of metals then current; he foresaw the necessity of a revision of its foundations and of applying, while so doing, Planck's quantum theory. This was accomplished, many years later, by Sommerfeld.

This successful period of his life was interrupted by his departure for Lwów and by the events of World War I. On coming back to Kraków, this time definitively, Zakrzewski endeavours to create his own «school», so indispensable in the Polish conditions of that time. The foundation of such a school at that period was most important because of the arrears in Polish scientific culture, due to the years of occupation by foreign powers, which had to be made up. Disposing of extremely modest means, he organizes a most important research work on dielectric constants, known and appreciated in Poland and abroad — it concentrated a large number of young scientists.

In this laboratory were started and carried through many investigations on the dielectric behaviour of liquids and solids, which enabled him to discover the up till now unexplained phenomenon of a «dipole-like» behaviour of some elements of the chlorine group. A new type of condenser was then constructed, generally in use nowadays, in which the substance under examination is separated from the electrodes by a dielectric layer; this renders possible the testing of bodies which attack the plates of the condenser. Besides the above mentioned problems, questions concerning the absorption of electromagnetic waves, diffraction of electrons, and viscosity were examined. Zakrzewski besides guiding the research work of his pupils pursued his investigations on the Kerr effect and the production and absorption of electromagnetic waves.

Prof. Zakrzewski's pedagogical activity did not consist entirely of his functions as lecturer and manager of the physical laboratory. The Polish scientific literature is indebted to him for several excellent reference books, which have not been surpassed till now. The whole contemporary generation of Polish physicists and naturalists was trained on Zakrzewski's manual «Elements of Physics» written in collaboration with A. Witkowski and reedited five times. His book on radioactivity for the use of university students, was, up till recent times, the only detailed reference book in Polish language on this topic; the manual of physics for secondary schools written in collaboration with W. Natanson contrasts largely with the customary standards. The numerous publications which he issued in different scientific periodicals attest not only of his talent in the way of expressing himself, but also of his literary abilities. The Polish Academy of Science conferred upon him in recognition of his scientific merits the title of correspondent member in 1920 and that of active member in 1930. He was decorated, in 1932, with the Commander Cross of the Polish order of «Polonia Restituta».

The part he played in the organization of Polish research work in physics forms a new chapter of his activity. He put all his soul and energy at the service of every endeavour in that domain. Before World War I he was for many years an active member of the Board of the Polish Copernicus Society of Naturalists; later on, during the re-organization of the Polish State, he helped to build up, as one of its leading members, the Polish Physical Society — he stood, for some time, at the head of the Kraków Section of that Society. He was one of the promoters of the idea of organizing a Commission of the Stations of Research on Cosmic Rays, attached to the Polish Academy of Science. In 1947, after its formation, he became its first chairman.

Zakrzewski's main feature was a great modesty, which led him to refuse any University honours or honorific functions. Once only, in 1927/28, he consented to accept the position of Dean of the Philosophical Faculty. In that capacity he is remembered by all as a man of great tact and justice, who had a way of treating official questions very leniently, with a heart full of understanding. His relations with his colleagues, co-workers, and pupils were friendly and excellent, his manners most winning. All those who have met him will never forget his personal charm, which claimed their sympathy.

During the German occupation, in spite of tragic personal experiences, he was always full of optimism and hope for the future. At the end of the war he started rebuilding the devastated Kraków Institute, putting all his energy at the service of that work, which he continued until the very last days of his life. He died suddenly, in the midst of his activities, on January 19, 1948.

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We regret to announce the following deaths of members of the Polish Physical Society:

Prof. Konstanty Zakrzewski, professor of experimental physicks at the Jagellonian University, Kraków, on January 19, 1948.

Prof. Jan Blaton, professor of theoretical mechanics at the Jagellonian University, Kraków, on Mai 17, 1948.

RELATIVISTIC INTENSITIES OF MULTIPOLE RADIATION IN THE LYMAN SERIES

By Jan RZEWUSKI, Institute of Theoretical Physics, University of Lublin.

(Received March 15, 1946).

In this paper the intensities of multipole radiation in the Lyman series are calculated on the basis of Dirac's relativistic theory of the electron. In the non-relativistic case the corresponding calculations were carried out by Rubinowicz. It was possible to work out the selection rules, the Zeeman effect, and the sum rules for the radiation considered. In the particular case of magnetic dipole and electric quadrupole radiation our formulae are in agreement with the formulae of Rubinowicz. Finally, an example is discussed: the probability is calculated of an inversion of the spin vector under the influence of a magnetic field, assuming that the electron is in the state $n = 1$.

§ 1. Introduction

In this paper we shall calculate the intensities of multipole radiation in the Lyman series. The calculation is based on the relativistic theory of the electron and on the theory of multipole radiation as given by J. Blaton *(1, 1937)*.

In a one-electron relativistic problem the state of an atom is given by three quantum numbers: n, k, m_j. (*2, 1939*, p. 28 and following); n is the main quantum number with possible values: 1, 2, 3,... etc. It is connected with the radial quantum number and with k, as follows

$$n = n_r + |k| . \tag{1}$$

The possible values of the quantum number k are: 1, ± 2, ± 3, $\ldots\ldots \pm (n-1)$, $\pm n$; k and the rotational quantum number l satisfy the equations

$$\text{for } k > 0 \quad k = l + 1, \quad \text{for } k < 0 \quad k = -l , \tag{2}$$

l corresponds to the orbital angular momentum of the electron, whereas the quantum number j corresponds to the total angular momentum of the electron. We have

$$j = |k| - \frac{1}{2}, \tag{3}$$

m_j is the projection of $\vec{j}$ on a certain direction and can vary within the limits

$$-j \leqslant m_j \leqslant j. \tag{4}$$

We have the vector equation: $\vec{j} = \vec{l} + \vec{s}$ and the algebraic equation: $m_j = m_l + m_s$, where m_j, m_l, m_s, are the projections of $\vec{j}$, $\vec{l}$, $\vec{s}$ respectively on a certain direction. $\vec{s}$ is the spin of the electron. m_s may in a one electron problem take the values $\pm\frac{1}{2}$.

The relativistic one-electron problem is fully described by Dirac's equations

$$\begin{aligned}
&\frac{i}{h}\left(\frac{E+eV}{c}+m_0c\right)\psi_1+\left(\frac{\partial}{\partial x}-i\frac{\partial}{\partial y}\right)\psi_4+\frac{\partial}{\partial z}\psi_3=0\\
&\frac{i}{h}\left(\frac{E+eV}{c}+m_0c\right)\psi_2+\left(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\right)\psi_3-\frac{\partial}{\partial z}\psi_4=0\\
&\frac{i}{h}\left(\frac{E+eV}{c}-m_0c\right)\psi_3+\left(\frac{\partial}{\partial x}-i\frac{\partial}{\partial y}\right)\psi_2+\frac{\partial}{\partial z}\psi_1=0\\
&\frac{i}{h}\left(\frac{E+eV}{c}-m_0c\right)\psi_4+\left(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\right)\psi_1-\frac{\partial}{\partial z}\psi_2=0,
\end{aligned} \tag{5}$$

where $i=\sqrt{-1}$; h is Planck's constant divided by 2π, E is the total energy of the system in the relativistic sense, e is the electric charge of the electron, c is the velocity of light, $-eV$ is the potential energy of the electron.

Equations (5) have two solutions which we shall denote with (a) and (b). Solution (a) corresponds to $k>0$ $(k=l+1)$, solution (b) to $k<0$ $(k=-l)$. These solutions are

$$\begin{array}{cc}
\text{(a)} & \text{(b)}\\
\left.\begin{aligned}
\psi_1&=-i\,\Phi^m_{l+1}R_1\\
\psi_2&=-i\,\Phi^{m+1}_{l+1}R_1\\
\psi_3&=(l+m+1)\,\Phi^m_l R_3\\
\psi_4&=-(l-m)\,\Phi^{m+1}_l R_3
\end{aligned}\right| e^{\frac{iE}{h}t}, &
\left.\begin{aligned}
\psi_1&=-i(l+m)\,\Phi^m_{l-1}R'_1\\
\psi_2&=i(l-m-1)\,\Phi^{m+1}_{l-1}R'_1\\
\psi_3&=\Phi^m_e R'_3\\
\psi_4&=\Phi^{m+1}_l R'_3
\end{aligned}\right| e^{\frac{iE}{h}t}.
\end{array} \tag{6}$$

The number m in (6) is connected to m_j by the equation

$$m+\frac{1}{2}=m_j \tag{7}$$

Φ_ν^μ is a spherical harmonic function according to Darwin's definition

$$\Phi_\nu^\mu=(\nu-\mu)!\,(1-\cos^2\vartheta)^{\frac{\mu}{2}}\left(\frac{d}{d\cos\vartheta}\right)^{\nu+\mu}\frac{(\cos^2\vartheta-1)^\nu}{2^\nu\,\nu!}e^{i\mu\varphi};\quad -\nu\leqslant\mu\leqslant+\nu. \tag{8}$$

R_1, R_3, and R'_1, R'_3 satisfy the equations (*3*, p. 48)

$$\begin{aligned}-\frac{1}{\hbar}\left[\frac{E+eV}{c}-m_0c\right]R_3+\left(\frac{d}{dr}+\frac{l+2}{r}\right)R_1=0\\ \frac{1}{\hbar}\left[\frac{E+eV}{c}+m_0c\right]R_1+\left(\frac{d}{dr}-\frac{l}{r}\right)R_3=0\end{aligned} \tag{9}$$

$$\begin{aligned}-\frac{1}{\hbar}\left[\frac{E+eV}{c}-m_0c\right]R'_3+\left(\frac{d}{dr}-\frac{l-1}{r}\right)R'_1=0\\ \frac{1}{\hbar}\left[\frac{E+eV}{c}+m_0c\right]R'_1+\left(\frac{d}{dr}+\frac{l+1}{r}\right)R'_3=0,\end{aligned} \tag{10}$$

Equations (9) correspond to the solutions (a) and therefore to $k=l+1$, equations (10) to the solutions (b) and therefore to $k=-l$. By putting $k-1$ instead of l in case (a) and $-k$ in case (b), we get identical equations for both cases

$$\begin{aligned}-\frac{1}{\hbar}\left[\frac{E+eV}{c}-m_0c\right]R_3+\left(\frac{d}{dr}+\frac{k+1}{r}\right)R_1=0\\ \frac{1}{\hbar}\left[\frac{E+eV}{c}+m_0c\right]R_1+\left(\frac{d}{dr}-\frac{k-1}{r}\right)R_3=0.\end{aligned} \tag{11}$$

The density of the electric charge and the electric current are given by (*3*, p. 458).

$$\rho=e\{\bar\psi_1\psi'_1+\bar\psi_2\psi'_2+\bar\psi_3\psi'_3+\bar\psi_4\psi'_4\}+\text{conj.} \tag{12}$$

$$\begin{aligned}j_x&=-ce\{\bar\psi_1\psi'_4+\bar\psi_2\psi'_3+\bar\psi_3\psi'_2+\bar\psi_4\psi'_1\}+\text{conj.}\\ j_y&=-ice\{-\bar\psi_1\psi'_4+\bar\psi_2\psi'_3-\bar\psi_3\psi'_2+\bar\psi_4\psi'_1\}+\text{conj.}\\ j_z&=-ce\{\bar\psi_1\psi'_3-\bar\psi_2\psi'_4+\bar\psi_3\psi_1-\bar\psi_4\psi'_2\}+\text{conj.},\end{aligned} \tag{13}$$

where ψ_l are solutions corresponding to the stationary state with the quantum numbers n, k, m, and ψ_l' solutions corresponding to the stationary state with the quantum numbers n', k', m'. The addition of the conjugate imaginary value in (12) and (13) is equivalent to multiplication by 2 (*2*, *1939*, p. 58). We shall therefore neglect this part of the sum and multiply our results by 2.

The total intensity of the electric and magnetic radiation respectively is given by

$$W^{el} = \sum_{\nu=1}^{\infty} \sum_{\mu=-\nu}^{+\nu} W_{\nu\mu}^{el} \tag{14}$$

$$W^{ma} = \sum_{\nu=1}^{\infty} \sum_{\mu=-1}^{+\nu} W_{\nu\mu}^{ma} \tag{15}$$

(cf. *1, 1937*), where $W_{\nu\mu}^{el}$ is the intensity of the μ-th component of the electric multipole radiation of the order 2^{ν} and $W_{\nu\mu}^{ma}$ is the intensity of the μ-th component of the magnetic multipole radiation of the order 2^{ν}. These intensities are

$$W_{\nu\mu}^{el} = 2ck'^2 \frac{\nu(\nu+1)}{2\nu+1} (\nu+\mu)!\,(\nu-\mu)!\,|p_{\nu\mu}|^2 \tag{16}$$

$$W_{\nu\mu}^{ma} = 2ck'^2 \frac{\nu(\nu+1)}{2\nu+1} (\nu+\mu)!\,(\nu-\mu)!\,|q_{\nu\mu}|^2, \tag{17}$$

where k' is the wave number of the emitted radiation ($k = \frac{2\pi\nu}{c}$, ν = frequency of the radiation); $p_{\nu\mu}$ and $q_{\nu\mu}$ are given by

$$p_{\nu\mu} = \frac{1}{ik'} \frac{2\nu+1}{(\nu+\mu)!\,(\nu-\mu)!} \frac{1}{\nu(\nu+1)} \left| \int \rho \frac{d\psi_\nu(k'r)}{dr} \bar{\Phi}_\nu^\mu d\tau - \frac{ik'}{c} \int j_r \psi_\nu(k'r) \bar{\Phi}_\nu^\mu d\tau \right| \tag{18}$$

$$q_{\nu\mu} = \frac{1}{2k'} \frac{2\nu+1}{(\nu+\mu)!(\nu-\mu)!} \frac{1}{\nu(\nu+1)} \int \left| \frac{\nu-\mu}{c} j_1 \bar{\Phi}_\nu^{\mu+1} + \frac{\nu+\mu}{c} j_2 \bar{\Phi}_\nu^{\mu-1} - \frac{2\mu}{c} j_3 \bar{\Phi}_\nu^\mu \right| \frac{\psi_\nu(k'r)}{r} d\tau, \tag{19}$$

where

$$\psi_\nu(k'r) = \sum_{x=0}^{\infty} \frac{(-1)^x}{2^x x!} \frac{(k'r)^{\nu+1+2x}}{1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2\nu+2x+1)}, \tag{20}$$

and j_r is the radial component of the current density. Further

$$j_1 = j_x + ij_y; \quad j_2 = j_x - ij_y; \quad j_3 = j_z. \tag{21}$$

It may be easily seen from these formulae that to calculate the intensities (14) and (15) we need only to know $p_{\nu\mu}$ and $q_{\nu\mu}$.

§ 2. Normalization factor

Before beginning the calculation of $p_{\nu\mu}$ and $q_{\nu\mu}$ we must know the normalization factor for the solutions of Dirac's equations. The normalization factor is calculated e. g. in Sommerfeld's *Atombau*

und Spektrallinien (2, *1939*, p. 293). We must only adapt his results to our definitions and notations. In case of the (a) solution, taking into account (1, 6), we get

$$N^2\int(\bar{\psi}_1\psi_1+\bar{\psi}_2\psi_2+\bar{\psi}_3\psi_3+\bar{\psi}_4\psi_4)d\tau=N^2\cdot 4\pi\cdot(l+m+1)!\,(l-m)!\int_0^\infty(R_1^2+R_3^2)\,r^2dr=1$$

and in case of the (b) solutions

$$N'^2\int(\bar{\psi}_1\psi_1+\bar{\psi}_2\psi_2+\bar{\psi}_3\psi_3+\bar{\psi}_4\psi_4)d\tau=N'^2\cdot 4\pi(l+m)!(l-m-1)!\int_0^\infty(R_1^2+R_3^2)r^2dr=1,$$

as may be easily seen from the following orthogonality relations for the $\Phi_{\nu\mu}$-functions

$$\int_0^{2\pi}d\varphi\int_0^{\pi}\sin\vartheta\,d\vartheta\,\Phi_\nu^\mu(\vartheta,\varphi)\bar{\Phi}_{\nu'}^{\mu'}(\vartheta,\varphi)=\delta_{\nu\nu'}\,\delta_{\mu\mu'}\cdot 4\pi\,\frac{(\nu+\mu)!\,(\nu-\mu)!}{2\nu+1}. \quad (1)$$

But in case (a), $l+1=|k|$ and in case (b), $l=|k|$. We get therefore the same result for both cases

$$N^2\cdot 4\pi\cdot(|k|+m)!\,(|k|-m-1)!\int_0^\infty(R_1^2+R_3^2)\,r^2dr=1.$$

Denoting $N^2=N_{\vartheta\varphi}^2\,N_r^2$ and

$$N_{\vartheta\varphi}^2=\frac{1}{4\pi\,(|k|+m)!\,(|k|-m-1)!}, \quad (2)$$

we get finally

$$N^2=\frac{N_r^2}{4\pi\,(|k|+m)!\,(|k|-m-1)!}, \quad (3)$$

where

$$\frac{1}{N_r^2}=\int_0^\infty(R_1^2+R_3^2)\,r^2\,dr.$$

We shall consider the last integral later, when dealing with the radial parts of the solutions. It should be remembered that N_r, as well as $N_{\vartheta\varphi}$, depend on k. We obtain the normalization factor for the (a) and (b) solutions by putting $k=l+1$ or $k=-l$ respectively.

§ 3. Calculation of $p_{\nu\mu}$ and $q_{\nu\mu}$. The angular parts of the integrals

Let us pass now to the calculation of $p_{\nu\mu}$ and $q_{\nu\mu}$ for transitions in the Lyman series. Here the lowest state is characterized by the quantum number $n=1$. From the remarks in § 1 we infer that k admits the only value $+1$. Therefore only the solution (a) is

admissible in the lowest state. For this state $l=0$, $j={}^1/_2$, $m_j=\pm{}^1/_2$, and according to (1, 7) $m=-1$ or $m=0$. Thus we obtain four possible types of transitions: $a(n,l,m)\to a(1,0,0)$, $b(n,l,m)\to a(1,0,0)$, $a(n,l,m)\to a(1,0,-1)$ and $b(n,l,m)\to a(1,0,-1)$. We put now the corresponding solutions of Dirac's equations in (1, 18) and (1, 19) (cf. (1, 6)), carry out the integration over the angles, using (2, 1) and the following relations

$$\begin{aligned}
&\bar{\Phi}^{\mu}_{\nu}\bar{\Phi}^{0}_{0}=\bar{\Phi}^{\mu}_{\nu}, \qquad \bar{\Phi}^{\mu}_{\nu}\bar{\Phi}^{0}_{1}=\frac{1}{2\nu+1}\left|\bar{\Phi}^{\mu}_{\nu+1}+(\nu-\mu)(\nu+\mu)\bar{\Phi}^{\mu}_{\nu-1}\right|,\\
&\bar{\Phi}^{\mu}_{\nu}\bar{\Phi}^{1}_{1}=\frac{1}{2\nu+1}\left|\bar{\Phi}^{\mu+1}_{\nu+1}-(\nu-\mu)(\nu-\mu-1)\bar{\Phi}^{\mu+1}_{\nu+1}\right|,
\end{aligned} \tag{1}$$

and insert finally the results in (1, 16) and (1, 17). The simple but rather tedious calculations yield the following selection and polarization rules:

(1) Electric multipole radiation

For the transition $a(n,l,m)\to a(1,0,0)$ there exists only the component $\mu=m$ of the multipole radiation of the order 2^{ν}, where $\nu=l$. The corresponding intensity is

$$W^{el}_{lm}\Big|^{a}_{0}=8ce^2\frac{l+m+1}{l(l+1)(2l+1)}\left|N^{a}_{r}N^{0}_{r}\sum_{x=0}^{\infty}\frac{(-1)^x}{2^x x!}\frac{k'^{l+1+2x}}{1\cdot3\cdots(2l+2x+1)}\right.$$
$$\left.[(l+1+2x)(I_1+I_2)-k'(I_3-I_4)]\right|^2. \tag{2}$$

For the transition $b(n,l,m)\to a(1,0,0)$ we get the same component and the same order as in (2). The intensity is

$$W^{el}_{lm}\Big|^{b}_{0}=8ce^2\frac{l-m}{l(l+1)(2l+1)}\left|N^{b}_{r}N^{0}_{r}\sum_{x=0}^{\infty}\frac{(-1)^x}{2^x x!}\frac{k^{l+1+2x}}{1\cdot3\cdots(2l+2x+1)}\right.$$
$$\left.[(l+1+2x)(I'_1+I'_2)-(k'(I'_3-I'_4)]\right|^2. \tag{3}$$

For the two other transitions $(n,l,m)\to a(1,0,-1)$ and $b(n,l,m)\to$ $\to a(1,0,-1)$ there appears only the component $\mu=m+1$ of the multipole radiation of the order 2 with $\nu=1$. The corresponding intensities are

$$W^{el}_{lm+1}\Big|^{a}_{-1}=8ce^2\frac{l-m}{l(l+1)(2l+1)}\left|N^{a}_{r}N^{0}_{r}\sum_{x=0}^{\infty}\frac{(-1)^x}{2^x x!}\frac{k'^{l+1+2x}}{1\cdot3\cdots(2l+2x+1)}\right.$$
$$\left.[(l+1+2x)(I_1+I_2)-k(I_3-I_4)]\right|^2 \tag{4}$$

and

$$W^{el}_{lm+1}\Big|^{b}_{-1} = 8ce^2\frac{l+m+1}{l(l+1)(2l+1)}\left|N^b_r N^0_r \sum_{x=0}^{\infty}\frac{(-1)^x}{2^x x!}\frac{k'^{l+1+2x}}{1\cdot 3\cdots(2l+2x+1)}\right.$$
$$\left.[(l+1+2x)(I'_1+I'_2)-k'(I'_3-I'_4)]\right|^2 \tag{5}$$

respectively.

In the above formulae, the indices «0» and «—1» on W indicate the value of m in the final state. The notations «a» and «b» denote transitions from states of type (a) or (b) respectively. «el» denotes «electric». The numbers l and m or m + 1 give the order and the component of the radiation. N^a_r and N^b_r denote the normalization factors belonging to the states a(n, l, m) and b(n, l, m) respectively. N^a_r is the common normalization factor for both final states a(1, 0, 0) and a(1, 0, — 1). The I's are the following integrals over the radial parts of the solutions

$$I_1+I_2=\int_0^\infty (R_1R^0_1+R_3R^0_3)r^{\nu+2x+2}dr\,; \qquad I_3-I_4=\int_0^\infty (R^0_1R_3-R^0_3R_1)r^{\nu+2x+3}dr\,;$$

$$I'_1+I'_2=\int_0^\infty R'_1R^0_3+R'_3R^0_3)r^{\nu+2x+2}dr\,; \qquad I'_3-I'_4=\int_0^\infty (R^0_1R'_3-R^0_3R'_1)r^{\nu+2x+3}dr.$$

(2) *Magnetic multipole radiation*

For the transition a(n, l, m) → a(1, 0, 0) there exists only the component μ = m of the radiation of the order 2^ν with ν = l + 1. The corresponding intensity is

$$W^{ma}_{l+1m}\Big|^{a}_{0}=8ce^2\frac{l+2}{l+1}\cdot\frac{l-m+1}{2l+3}\left|N^a_r N^0_r\sum_{x=0}^{\infty}\frac{(-1)^x}{2^x x!}\frac{k'^{l+2+2x}}{1\cdot 3\cdots(2l+2x+3)}[I_5+I_6]\right|^2. \tag{6}$$

For the transition b(n, l, m) → a(1, 0, 0) there appears only the component μ = m of the radiation of the order 2^ν, where ν = l — 1, with the intensity

$$W^{ma}_{l-1,m}\Big|^{b}_{0}=8ce^2\frac{l-1}{l}\cdot\frac{l+m}{2l-1}\left|N^b_r N^0_r\sum_{x=0}^{\infty}\frac{(-1)^x}{2^x x!}\frac{k'^{l+2x}}{1\cdot 3\cdots(2l+2x-1)}[I'_5+I'_6]\right|^2. \tag{7}$$

For the transition a(n, l, m) → a(1, 0, — 1) there appears only the component μ = m + 1 of the radiation of the order 2^ν, where ν = l + 1, with the intensity

$$W^{ma}_{l+1\ m+1}\Big|^{a}_{-1}=8ce^2\frac{l+2}{l+1}\cdot\frac{l+m+2}{2l+3}\left|N^a_r N^0_r\sum_{x=0}^{\infty}\frac{(-1)^x}{2^x x!}\frac{k'^{l+2+2x}}{1\cdot 3\cdots(2l+2x+3)}[I_5+I_6]\right|^2. \tag{8}$$

Finally, in the transition $b(n,l,m) \to a(1,0,-1)$ there exists only the component $\mu = m+1$ of the radiation of the order 2^ν with $\nu = l-1$. The corresponding intensity is

$$W^{ma}_{l-1,\,m+1}\Big|^{b}_{-1} = 8ce^2\frac{l-1}{l}\cdot\frac{l-m-1}{2l-1}\left|N^b_r N^0_r \sum_{x=0}^{\infty}\frac{(-1)^x}{2^x x!}\,\frac{k'^{l+2x}}{1\cdot 3\cdots(2l+2x-1)}[I'_5+I'_6]\right|^2. \quad (9)$$

The indices on W and N have the same meaning as for electric radiation; «ma» denotes «magnetic». The I-s are the following integrals over the radial parts of the solutions:

$$I_5+I_6=\int_0^\infty (R^0_1 R_3+R^0_3 R_1)r^{\nu+2x+2}dr\,; \qquad I'_5+I'_6=\int_0^\infty (R^0_1 R'_3+R^0_3 R'_1)r^{\nu+2x+2}dr.$$

Summing up, we come to the following results:

(1) For electric radiation

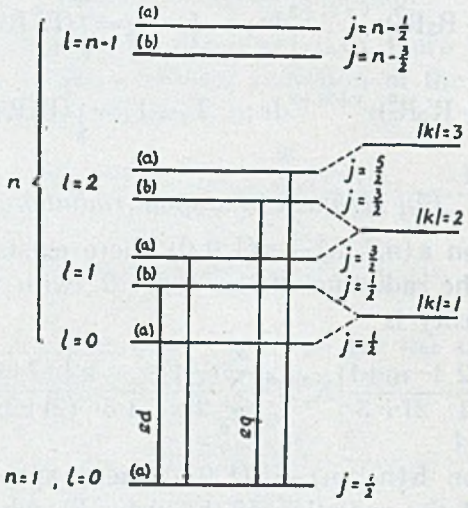

Fig. 1

To every state n, l, m corresponds a multipole radiation of the order 2^l. The order is independent of the value of k, which may be either $l+1$ or $-l$. For $l=0$ there is no radiation at all. The corresponding transition $l=0 \to l=0$ is forbidden. For $l=1$ we have simple dipole radiation from both states (a) and (b), (both possible values for k). For $l=2$ we have quadrupole radiation, etc.

In Fig. 1 the splitting of the terms in a magnetic field is not marked. We shall consider this question later. Here it may be observed that the transition from any state m_1 to the state $m_2=0$ is connected

with the component $\mu = m_1$ of the radiation and the transition from any state m_1 to the state $m_2 = -1$ is connected with the component $\mu = m+1$ of the radiation. The formulae (2), (3), (4) and (5) include therefore the polarization rules of the radiation emitted.

(2) For magnetic radiation

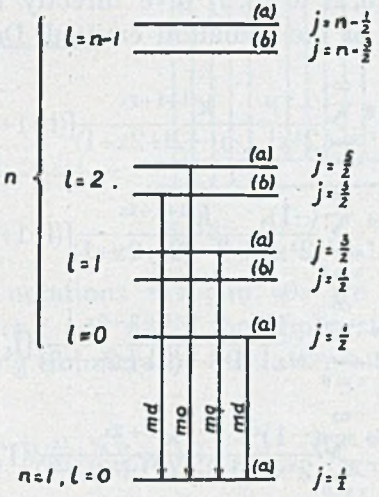

Fig. 2

The transition from an arbitrary state n, l, m to the state n = 1 is accompanied by a magnetic multipole radiation of the order 2^{l+1} if the transition occurs from the level (a) ($j = l + {}^1/_2$) and of the order 2^{l-1} if it occurs from the level (b) ($j = l - {}^1/_2$). Thus, there is a difference between the magnetic and the electric radiation, the order of the latter depending only on l; the correspondence between the components μ of the radiation and the magnetic quantum number m is the same for both radiations.

Finally, some remarks have to be done concerning Fig. 1. All energies with the same $|k|$ and n_r are identical, as follows from (2, *1939*), p. 279, formula (50):

$$\frac{E}{m_0 c^2} = \left| 1 + \frac{\alpha^2 Z^2}{(n_r + \sqrt{k^2 - \alpha^2 Z^2})^2} \right|^{-\frac{1}{2}}. \tag{10}$$

The real levels are drawn on the right. The same remark concerns the next figures.

It is important to emphasize that all our considerations up to now are valid also for an arbitrary potential function depending only

on r: $V = V(r)$. Only when calculating the radial parts of the integrals, we shall have to assume a special dependence of V on r, namely $V = Ze/r$.

§ 4. Intensities of the Zeeman components. Sum rules

The formulae (3,2) to (3,9) give directly the intensities of the Zeeman components of the radiation emitted. Denoting

$$C = \frac{8ce^2}{l(l+1)(2l+1)}\left|N_r^a N_r^0 \sum_{x=0}^{\infty} \frac{(-1)^x}{2^x x!} \frac{k'^{l+1+2x}}{1\cdot 3\cdots(2l+2x+1)}[(l+1+2x)(I_1+I_2)-k'(I_3-I_4)]\right|^2 \quad (1)$$

$$C' = \frac{8ce^2}{l(l+1)(2l+1)}\left|N_r^b N_r^0 \sum_{x=0}^{\infty} \frac{(-1)_x}{2^x x!} \frac{k'^{l+1+2x}}{1\cdot 3\cdots(2l+2x+1)}[(l+1+2x)(I'_1+I'_2)-k'(I'_3-I'_4)]\right|^2 \quad (2)$$

$$C_1 = \frac{8ce^2(l+2)}{(l+1)(2l+3)}\left|N_r^a N_r^0 \sum_{x=0}^{\infty} \frac{(-1)^x}{2^x x!} \frac{k'^{l+2+2x}}{1\cdot 3\cdots(2l+2x+3)}[I_5+I_6]\right|^2 \quad (3)$$

$$C'_1 = \frac{8ce^2(l-1)}{l(2l-1)}\left|N_r^b N_r^0 \sum_{x=0}^{\infty} \frac{(-1)^x}{2^x x!} \frac{k'^{l+2x}}{1\cdot 3\cdots(2l+2x-1)}[I'_5+I'_6]\right|^2, \quad (4)$$

we obtain

$$W^{el}_{lm}\Big|^a_0 = (l+m+1)C \qquad W^{el}_{lm}\Big|^b_0 = (l-m)C' \quad (5)$$

$$W^{el}_{l,m+1}\Big|^a_{-1} = (l-m)C \qquad W^{el}_{l,m+1}\Big|^b_{-1} = (l+m+1)C' \quad (6)$$

$$W^{ma}_{l+1,m}\Big|^a_0 = (l-m+1)C_1 \qquad W^{ma}_{l-1,m}\Big|^b_0 = (l+m)C'_1 \quad (7)$$

$$W^{ma}_{l+1,m+1}\Big|^a_{-1} = (l+m+2)C_1 \qquad W^{ma}_{l-1,m+1}\Big|^b_{-1} = (l-m-1)C'_1, \quad (8)$$

where the C-s do not depend on m. Thus the formulae (5), (6), (7) and (8) give the dependence of the intensities on m.

As an example we shall calculate the special case of the electric quadrupole radiation ($\nu=2$) for transitions from the energy levels (a). We know that transitions may occur only from the energy levels with $l=2$, $j=5/2$. In a magnetic field, according to (1,4) the energy level splits into $2j+1=6$ levels (cf. Fig. 3).

From the first of the formulae (5) we have

$$W^{el}_{2m}\Big|^a_0 = (m+3)C \quad (9)$$

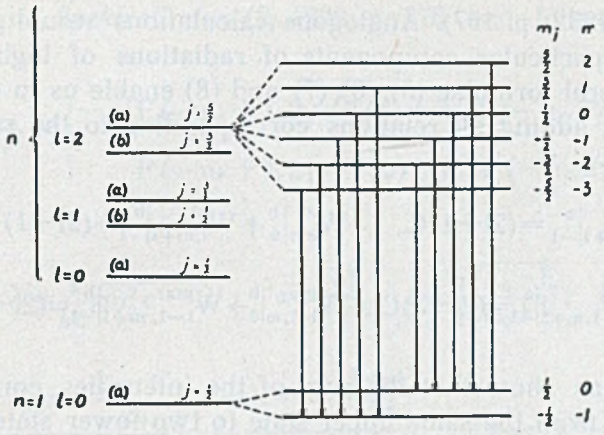

Fig. 3

Instead of the notations «a» and «0» we will introduce now the quantum numbers j and m_j of the upper state as upper indices and the corresponding numbers of the lower state as lower indices. The intensities are

$$W^{el}_{22}\left|{}^{5/2\ 5/2}_{1/2\ 1/2}\right. = 5C, \qquad W^{el}_{21}\left|{}^{5/2\ 3/2}_{1/2\ 1/2}\right. = 1C, \qquad W^{el}_{20}\left|{}^{5/2\ 1/2}_{1/2\ 1/2}\right. = 3C,$$
$$W^{el}_{2-1}\left|{}^{5/2-1/2}_{1/2\ 1/2}\right. = 2C, \qquad W^{el}_{2-2}\left|{}^{5/2-3/2}_{1/2\ 1/2}\right. = C, \qquad W^{el}_{2-3}\left|{}^{5/2-5/2}_{1/2\ 1/2}\right. = 0, \tag{10}$$

in conformity with the limitation of m within the range $-l \leqslant m \leqslant +l$. This limitation can be deduced from the fact that the radiation of the order 2^ν has $2\nu+1$ components only, because of the restriction $-\nu \leqslant \mu \leqslant +\nu$ (cf. *I*, *1937*).

We have considered the transition to the level $m = {}^1/_2$. For the transition to the level $m = -{}^1/_2$ we get from the first of the formulae (6)

$$W^{el}_{2m+1}\left|{}^{a}_{-1}\right. = (2-m)\,C, \tag{11}$$

and for the particular transitions:

$$W^{el}_{22}\left|{}^{5/2\ 3/2}_{1/2-1/2}\right. = C, \qquad W^{el}_{21}\left|{}^{5/2\ 1/2}_{1/2-1/2}\right. = 2C, \qquad W^{el}_{20}\left|{}^{5/2-1/2}_{1/2-1/2}\right. = 3C;$$
$$W^{el}_{2-1}\left|{}^{5/2-3/2}_{1/2-1/2}\right. = 4C, \qquad W^{el}_{2-2}\left|{}^{5/2-5/2}_{1/2-1/2}\right. = 5C. \tag{12}$$

Here the transition $m_j = {}^5/_2 \to m_j = -{}^1/_2$ is forbidden according to the same restrictions as above for the transition $m_j = -{}^5/_2 \to m_j = {}^1/_2$. We thus get the well known selection rules for electric quadrupole radiation: $\triangle m_j \leqslant 2$ (cf. *5*, *1932*, p. 195, formula (41)). The intensities (10) and (12) are in agreement with those calculated by Rubi-

nowicz (*5, 1932*, p. 197). Analogous calculations would give us the intensities of particular components of radiations of higher orders.

The general formulae (5), (6), (7) and (8) enable us to deduce the sum rules. By adding all relations corresponding to the same upper state, we get

$$W^{el}_{lm}\Big|^{a}_{0}+W^{el}_{l,m+1}\Big|^{a}_{-1}=(2l+1)C, \qquad W^{el}_{em}\Big|^{b}_{0}+W^{el}_{l,m+1}\Big|^{b}_{-1}=(2l+1)C', \tag{13}$$

$$W^{ma}_{l+1,m}\Big|^{a}_{0}+W^{ma}_{l+1,m+1}\Big|^{a}_{-1}=(2l+3)C_1, \quad W^{ma}_{l-1,m}\Big|^{b}_{0}+W^{ma}_{l-1,m+1}\Big|^{b}_{-1}=(2l-1)C_1. \tag{14}$$

These formulae show that the sum of the intensities corresponding to transitions from the same upper state to two lower states differing only in m does not depend on the quantum number m of the upper state. Formulae (13) and (14) give directly the total intensities W connected with the transition probabilities A between two unsplit states through the formula $W = Ah\nu$.

For quadrupole radiation the formulae (13) and (14) are particular cases of formulae given by Rubinowicz (*5, 1932*, p. 198).

§ 5. The radial parts of the integrals

The only unknown quantities in the intensities (3,2) to (3, 9) are the integrals I_1, I_2, I_3, I_4, I_5, I_6 and I'_1, I'_2, I_3, I'_4, I'_5, I'_6. The second set does not differ from the first one when l is expressed in terms of k. We shall therefore make use in general of the number k and obtain the first set by putting $k = l+1$ according to the solutions (a) and the second set by putting $k = -l$ according to the solutions (b).

The radial parts of the solutions (a): R_1 and R_3, and of the solutions (b): R'_1 and R_3 satisfy the same differential equations (1, 11). We rewrite here these equations putting $eV = Ze^2/r$ (Ze = electric charge of the nucleus)

$$\begin{aligned}\left(\frac{d}{dr}+\frac{k+1}{r}\right)R_1 &= \frac{1}{\hbar c}\left(E+\frac{Ze^2}{r}-m_0c^2\right)R_3\\ \left(\frac{d}{dr}-\frac{k-1}{r}\right)R_3 &= -\frac{1}{\hbar c}\left(E+\frac{Ze^2}{r}+m_0c^2\right)R_1.\end{aligned} \tag{1}$$

From now on we shall have to do with the special case of a Coulomb potential. These equations are solved for instance in Sommerfelds

Atombau und Spektrallinien (2, *1939*, p. 276 and following). The solutions are

$$\left.\begin{matrix} R_1 \\ R_3 \end{matrix}\right\} = e^{-\frac{\rho}{2}} \rho^{\gamma-1} \left| \begin{matrix} -\varepsilon\,[F(-n_r+1, 2\gamma+1; \rho) + A \cdot F(-n_r, 2\gamma+1; \rho)] \\ F(-n_r+1, 2\gamma+1; \rho) - A \cdot F(-n_r, 2\gamma+1; \rho) \end{matrix} \right|, \quad (2)$$

where

$$\rho = 2\lambda r, \quad \lambda = \frac{1}{hc}\sqrt{m_0^2 c^4 - E^2}, \quad \alpha = \frac{e^2}{hc}, \quad \varepsilon = \sqrt{\frac{E_0 - E}{E_0 + E}}, \quad E_0 = m_0 c^2, \quad (3)$$

$$A = \frac{\gamma + \alpha Z \cdot \frac{1-\varepsilon^2}{2\varepsilon}}{-k + \alpha Z \frac{1+\varepsilon^2}{2\varepsilon}} = \frac{-k - \alpha Z \frac{1+\varepsilon^2}{2\varepsilon}}{\gamma - \alpha Z \frac{1-\varepsilon^2}{2\varepsilon}}, \qquad \gamma = \sqrt{k^2 - \alpha^2 Z^2}, \quad (4)$$

$$n_r + \gamma = \frac{1}{2} \alpha Z \left(\frac{1}{\varepsilon} - \varepsilon \right), \quad (5)$$

$$F(a, c; \rho) = 1 + \frac{a}{c} \cdot \frac{\rho}{1!} + \frac{a(a+1)}{c(c+1)} \cdot \frac{\rho^2}{2!} + \frac{a(a+1)(a+2)}{c(c+1)(c+2)} \cdot \frac{\rho^3}{3!} + \cdots \quad (6)$$

We remind that the solutions (2) correspond respectively to the general solutions (a) and (b) depending on whether $k = l+1$ or $k = -l$. For the lowest state $n = 1$ we have (2, *1939*, p. 290):

$$\left.\begin{matrix} R_1^0 \\ R_3^0 \end{matrix}\right| = e^{-\frac{\rho}{2}} \rho^{\gamma_0 - 1} \left| \begin{matrix} -\varepsilon_0 A_0 \\ -A_0 \end{matrix} \right|, \quad \text{where } \gamma_0 = (\gamma)_{k=1} = \sqrt{1 - \alpha^2 Z^2}. \quad (7)$$

Here the factor A_0 is infinite ((4) and (5)), but when multiplied by the normalization factor it becomes finite. The normalization factor of the radial parts of the solutions was calculated by Bechert (*4*, *1930*). We cannot however use his results in their original form since his definition of the radial parts differs from that accepted in this paper. The normalization factor adapted to our functions is:

$$\frac{1}{N_r^2} = \frac{1}{(2\lambda)^3} \frac{2E_0}{E_0 + E} \binom{2\gamma + n_r - 1}{n_r - 1}^{-2} \frac{\Gamma(2\gamma + n_r) \cdot 2\alpha' \cdot E_0 (k + \alpha' E_0)}{(2\gamma + n_r) \cdot n_r!}. \quad (8)$$

Using (3), (4) and (5) and taking into account that $\alpha' = \alpha Z / \sqrt{E_0^2 - E^2}$ we can give to this factor the form obtained previously by Sommerfeld (2, *1939*, p, 760).

Let us pass now to the calculation of I_1+I_2, I_3-I_4 and I_5+I_6. From (2) and (7) we get

$$I_1+I_2=\int(R_1R_1^0+R_3R_3^0)\,r^{\nu+2x+2}\,dr=$$
$$(2\lambda_0)^{\gamma_0-1}(2\lambda)^{\gamma-1}\int e^{-(\lambda+\lambda_0)r}r^{\gamma+\gamma_0+\nu+2x}A_0\{(\varepsilon\varepsilon_0-1)F(-n_r+1,2\gamma+1;2\lambda r)+$$
$$+(\varepsilon\varepsilon_0+1)\cdot A\cdot F(-n_r,2\gamma+1;2\lambda r)\}dr$$

and similar expressions for I_3-I_4 and I_5+I_6, where the index «0» indicates the lowest state. Using the formula

$$\int_0^\infty e^{-\alpha r}r^p F(-n,c;\beta r)\,dr=\frac{\Gamma(p+1)}{\alpha^{p+1}}F\left(-n,p+1,c;\frac{\beta}{\alpha}\right),\qquad(9)$$

where

$$F(a,b,c;\delta)=1+\frac{a\cdot b}{1\cdot c}\delta+\frac{a(a+1)}{1\cdot 2}\cdot\frac{b(b+1)}{c(c+1)}\cdot\delta^2+\cdots$$

and denoting

$$b=\gamma+\gamma_0+\nu+2x+1;\quad c=2\gamma+1;\quad \delta=\frac{2\lambda}{\lambda+\lambda_0},\qquad(10)$$

we obtain finally

$$I_1+I_2=\frac{(2\lambda)^{\gamma-1}(\lambda_0)^{\gamma_0-1}\Gamma(b)}{(\lambda+\lambda_0)b}A_0\{(\varepsilon\varepsilon_0-1)F(-n_r+1,b,c;\delta)+$$
$$+(\varepsilon\varepsilon_0+1)AF(n_r,b,c;\delta)\},\qquad(11)$$

$$I_3-I_4=\frac{(2\lambda)^{\gamma-1}(2\lambda_0)^{\gamma_0-1}\Gamma(b+1)}{(\lambda+\lambda_0)^{b+1}}A_0\{(\varepsilon_0-\varepsilon)A\cdot F(-n_r,b+1,c;\delta)-$$
$$-(\varepsilon_0+\varepsilon)F(-n_r+1,b+1,c;\delta)\},\qquad(12)$$

$$I_5+I_6=\frac{(2\lambda)^{\gamma-1}(2\lambda_0)^{\gamma_0-1}\Gamma(b)}{(\lambda+\lambda_0)^b}A_0\{(\varepsilon-\varepsilon_0)F(-n_r+1,b,c;\delta)+$$
$$+(\varepsilon+\varepsilon_0)AF(-n_r,b,c;\delta)\}.\qquad(13)$$

If these expressions are to be used in the formulae concerning transitions from a state (a) we must replace everywhere k by $l+1$. Otherwise k has to be replaced by $-l$.

At last, we shall apply our formulae to the transition from the energy level $n=1$, $l=0$, $m=-1$ to the level $n=1$, $l=0$, $m=0$. This is a transition where only the magnetic quantum number changes. As we already know, such a transition is permitted only for

magnetic dipole radiation. In our theory (cf. (3, 10)) states with identical n and |k| belong to the same energy level. Therefore the transition probability must be 0. This follows from $k' = 0$. In a magnetic field, however, an energy level splits into $2j + 1$ levels. In this case for the considered transition k' is very small indeed but does not fall to nought. With this k' we will calculate the intensities.

Our transition is characterized by $\nu = l + 1 = 1$, $\mu = m = -1$, $k = l + 1 = 1$, $n_r = 0$. With these values we get from (11)

$$I_5 + I_6 = A_0^2 \cdot 2\varepsilon_0 \frac{\Gamma(2\gamma_0 + 2x + 2)}{(2\lambda_0)^{2x+4}} \tag{14}$$

and from (8)

$$\left(\frac{1}{N_r^0}\right)^2 = \frac{1}{(2\lambda_0)^3} \frac{2E_0}{E_0 + E} \binom{2\gamma_0 + n_r - 1}{n_r - 1}^{-2}_{n_r = 0} \cdot \frac{1}{2\gamma} \Gamma(2\gamma_0) \frac{2\alpha Z E_0}{\sqrt{E_0^2 - E^2}} \left(k + \frac{\alpha Z E_0}{\sqrt{E_0^2 - E^2}}\right)_{k=1}. \tag{15}$$

It follows from (3), (4) and (5) that

$$An_r = k + \alpha Z \frac{1 + \varepsilon^2}{2\varepsilon}, \qquad 1 + \varepsilon_2 = \frac{2E_0}{E_0 + E}, \qquad \frac{1}{\varepsilon} + \varepsilon = \frac{2E_0}{\sqrt{E_0^2 - E_2}}$$

and

$$\left[\binom{2\gamma_0 + n_r - 1}{n_r - 1}^{-2} n_r^2\right]_{n_r = 0} = (2\gamma_0)^2.$$

Inserting this in (15) we obtain

$$\left(\frac{1}{N_r^0}\right)^2 = \frac{1}{(2\lambda_0)^3}(1 + \varepsilon^2)\, 2\gamma_0\, \Gamma(2\gamma_0) \alpha Z \frac{1 + \varepsilon^2}{\varepsilon} \frac{A_0^2}{1 + \alpha Z \frac{1 + \varepsilon^2}{2\varepsilon}}.$$

Because of (3) and (3, 10) we get for the special case $n_r = 0$, $k = 1$ (cf. 2, *1939*, formula (98))

$$\frac{E}{E_0}\gamma_0, \qquad \frac{1}{\varepsilon} - \varepsilon = \frac{2}{\alpha Z}(n_r + \sqrt{k^2 - \alpha^2 Z^2}),$$

$$\frac{1}{\varepsilon} - \varepsilon = \frac{2}{\alpha Z} \cdot \gamma_0, \qquad \alpha Z \cdot \frac{1 + \varepsilon^2}{\varepsilon} = \alpha Z \frac{1 - \varepsilon^2}{\varepsilon} \cdot \frac{E_0}{E} = 2,$$

and finally

$$\left(\frac{1}{N_r^0}\right)^2 = \frac{A_0^2}{(2\lambda_0)^3}(1 + \varepsilon^2)\, \Gamma(2\gamma_0 + 1). \tag{16}$$

(16) and (14) yield

$$(N_r^0)^2 (I_5 + I_6) = \frac{1}{(2\lambda_0)^{2x+1}} \cdot \frac{2\varepsilon_0}{1 + \varepsilon_0^2} \frac{\Gamma(2\gamma_0 + 2x + 2)}{\Gamma(2\gamma_0 + 1)}. \tag{17}$$

We could have obtained the same results much easier by calculating N_r^0 and $I_5 + I_6$ directly for the particular case (7).

Inserting (17) in (3, 6) and considering that $2\varepsilon_0/(1+\varepsilon_0^2)=\alpha Z$, we obtain

$$W^{\text{ma}}_{1,-1|0}=\frac{8}{27}ce^2k'^2\left(\frac{k'}{K}\right)^2(2\gamma_0+1)^2\left|\sum_{x=0}^{\infty}\frac{(-1)^x}{2^x x!}\frac{3}{1.3...(2x+3)}\left(\frac{k'}{2\lambda_0}\right)^{2x}\frac{\Gamma(2\gamma_0+2x+2)}{\Gamma(2\gamma_0+2)}\right|^2. \quad (18)$$

Let us consider the first term of the series (18) neglecting αZ against 1. We get

$$W=\frac{8}{3}ce^2k'^2\left(\frac{k'}{K}\right)^2. \quad (19)$$

To obtain the probability per unit of time for the transition between the states considered, we must divide (19) by $h\nu$.

Now let us calculate k′ on the basis of the theory of the Zeeman effect

$$k'=\frac{2\pi\nu}{c}=\frac{2\pi}{c}\cdot\frac{eH}{2\pi mc}=\frac{eH}{mc^2};$$

on the other hand $K=mc/h$. We have therefore

$$P=\frac{8}{6\pi}\left(\frac{e}{m}\right)^5\cdot\frac{h}{c^8}\cdot H^3\cong 1,75\,.\,10^{-22}\,.\,H^3$$

The next approximation contains

$$\left(\frac{k'}{2\lambda_0}\right)^2=\left(\frac{k'}{K}\right)^2\frac{1}{4\alpha^2Z^2}=\left(\frac{h}{2\pi mc}\right)^4\left(\frac{1}{2e}\right)^2\left(\frac{H}{Z}\right)^2$$

and can be neglected for fields experimentally obtainable. For fields of the order 10^4 Gauss we have

$$P\cong 2\cdot 10^{-10}\,\text{sec}^{-1}$$

and for fields of the order 10^6 Gauss we have

$$P\cong 2\cdot 10^{-4}\,\text{sec}^{-1}.$$

The life-time of an electron in the state $n=1$, $m_j={}^1/_2$ is therefore in such a field $5\cdot 10^3$ sec or about 83 min.

The present work was suggested by Professor Blaton, to whom the author is also indebted for helpful advice and discussions.

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ON THE DIVERGENCE PROBLEM IN THE THEORY OF QUANTIZED FIELDS *

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(Received April 10, 1947)

By a suitable change of the definition of the four-vector of charge and current density a Lorentz-invariant formfactor has been introduced which removes the well known divergencies in the theory of quantized fields. Physically, this amounts to the introduction of an elementary length connected with the finite dimensions of particles. Connections are shown to the problem of field equations with higher derivatives.

It is well known that quantum mechanics of fields yields divergent results when higher degrees of perturbation calculus are applied. For example, the self-energy of elementary particles becomes infinite whatever reasonable coupling term between the two fields has been assumed.

Let us consider the interaction between electrons and the electro-magnetic field. The energy density is

$$H = H^{(e)} + H^{(p)} + H^{(e,p)}, \tag{1}$$

where $H^{(e)}, H^{(p)}$ are the well known energy densities of the electron field and the el-magn. field and $H^{(e,p)}$ is the energy density of interaction

$$H^{(e,p)} = -h\varepsilon\psi^*\{(\vec{\alpha}\vec{\varphi}) - \varphi_0\}\psi = -\left(\frac{(s_k\varphi_k)}{c} - \rho\varphi_0\right), \tag{1'}$$

where φ_k, φ_0 are the e. m. potentials and s_k, ρ are the current and charge densities. The corresponding field equations are the Dirac equation

$$\frac{h}{i}\left(\frac{\partial}{\partial t} + i\varepsilon\varphi_0\right) + \frac{ch}{i}\left\{\vec{\alpha}\left(\text{grad} - \frac{i\varepsilon}{c}\vec{\varphi}\right)\right\} + mc^2\beta\psi = 0 \tag{2}$$

and the equation for the electro-magnetic potential

$$\Box\varphi_\mu = -s_\mu. \tag{3}$$

The expression for the coupling of the two fields has been taken over from the classical theory of the movement of a charged p o i n t —

* Presented ad a meeting of the Warsaw Section of the Polish Physical Society on May 4, 1946.

particle in the e. m. field φ_μ which leads to the well known rule that the momentum of a particle in the field φ_μ has to be replaced by

$$p_\mu - \frac{h\varepsilon}{c}\varphi_\mu(x), \tag{4}$$

where $\varphi_\mu(x)$ is taken at the point actually occupied by the particle. The theory of fields in vacuo yields satisfying results and is supposed to be correct. Thus, we have to pay attention to the coupling term (1′) which seems to be responsible for the divergencies. But every modification of (1′) will generally cause some modification of the corresponding field equations. Equations (2) or (3) or both of them will thus be modified. It may be desirable to introduce such a modification of the interaction term, which would change only (2) without modifying (3) and the Maxwell equations. The beauty and advantages of the latter are generally acknowledged, while classical ideas concerning the movement of a point particle in the field, leading to (4), seem suspicious.

It has been obvious from the very outset that this problem is connected with the question of the radius of elementary particles. In order to take account of this radius one used to introduce (*1, 1933*) a factor $D(x-x')$ different from zero for x' in the neighbourhood of x (expressing the proper density of mass of the particle) and to replace $\varepsilon\varphi_\mu(x)$ by $\varepsilon\int D(x-x')\varphi_\mu(x')\,dx'$. Such a factor diminishes the action of the field on the particle and removes the divergencies but on the other hand it spoils the relativistic invariance of the formalism; besides, it means an inconsistency, since in the expression for the interaction (1′) there appears already a density function s_μ. Thus, we had to deal with two sorts of densities: an «external» and an «internal» one ($\rho(x)$ and $D(x-x')$), which does not seem convincing at all. Sometimes one used to introduce «*par force*» a formfactor $e^{-\alpha k}$ or $e^{-\alpha^2 k^2}$ when applying the perturbation calculus. This is in principle equivalent to the introduction of a factor $D(x-x')$ and it spoils also the relativistic invariance of the formalism. We shall discuss here the possibility of introducing Lorentz invariant formfactors.

1. New Definitions of Densities of Charge, Current etc.

In order to simplify our formulas we shall deal in this Section with the scalar meson field in vacuo. Such a field is usually described by the aid of the Lagrangian function

$$L = -c^2\left|\sum_\nu \frac{\partial\psi^*}{\partial x_\nu}\frac{\partial\psi}{\partial x_\nu} + \mu^2\psi^*\psi\right|. \tag{5}$$

The corresponding Lagrangian equations are the Schrödinger-Gordon equations

$$\Box - \mu^2, \psi = 0, \qquad \Box - \mu^2, \psi^* = 0. \tag{6}$$

The energy-impulse tensor is in this case

$$T_{\mu\nu} = c^2 \left(\frac{\partial \psi^*}{\partial x_\mu} \cdot \frac{\partial \psi}{\partial x_\nu} + \frac{\partial \psi^*}{\partial x_\nu} \cdot \frac{\partial \psi}{\partial x_\mu} \right) + L \delta_{\mu\nu} \tag{7}$$

The current density is

$$s_\mu = - i \varepsilon c^2 \left(\psi \frac{\partial \psi^*}{\partial x_\mu} - \psi^* \frac{\partial \psi}{\partial x_\mu} \right). \tag{8}$$

The quantization of the field is introduced by the well known commutation relations

$$[\psi^*(x', t'), \psi(x, t)] = \frac{h}{i} D(x' - x, t' - t). \tag{9}$$

In order to obtain the eigenvalues of energy, momentum, and charge, we may represent the field functions as a superposition of plane waves[1]

$$\psi = \frac{1}{V^{1/2}} \sum_k \sqrt{\frac{h}{2 k_0 c}} \left| a_k e^{i(\vec{k}\vec{x} - k_0 ct)} + b_k^* e^{-i(\vec{k}\vec{x} - k_0 ct)} \right|, \tag{10}$$

where a_k, b_k are the well known matrices verifying

$$[a_k, a^*_k] = \delta_{kl}, \qquad [b_k, b^*_k] = \delta_{kl}, \tag{11}$$

all other commutators of a_k, a^*_k b_k, b^*_k vanishing. The impulse-energy tensor and the current four-vector satisfy the continuity equations

$$\sum_\mu \frac{\partial \delta_{\nu\mu}}{\partial x_\mu} = 0, \qquad \sum_\mu \frac{\partial s_\mu}{\partial x_\mu} = 0. \tag{12}$$

Little attention has been paid as yet to the fact that there exist plenty of other expressions which satisfy also continuity equations and yield the same eigenvalues for energy, charge etc. as $\delta_{\mu\nu}$, s_μ. If we take an operator O which is a scalar with regard to Lorentz transformations, commuting with the operator Div, then the new expression

$$s'_\mu = O s_\mu \tag{13}$$

preserves the vector character and satisfies the continuity equation. Particularly interesting are the following operators

$$e^{\alpha^2 \Box} \ (\alpha), \quad e^{-\alpha^4 \Box^2} \ (\beta), \quad \frac{1}{1 - \alpha^2 \Box} \ (\gamma), \quad \frac{1}{1 + \alpha^4 \Box^2} \ (\delta), \tag{14}$$

[1] Which means simply, working in the Heisenberg picture in a system where the energy (without interaction) is diagonal.

where $\Box$ is the d'Alembert operator, α is a constant (dimension length).

$$e^{\alpha^2\Box} \underset{\text{def}}{=} 1 + \alpha^2\Box + \frac{\alpha^4\Box^2}{2!} + \ldots \tag{15}$$

and $\frac{1}{1-\alpha^2\Box}$ may be defined as the reciprocal of $1-\alpha^2\Box$. The vector s_μ is quadratic in the field variables ψ, ψ^* and their derivatives. By developing the field functions into plane waves it becomes

$$s_\mu = \frac{1}{V}\sum_{kl} a_{kl}^{(\mu)} e^{i\left\{(\vec{k}-\vec{l})\vec{x}-(k_0-l_0)ct\right\}} \tag{16}$$

As

$$\Box e^{i\left\{(\vec{k}-\vec{l})\vec{x}-(k_0-l_0)ct\right\}} = -[(\vec{k}-\vec{l})^2-(k_0-l_0)^2]\, e^{i\left\{(\vec{k}-\vec{l})\vec{x}-(k_0-l_0)ct\right\}} \tag{17}$$

we get

$$e^{\alpha^2\Box} s_\mu = \frac{1}{V}\sum_{kl} e^{-\alpha^2\left\{(\vec{k}-\vec{l})^2-(k_0-l_0)^2\right\}} \cdot a_{kl}^{(\mu)} e^{i\left\{(\vec{k}-\vec{l})\vec{x}-(k_0-l_0)ct\right\}} \tag{18}$$

Thus, all the coefficients a_{kl} in (16) have been multiplied by the exponential factor $e^{-\alpha^2\left\{(\vec{k}-\vec{l})^2-(k_0-l_0)^2\right\}}$. Similarly, when using the operator (14 β) or (14 γ) we get multiplication by the factor $e^{-\alpha^4\left\{(\vec{k}-\vec{l})^2-(k_0-l_0)^2\right\}^2}$ or $\frac{1}{1-\alpha^2\left\{(\vec{k}-\vec{l})^2-(k_0-l_0)^2\right\}}$. If we calculate with the aid of these modified expressions the values of the corresponding observables by integrating over the volume V, then all terms with $\vec{k} \neq \vec{l}$ vanish as usual and there remain only the terms with $\vec{k} = \vec{l}$. But in this case the factors $e^{-\alpha^2\left\{(\vec{k}-\vec{l})^2-(k_0-l_0)^2\right\}}$ etc... become unity and we get the same result as in the case of the usual definitions of densities:

$$\int \rho' dV = \int \rho dV\,. \tag{19}$$

Thus, the eigenvalues of the charge remain the same as in case of the usual definition (8). It is easily seen that there exists a lot of operators with the desired qualities: each operator O which yields

$$O\, e^{i\left\{(\vec{k}-\vec{l})\vec{x}-(k_0-l_0)ct\right\}} = f\left\{(\vec{k}-\vec{l})^2-(k_0-l_0)^2\right\} \cdot e^{i\left\{(\vec{k}-\vec{l})\vec{x}-(k_0-l_0)ct\right\}} \tag{20}$$

will give correct eigenvalues of the observables (if only $f\{0\}=1$) and thus the modified theory of fields in vacuo will give identical results with the usual one. We could apply the same modification of density definition to other observables (namely the energy-impulse tensor $T_{\mu\nu}$

and the angular momentum vector $\vec{M}$) without modifying the eigenvalues of energy (without interaction), momentum and spin. However, this modification is not essential and seems to have no physical meaning. What we really need is only a modification of the charge and current density four-vector.

2. On the physical properties of the introduced modification

Suppose there is given a certain function $\rho(x)$ which may be regarded as the classical density of charge. What will be the influence of introducing the operator (14 β) ? Let us develop $\rho(x)$ as a Fourier series

$$\rho = \sum_k a_k e^{ikx}.$$

In case $a \neq 0$ for $k_1 < k < k_2$ the density may be made different from zero in the region $\triangle x = \frac{1}{|k_1 - k_2|}$ (we have taken the one dimensional case for simplicity). It is well known that for $|k_1 - k_2| \to \infty$ we may build a point charge: the density function $\rho(x)$ tends to infinity in a certain point ($x = o$) and is zero elsewhere; we have only to take $a(k) = \frac{1}{2\pi}$ and to integrate from $-\infty$ to $+\infty$

$$\lim \rho = \lim_{k \to \infty} \frac{1}{2\pi} \int_{-k}^{+k} e^{ikx}\, dk = \delta(x),$$

where $\delta(x)$ is the Dirac function. The corresponding $\rho'(x)$ is in this case

$$\lim \rho' = \lim_{k \to \infty} e^{-\alpha^4 \Box^2} \frac{1}{2\pi} \int_{-k}^{k} e^{ikx}\, dk = \int_{-\infty}^{+\infty} e^{-\alpha^4 \Box^2} e^{ikx}\, dk.$$

Assuming (what is certainly allowed in the classical case) that the density $\rho(x)$ is independent of time, we get

$$e^{-\alpha^4 \Box^2} e^{ikx} = e^{-\alpha^4 \frac{\partial^4}{\partial x^4}} e^{ikx} = e^{-\alpha^4 k^4} e^{ikx},$$

whence

$$\lim \rho' = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-\alpha^4 k^4} e^{ikx}\, dk,$$

which is a regular function of x. Since $e^{-\alpha^4 k^4}$ is a damping factor, we may put also with sufficient accuracy

$$\lim \rho = \frac{1}{2\pi} \int_{-\frac{1}{\alpha}}^{\frac{1}{\alpha}} e^{-\alpha^4 k^4} e^{ikx}\, dk$$

that is $\triangle k \cong \frac{2}{\alpha}$. Thus, the density is surely different from zero in the region $\triangle x \cong \frac{1}{\triangle k} \cong \frac{\alpha}{2}$.

The introduced modification causes a softening of the singularities and the constant α determines the region where the density is different from zero.

Let us now consider a particle in a (one dimensional) box of lenght L. Its momentum be $k = \frac{\pi n}{L}$. The wave function is in this case $\psi = \sin kx\, e^{ik_0 ct}$ and the «density» is

$$\rho = \psi^* \psi = \sin^2 kx = \frac{1}{2}(1 - \cos 2kx).$$

It represents a «frozen» wave with the ratio of the amplitude to the wave length increasing to infinity for $k \to \infty$. By modifying this density e. g. with the aid of the operator (14α) we get

$$e^{\alpha^2 \Box} \rho = e^{\alpha^2 \frac{\partial^2}{\partial x^2}} \rho = \frac{1}{2}\left(1 - e^{-\alpha^2 (2k)^2} \cos 2kx\right).$$

The above mentioned ratio has in this case a maximum for $8\alpha^2 k^2 = 1$. The wave length $\lambda = \frac{2\pi}{k}$ corresponding to this maximum is thus of the order of α. The frozen wave representing the density is the result of interference of the waves of probability. Since the interfering waves are represented by the wave vectors k and $-k$, we see that in the modified theory the effects of the interference diminish with increasing difference $k - (-k) = 2k$. If this difference is $>> \frac{1}{\alpha}$ the waves practically cease to interfere.

3. On the interaction between electrons and the el.-magn. field

For the energy density of the el.-magn. field interacting with the electrons we assume the usual expresion (1) but, according to the definitions proposed in section 1, we introduce the modified density (1'). Assuming that the Fourier coefficients $a_k(t)$ of the expansion

$$\psi_b = \sum_k a_k(t)\, u_b\, e^{i(\vec{k}\vec{x} - k_0 ct)}$$

are slowly varying functions of time, we may omit the time differentiations of the a_k (in the first approximation) and replace $s'_\mu = e^{-\alpha^4\Box^2} s_\mu$ by

$$s^{(\mu)'} = \sum_{kl} s^{(\mu)'}_{kl} \cong \sum_{kl} e^{-\alpha^4\{(\vec{k}-\vec{l})^2-(k_0-l_0)^2\}^2} s^{(\mu)}_{kl}(t), \tag{21}$$

where $s^{(\mu)}_{kl}$ are the Fourier terms of the non-modified charge and current densities. We are certainly allowed to omit the time differentiations provided by the introduction of the operator $e^{-\alpha^4\Box^2}$ for those $a_k a_l$ (or s_{kl}), where $(k_\mu - l_\mu)^2$ is small compared with $\frac{1}{\alpha^2}$, since the effect of the $e^{-\alpha^4\Box^2}$ is negligible in these cases. But, as we shall see later, we may also omit the time derivatives of $a_k a_l$ for $(k_\mu - l_\mu)^2$ very large, since in the latter cases the exponential factor $e^{-\alpha^4\{(\vec{k}-\vec{l})^2\}}$ will sensibly diminish the probability of transition between the states k and l. The approximation is poor only for $(k_\mu - l_\mu)^2$ of the order of $\frac{1}{\alpha^2}$. However, this cannot affect the general argument of the convergence of this formalism since this convergence depends only upon the terms with $(k_\mu - l_\mu)^2$ which tend to infinity.

We see that the problem has remained unchanged with respect to the variables Φ_μ. We may thus take over the whole electromagnetic part of the usual theory: define the Lagrangian function of the el.-magn. field as a function of the variables Φ_μ, deduce the Lagrangian equations

$$\Box \Phi_\mu = - s'_\mu, \tag{22}$$

and, with the aid of the well known initial conditions, we may get the Maxwell equations in the operator form. The only difference is that the current-density four-vector appearing in (22) is defined according to (21). The problem is quite different, however, with regard to the variables ψ, ψ^* of the electronic field. In the interaction energy (1') there appear terms of the type (21) multiplied by the function $\Phi_\mu(x)$. If we integrate over the volume V, the terms with $\vec{k} \neq \vec{l}$ do not vanish and the factors $e^{-\alpha^4\{(\vec{k}-\vec{l})^2-(k_0-l_0)^2\}^2}$ etc... appearing in consequence of the modified definitions of current and charge densities remain essential. These factors cause here important complications: it may be seen that in this case the Dirac equations in the presence of the field Φ_μ are no more valid. We are obliged to consider the equations (2) as only approximatively valid, namely in cases when with sufficient approximation we may put $\alpha = 0$. Fortunately, the existence

of some equations of field like (2) is not indispensable for obtaining definite quantitative results. In order to get definite answers we need only to know (a) the Schrödinger equation (i. e. the energy function given by (1) and (1′) together with the corresponding definition of s'_μ), (b) the zero-order functions (i. e. the field without interaction, which is identical with the one in the non-modified theory γ), (c) a perturbation theory enabling us to find higher approximations. We shall show now that in the modified theory (at least in the cases corresponding to the operators (14 β) and (14 δ)) the coupling energy cannot increase indefinitely.

According to (21) the interaction energy splits into a sum of terms H'_{kl} which differ from the terms H_{kl} of the usual theory approximatively by the factors

$$e^{-\alpha^4\{(\vec{k}-\vec{l})^2-(k_0-l_0)^2\}^2}. \tag{23}$$

In the system of coordinates in which $l_i=0$ $(i=1,2,3)$ we get

$$(\vec{k}-\vec{l})^2-(k_0-l_0)^2=2\mu\,(\sqrt{\mu^2+k^2}-\mu)\geqslant 0.$$

This corresponds to the case of an electron at rest $(\vec{l}=0, l_0=\mu)$ which gets the momentum k by absorbing or emitting a quantum of light (with momentum k). If k is very large $(k^2>>\mu^2)$ then we may omit μ and get for the exponential factor (23)

$$e^{-4\alpha^4\mu^2k^2}. \tag{24}$$

We see that in the system of rest the coupling term has been multiplied by a factor which decreases exponentially with increasing energy of the interacting photon. Hence, the modified theory yields a form factor which causes a damping of high energy interactions [2].

As yet we have limited our considerations to positive energy states only. The transitions between two negative states are similar to those between two positive ones, but transitions between a positive and a negative state are different. In this case $k_\mu-l_\mu$ is not a space-like but a time-like four-vector

$$k_\mu=\{\vec{k},\ i\sqrt{\mu^2+k^2}\},\quad l_\mu=\{\vec{l},\ -i\sqrt{\mu^2+l^2}\},\quad \sum_\mu(k_\mu-l_\mu)^2=(\vec{k}-\vec{l})^2-(\sqrt{\mu^2+k^2}+\sqrt{\mu^2+l^2})^2.$$

For large impulses we get (in a system where $\vec{l}=0$)

$\sum_\mu(k_\mu-l_\mu)^2=-2\mu k-2\mu^2$, whence (14 β) yields $e^{-\alpha^4(2\mu k+2\mu^2)}$.

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[2] A first attempt to introduce a Lorentz-invariant damping factor was made by Wathagin (Zeits f. Physik. 88 (1934)), but his article has been completely overlooked.

where we may neglect μ^2 in comparison with μk and get approximately the former result (24). But it would be quite different if we had used the operator (14 α). In that case we obtain the damping factor $e^{-2\alpha^2\mu k}$ for transitions between two states with equal signs of energy but an increased interaction (by $e^{+2\alpha^2\mu k}$) for transitions between two states of different signs. Such a theory would lead to increased probabilities of annihilation of positrons and of creation of pairs. This may be an interesting result, not without some importance perhaps for the theory of showers. On the other hand, the introduction of a non-definite exponent increases the well known difficulties connected with the hole theory, while the introduction of a definite one removes these difficulties by diminishing the probabilities of all transitions.

4. Self energy of an electron in an electro-magnetic field [3]

Let us calculate the self energy of an electron in the case corresponding to the operator (14 β). We assume α to be small in comparison to $\frac{1}{\mu}$. The result obtained will confirm that we really may put $\alpha\mu \leqslant \frac{1}{137}$.

As the electro-magnetic part of the problem has not been actually changed, we may — as in the usual theory — split the el.-magn. field into a longitudinal and a transverse part. The longitudinal part is the Coulomb energy

$$E^S = \frac{1}{2}\int \frac{\rho'(x)\cdot\rho'(x')}{|x-x'|}\,dx\,dx'. \tag{25}$$

the other part of the energy depends only on the transverse field $\Phi_\perp$

$$E^D = -\frac{1}{2}\int \sum_{k/_\perp}^{3} s'_k(x)\,\Phi_{\perp k}(x)\,dx. \tag{26}$$

The proof of the formulae (25), (26) depends only upon the form of the interaction energy $H^{(e,\,p)}$ (1') and on the fact that the Maxwell equations are valid, but it is quite independent of the special definition of the four-vector of charge and current density. Hence, we may take over literally the well known calculations of the self-energy, introducing only the new definitions of charge and current densities. For an electron at rest this amounts to the introduction of the

[3] Cf. reference *(2, 1934)*.

form-factor (24) for virtual transitions to positive as well as to negative states for $k >> \mu$, and to the omission of the form-factor for $k \sim \mu$ or $k < \mu$ (as α is very small). Thus, instead of the integral

$$E^S = 2e^2\mu \int_0^\infty \frac{1}{\sqrt{\mu^2+k^2}}\,dk,$$

which is known to diverge, we obtain approximatively

$$E^S \cong 2e^2\mu \int_0^\infty \frac{e^{-8\alpha^4\mu^2k^2}}{\sqrt{\mu^2+k^2}}\,dk.$$

If we put e. g. $\frac{1}{\alpha} = 137\,\mu$, this integral yields $E^S = 0{\cdot}13\ mc^2$. The dynamical part of the self-energy is $E^D = \frac{1}{2}E^S$ for an electron at rest. Hence the el-magn. self-energy is about 20% of the total self-energy of the electron. This result shows that α may be assumed much smaller than 10^{-13} cm.

5. Some final remarks

This paper may be regarded as an attempt to introduce an elementary length α into the theory of quantized fields. In order to obtain a linear theory which in the limit of vanishing interaction constant $e \to 0$ yields the usnal Dirac and Maxwell equations in vacuo, we were obliged to modify suitably the charge and current density four-vector. This is equivalent to the introduction of equations of an infinitely high order, which cannot be solved exactly. This result is not surprising: the equations of finite order yield connections between points the distances of which may be taken arbitrarily small, so that the need of equations of infinite order (or some integral equations) is obvious if we want to introduce an elementary length into a linear field theory.

Besides the difficulties in formulating a field theory involving derivatives of any order we meet also other difficulties: there exist several operators (besides (14β)) which also remove the divergencies, so that our results are not unique.

Let us consider briefly the case (14γ), which yields the factor $\frac{1}{2+\alpha^2\sum_\mu(k_\mu-l_\mu)^2}$. In the system of reference where $\vec{1}=0$ it gives $\frac{1}{1-2\alpha^2\mu^2\pm 2\alpha^2\mu\sqrt{\mu^2+k^2}}$ with the sign $+$ (or $-$) for transitions between

states with equal (or opposite) energy signs. In the first case this formula gives damping, but in the second one the denominator may even become zero unless α is sufficiently large. If we put $2\alpha^2\mu^2 = 1$, i. e. α of the order of the Compton wave-length, the damping factor becomes $\frac{1}{\pm\sqrt{1+\frac{k^2}{\mu^2}}}$. In this case we obtain always damping, but, unfortunately, it becomes appreciable already for $k = \alpha$, what seems to be in disagreement with experiment. The damping operator (14 γ) stands in close connection with the theory Bopp and Podolsky (3). From. (14 γ) and (22) we get

$$\Box \Phi_\mu = \frac{-1}{1-\alpha^2\Box} \cdot s_\mu .$$

Operating on both sides with the inverse operator O^{-1} we get

$$-\alpha^2 \Box^2 \Phi_\mu + \Box \Phi_\mu = -s_\mu . \tag{27}$$

We see that the assumption of the new definitions of densities is, to some extent, equivalent with the introduction of more complicated equations of el.-magn. potentials involving the elementary length α together with higher derivatives. The last operator is (to the second power of α) equivalent with the operator (14 α) which leads to difficulties whenever virtual transitions to or from negative energy states are involved. The operator (14 δ) is equivalent (to the fourth power of α) with the operator (14 β) and yields damping of the interaction in any case. Hence instead of (27) the equation

$$\alpha^4 \Box^3 + \Box, \Phi_\mu = -s_\mu \tag{28}$$

seems more promising as the generalization of the equation of the el.-magn. potentials. The equation (28) has the following central symmetrical static solutions without singularities

$$\frac{1}{r}\left(1 - e^{-\frac{r}{\alpha}} \cdot \cos\frac{r}{\alpha}\right) \text{ and } \frac{1}{r} e^{-\frac{r}{\alpha}} \sin\frac{r}{\alpha}$$

where $\alpha = \sqrt{2}\,\alpha$. No simple equations liks (27) or (28) correspond to the case of exponential factors, since (14 α) or (14 β) involve derivatives of any order. There are, however, some reasons in favour of (14 α) or (14 β), namely their connections with the Gaussian error

function. The usual Gaussian function $e^{-\frac{\alpha^2}{2}(\vec{k}-\vec{l})^2}$ which describes the probability of obtaining the value $\vec{k}$ in case the mean (or initial) value has been $\vec{l}$, within the allowance $\frac{1}{\alpha}$, is not invariant. But there exist two immediate possibilities of relativising it: we may complete the three-vectors $\vec{k}$, $\vec{l}$ to four-vectors:

$$(\vec{k}-\vec{l})^2 \to (\vec{k}-\vec{l})^2-(k_0-l_0)^2 \text{ or } (\vec{k}-\vec{l})^2=|(\vec{k}-\vec{l})^2| \to |(\vec{k}-\vec{l})^2-(k_0-l_0)^2|.$$

In both these cases the relativised function becomes the usual one for small values of $\vec{k}$ and $\vec{l}$. But also the function $\exp\{(k-l)^2-(k_0-l_0)^2\}^2$ may be regarded as a relativised Gaussian error function: if $\vec{l}$ means the initial state of the electron, then, in the system of rest $(\vec{l}=0,\ l_0=\mu)$, we get

$$\left\{(\vec{k}-\vec{l})^2-(k_0-l_0)^2\right\}^2 = 4\mu^2(\sqrt{\mu^2+k^2}-\mu)^2 = \left(\frac{2\mu}{hc}\right)^2(E_k-E_0)^2,$$

where E_k is the energy of the electron with momentum $\vec{k}$ and E_0 is the energy at rest. In this case we obtain a Gaussian function for the energy in the system of rest of the particle.

I should like to express my thanks to Professor Rubinowicz for helpful discussion and encouraging interest in this work.

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RELATIVISTIC EQUATIONS OF MOTION OF FREE DIPOLE AND QUADRUPOLE PARTICLES*

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(Received April 22, 1947)

1. Recapitulation of Mathisson's variational principle for deriving the equations of motion of multipole particles. 2. Equations of motion of a dipole particle characterized by a spin bivector $s^{\alpha\beta}$ and a dipole moment n^{α}. 3. Solutions of the equations of motion of a dipole particle of the second kind (characterized by n^{α} only). 4. Equations of motion of a dipole-quadrupole particle. 5. Solution of the equations of motion of a quadrupole particle.

Introduction

In a paper entittled «A New Mechanics of Material Systems» (*1, 1937*) M. Mathisson introduced a variational principle by help of which he obtained the equations of motion of multipole particles considered as singularities in a gravitational field. Here, we shall be only concerned with the special case of vanishing gravitational field.

Mathisson's principle contains an infinite series deriving from a development of a certain tensor field in a series of multipoles. Leaving out of account all the terms of this series except the first one, one obtains the well known equations of geodetic lines. The two first terms lead after some simplifications to Mathisson's equations of motion of a dipole particle characterized by a spin bivector $s^{\alpha\beta}$, which we shall call in the sequel «dipole particle of the second kind». These equations were found previously by Frenkel (*2, 1926*) and afterwards derived by Lubański (*4, 1937*) with the help of gravitational retarded potentials and by J. Weyssenhoff and A. Raabe (*7, 1947*) in a simplified form from their theory of spin fluids. We shall call «dipole particle of the second kind» a particle characterized by a 4-vector of dipole moment n^{α} rather than by the spin bivector $s^{\alpha\beta}$. Such particles have been considered by Höni and Pa-

* Extract of dissertation for the degree of Ph. D.

papetrou (*5, 1939*) who applied to them the method of Lubański. They did not find, however, the general solutions of these equations, but showed only that a suitably chosen uniform circular motion fulfils these equations (for a free particle).

In this paper the equations of motion of a general dipole particle are first derived in a somewhat simplified manner from Mathisson's variational principle, they are afterwards specialized to the Hönl and Papapetrou case and solved generally under the assumption $n^\alpha n_\alpha = \text{const}$. In the last two sections the equations of motion of a dipole-quadrupole particle are established anew (as Mathisson dropped in his derivation just the term which differentiate them from the corresponding equations for a dipole particle) and solved generally for a quadrupole particle with $s^{\alpha\beta} = 0$ and $n^\alpha = 0$.

1. Recapitulation of Mathisson's variational principle for deriving the equation of motion of multipole particles

Mathisson's variational principle (*6, 1940*) on the background of special relativity-theory has the form

$$\int_{\tau_1}^{\tau_2} (m^{\alpha\beta}\partial_\beta \xi_\alpha + d^{\lambda\alpha\beta}\partial_\lambda \partial_\beta \xi_\alpha + q^{\lambda\mu\alpha\beta}\partial_\lambda \partial_\mu \partial_\beta \xi_\alpha + \cdots\cdots) d\tau = 0, \qquad (1)$$

where ξ^α is an arbitrary vector field vanishing with its derivatives at τ_1 and τ_2 and

$$\begin{aligned} &q^{\lambda\mu\alpha\kappa} = q^{(\lambda\mu)\alpha\beta}, \qquad q^{\lambda\mu\nu\alpha\beta} = q^{(\lambda\mu\nu)\alpha\beta} \quad \cdots\cdots \\ &d^{\lambda\alpha\beta} u_\lambda = q^{\lambda\mu\alpha\beta} u_\lambda = q^{\lambda\mu\alpha\beta} u_\mu = \quad \cdots\cdots = 0 \qquad (2) \\ &m^{\alpha\beta} = m^{(\alpha\beta)}, \quad d^{\lambda\alpha\beta} = d^{\lambda(\alpha\beta)}, \quad q^{\lambda\mu\alpha\beta} = q^{\lambda\mu(\alpha\beta)}. \end{aligned}$$

The integral is taken along a time-like world line $L: x^\alpha = x^\alpha(\tau)$. Let Σ_0 be a frame of momentary rest, with the point $P(\tau)$ as centre, in which the 4-velocity u^λ has the components $(0,0,0,1)$, and let us denote by $\Pi(\tau)$ the hyperplane perpendicular to L in $P(\tau)$ (containing the three space-like axes of Σ_0). In the coordinate system Σ_0

$$\begin{aligned} &m^{\alpha\beta} = \int_{\pi(\tau)} T^{\alpha\beta}\, dV \\ &d^{i\alpha\beta} = \int_{\pi(\tau)} x^i T^{\alpha\beta}\, dV, \qquad d^{4\alpha\beta} = 0 \qquad (3) \\ &q^{ij\alpha\beta} = \int_{\pi(\tau)} x^i x^j T^{\alpha\beta}\, dV, \qquad q^{4\nu\alpha\beta} = q^{\mu 4\alpha\beta} = 0. \end{aligned}$$

In all other coordinate systems $m^{\alpha\beta}$, $d^{\lambda\alpha\beta}$,... are defined by the laws of transformation of tensors.

2. Equations of motion of a dipole particle

Taking into account the two terms in (1) Mathisson derived the equations of motion of a dipole particle. However, his equations are not the most general possible, as he put the vector n^α defined by (3) below equal to nought. By so doing he excluded for instance the case treated afterwards by Hönl and Papapetrou (*5, 1939*). We begin by giving a sketch of a somewhat simplified derivation of the equations of motion, differing also from Mathisson's work by the explicite introduction (Weyssenhoff and Raabe (*7, 1947*)) of the 4-vector of momentum and energy.

Following Mathisson we break up the tensor $d^{\lambda\alpha\beta}$ with regard to the 4-velocity u^α, and writing out only the nonvanishing terms, which are perpendicular in all indices to u^α, we get [1]

$$d^{\lambda\alpha\beta} = \underset{\perp}{d}^{\lambda\alpha\beta} + \frac{1}{2}(s^{\lambda\alpha} u^\beta + s^{\lambda\beta} u^\alpha) + n^\lambda u^\alpha u^\beta. \tag{4}$$

Denoting $n^\lambda u^\alpha$ by $N^{\lambda\alpha}$, substituting (3) in (1), and integrating by parts, we obtain

$$\int (m^{\alpha\beta} - \frac{1}{2}\dot{s}^{\beta\alpha} - \dot{N}^{\beta\alpha})\, \partial_\beta \xi_\alpha \, d\tau + \int (\underset{\perp}{d}^{\lambda\alpha\beta} + \frac{1}{2} s^{\lambda\beta} u^\alpha)\, \partial_\lambda \partial_\beta \xi_\alpha \, d\tau = 0. \tag{5}$$

As the value of the second integral in (4) depends only on the derivatives of $\partial_\lambda \partial_\beta \xi_\alpha$ in directions perpendicular to L and these derivatives are quite arbitrary, we conclude that

$$\underset{\perp}{d}^{\lambda\alpha\beta} = 0, \qquad s^{(\lambda\beta)} = 0,$$

and (5) becomes

$$\int (m^{\alpha\beta} - \frac{1}{2}\dot{s}^{\beta\alpha} - \dot{N}^{\beta\alpha})\, \partial_\beta \xi_\alpha \, d\tau = 0.$$

Breaking up the tensor in brackets under the integration sign, we obtain

$$m^{\alpha\beta} = \underset{\perp}{m}^{\alpha\beta} + M^\alpha u^\beta + M^\beta u^\alpha + M u^\alpha u^\beta, \tag{6}$$

$$\frac{1}{2}\dot{s}^{\beta\alpha} + \dot{N}^{\beta\alpha} = \underset{\perp}{D}^{\alpha\beta} + E^\beta u^\alpha + F^\alpha u^\beta + H u^\alpha u^\beta. \tag{7}$$

Proceeding in the same manner as before, we get

$$\int \{(M - H) u^\alpha + M^\alpha u^\beta - F^\alpha u^\beta\}\, \partial_\beta \xi_\alpha \, d\tau = 0, \tag{8}$$

$$\underset{\perp}{m}^{\alpha\beta} - \underset{\perp}{D}^{\alpha\beta} = 0 \qquad (8') \qquad M^\beta = E^\beta \qquad (8').$$

[1] The notations are the same as in (*7, 1947*) and (*8, 1947*).

Integrating by parts and taking account of the arbitrariness of ξ^α we get

$$d_\tau\{(M-H)u^\alpha + M^\alpha - F^\alpha\} = 0. \tag{9}$$

Substituting

$$M - H = M + \dot{n}^\lambda u_\lambda = m_0 \tag{10}$$

into (8), we get on account of (8″) and (7)

$$d_\tau\{(m_0 + \dot{n}^\lambda u_\lambda)u^\alpha + \dot{s}^{\alpha\beta}\dot{u}_\beta + \dot{n}^\alpha\} = 0. \tag{11}$$

We see that the tensor under the sign of integration plays the role of the momentum-energy vector of the particle. We denote it by

$$G^\alpha = (m_0 + \dot{n}^\lambda u_\lambda)u^\alpha + s^{\alpha\beta}\dot{u}_\beta + \dot{n}^\alpha \tag{12}$$

and (11) becomes

$$\dot{G}^\alpha = 0. \tag{13}$$

If we substitute now $m^{\alpha\beta}_{\perp}$ and $D^{\beta\alpha}_{\perp}$ in (8′) from (6) and (7) we get with the help of (8″) and (12)

$$d_\tau(s^{\alpha\beta} + n^\alpha u^\beta - n^\beta u^\alpha) = G^\alpha u^\beta - G^\beta u^\alpha. \tag{14}$$

The last three equations are the equations of motion of the dipole particle. Dropping the terms with n^α we get Mathisson's and Frenkel's equations of motion in the form given by Weyssenhoff and Raabe (7, *1947*) for a free particle with $n^\alpha = 0$, $s^{\alpha\beta} \neq 0$ (which we shall call further on dipole particle of the first kind).

3. Solutions of the equations of motion of a free dipole particle of the second kind

In (*1, 1937* p. 180) Mathisson expounds the reasons which induced him to put $n^\lambda = 0$ and which are intimately connected with the assumption of positive mass density. Hönl and Papapetrou (5, 1939) put $s^{\alpha\beta} = 0$, $n^\lambda \neq 0$ explaining the appearance of n^λ by the possibility of the existence of negative mass density.

Putting $s^{\alpha\beta} = 0$, (12), and (13) becomes

$$\dot{G}^\alpha = 0, \tag{15}$$

$$G^\alpha = (m_0 + \dot{n}^\lambda u_\lambda)u^\alpha + \dot{n}^\alpha. \tag{16}$$

Substituting further (12) in (14), we get

$$n^\alpha \dot{u}^\beta - n^\beta \dot{u}^\alpha = 0 \tag{17}$$

and therefore

$$n^\alpha = k\dot{u}^\alpha. \tag{18}$$

where k is a scalar.

As n^λ has to be a quantity characterizing the given kind of particles it must be either constant or at least its magnitude must be constant. The first alternative would bring us back to ordinary relativistic dynamics without spin, we choose therefore the second one and put

$$n^\lambda n_\lambda = \text{const.} \tag{19}$$

It seems that (19) is neither a consequence of the equations of motion nor of the variational principle of Mathisson. Hönl and Papapetrou do not make this assumption explicitly, but it is fulfilled in the particular case considered by them. Because of (19) the general solution of the equations of motion of a dipole particle of the second kind (without external forces) may be found.

Differentiating (19) we have thanks to (18)

$$\dot{n}^\alpha \dot{u}_\alpha = 0. \tag{20}$$

Multiplying (16) by n_α and taking into account (18) and (19), we get

$$\dot{u}^\alpha G_\alpha = 0. \tag{21}$$

According to (20), (21) and (16) we obtain

$$m_0 = -G^\alpha u_\alpha = \text{const.} \tag{22}$$

Now, we shall prove that $k = \text{const}$. Differentiating twice the relation

$$n^\lambda u_\lambda = 0,$$

we get in virtue of (20) and (19)

$$2k \dot{u}_\lambda \ddot{u}^\lambda + \dot{k} \dot{u}_\lambda \dot{u}^\lambda = 0, \tag{23}$$

Differentiating (18) and substituting in (20), we have

$$k \dot{u}_\lambda \ddot{u}^\lambda + \dot{k} \dot{u}_\lambda \dot{u}^\lambda = 0.$$

By comparison of (23) and (21) we see that

$$\dot{k} = 0, \qquad k = \text{const}$$

and (19) yields

$$\dot{n}^\lambda u_\lambda = \text{const},$$

so that the coefficient of u^α in the formula for G^α is constant:

$$m_0 + \dot{n}^\lambda u_\lambda = \mu = \text{const}.$$

Now let us assume, as usual, that the vector G^α is a time-like vector. Since it is constant we may choose a coordinate system Σ_c in which

$$G^1 = 0, \qquad G^4 = \text{const} \neq 0.$$

Hence we conclude from (22) (cf 7, *1947*, p. 17) that in Σ_c the magnitude of the ordinary (3-dimensional) velocity v is constant:

$$\sqrt{v^i v_i} = v = \text{const.} \tag{27}$$

Assuming $m_0 > 0$ it can be proved (*5, 1939* p. 527) that

$$0 < \mu < m_0.$$

From (18), (27), (28) it follows that

$$k > 0.$$

Using (18) the three first equations (16) take in Σ_c the form

$$\mu u^i + k \ddot{u}^i = 0$$

with two positive constants μ and k. Transforming from τ and u^i to t and v^i and remembering that v is constant we get for the solution of (29) a uniform circular motion with the angular velocity

$$\omega = \frac{v\mu}{n}.$$

The radius of the circle is

$$r = \frac{n}{\mu}$$

(Cf. 7, *1947* p. 15).

4. Equations of motion of a dipole-quadrupole particle

We consider now Mathisson's variational principle leaving in the integral (3) the three first terms. Using the same method as in the case of the dipole particle we break up $q^{\lambda\mu\alpha\beta}$ with respect to u^α. Writing out the nonvanishing terms only, we have

$$q^{\lambda\mu\alpha\beta} = q_{\perp}^{\lambda\mu\alpha\beta} + q^{\lambda\mu\alpha} u^\beta + q^{\lambda\mu\beta} u^\alpha + q^{\lambda\mu} u^\alpha u^\beta. \tag{30}$$

The terms on the right-hand side are perpendicular to u^α. Substituting in (3) from (30) and integrating by parts (*1, 1937* eqs (5, 2) — (5, 5)) we obtain

$$\int m^{\alpha\beta} \partial_\beta \xi_\alpha \, d\tau + \int (d^{\lambda\alpha\beta} - \dot{q}^{\lambda\beta\alpha} - \dot{q}^{\lambda\beta} u^\alpha - q^{\lambda\beta} \dot{u}^\alpha) \partial_\lambda \partial_\beta \xi_\alpha \, d\tau = 0 \tag{31}$$

$$q^{(\lambda\mu\alpha)} = 0, \qquad q_{\perp}^{\lambda\mu\alpha\beta} = 0. \tag{32}$$

Now, we break up in the same manner the tensors $d^{\lambda\alpha\beta}$, $\dot{q}^{\lambda\beta\alpha}$ and $\dot{q}^{\lambda\beta}$ and get

$$d^{\lambda\alpha\beta} = d_{\perp}^{\lambda\alpha\beta} + d^{\lambda\alpha} u^\beta + d^{\lambda\beta} u^\alpha + n^\lambda u^\alpha u^\beta, \tag{33}$$

$$\dot{q}^{\lambda\beta\alpha} = \dot{q}_{\perp}^{\lambda\beta\alpha} + Q^{\lambda\alpha} u^\beta + Q^{\beta\alpha} u^\lambda + R^{\lambda\beta} u^\alpha + L^\lambda u^\alpha u^\beta + L^\beta u^\lambda u^\alpha + L^\alpha u^\beta u^\lambda + \bar{L} u^\alpha u^\beta u^\lambda, \tag{34}$$

$$\dot{q}^{\lambda\beta} = \dot{q}_{\perp}^{\lambda\beta} + Q^\lambda u^\beta + Q^\beta u^\lambda + Q u^\lambda u^\beta, \tag{35}$$

where all tensors on the right-hand side are perpendicular to u^α in all indices. In this decomposition the following terms vanish

$$L^\lambda, \quad \bar{L}^\alpha, \quad L, \quad Q^\lambda, \quad Q.$$

We shall prove it for Q only, as the proofs for all remaining terms are similar. We infer from (35) that $Q = \dot{q}^{\lambda\beta} u_\lambda u_\beta$, hence in the system of momentary rest Σ_0

$$Q = \dot{q}^{44}. \tag{36}$$

But from (30) it follows that

$$q^{\lambda\mu} = q^{\lambda\mu\alpha\beta} u_\alpha u_\beta \text{ and in } \Sigma_0 \; q^{\lambda\mu} = q^{\lambda\mu 44}.$$

Taking into account (3) we see that in Σ_0 : $= q^{4444} = q^{44} = 0$ and (35) yields $Q = 0$.

Following Mathisson we put $n^\lambda = 0$ as in the case of the dipole particle of the first kind. Substituting (33)—(35) in (31) and integrating by parts, we get

$$\begin{aligned} &\int (m^{\alpha\beta} + \dot{d}^{\beta\alpha} + 2\dot{Q}^{\beta\alpha}) \partial_\beta \xi_\alpha \, d\tau = 0, \\ &d^{(\lambda\beta)} - R^{(\lambda\beta)} - q_{\perp}^{(\lambda\beta)} = 0, \\ &d^{(\lambda\beta)\alpha} - q_{\perp}^{(\lambda\beta)\alpha} - q^{(\lambda\beta)} \dot{u}^\alpha = 0. \end{aligned} \tag{37}$$

Breaking up the subject of integration of (33) with respect to u^α, we get

$$m^{\alpha\beta} + \dot{d}^{\beta\alpha} + 2\dot{Q}^{\beta\alpha} = S_{\perp}^{\alpha\beta} + S^\alpha u^\beta + T^\beta u^\alpha + m u^\alpha u^\beta. \tag{38}$$

Substituting (38) in (37) and integrating by parts we have

$$\int d_\tau (m u^\alpha + T^\alpha) \xi_\alpha \, d\tau = 0, \tag{39}$$

$$S^\beta = 0, \qquad S_{\perp}^{\alpha\beta} = 0. \tag{40}$$

From (39) we infer that

$$d_\tau (m u^\alpha + T^\alpha) = 0. \tag{41}$$

Therefore if we put

$$G^\alpha = m u^\alpha + T^\alpha, \tag{42}$$

G^α will play the role of the 4-vector of momentum and energy of the particle and the equation (41) becomes

$$\dot{G}^\alpha = 0. \tag{43}$$

From (42) we have

$$G^\alpha u^\beta - G^\beta u^\alpha = T^\alpha u^\beta - T^\beta u^\alpha, \tag{44}$$

and on account of (38), (40), and the symmetry of $m^{\alpha\beta}$ we can write

$$G^\alpha u^\beta - G^\beta u^\alpha = d_\tau (d^{\alpha\beta} - d^{\beta\alpha} + 2Q^{\alpha\beta} - 2Q^{\beta\alpha})$$

and consequently, thanks to (33), (34) and the perpendicularity of $Q^{\alpha\beta}$ to u_α,

$$G^\alpha u^\beta - G^\beta u^\alpha = d_\tau \left\{ (d^{\beta\alpha\lambda} - d^{\alpha\beta\lambda}) u_\lambda + 2 (q^{\lambda\beta\alpha} - q^{\lambda\alpha\beta}) \dot{u}_\lambda \right\}.$$

We introduce now the two following tensors

$$s^{\alpha\beta} = s^{[\alpha\beta]} = (d^{\beta\alpha\lambda} - d^{\alpha\beta\lambda}) u_\lambda, \tag{45}$$

$$k^{\lambda\alpha\beta} = k^{\lambda[\alpha\beta]} = 2 (q^{\lambda\beta\alpha} - q^{\lambda\alpha\beta}), \tag{46}$$

playing respectively the role of the dipole and quadrupole moments of the particle. From (30) and (33), as $n^\lambda = 0$, we see that both $s^{\alpha\beta}$ and $k^{\lambda\alpha\beta}$ are perpendicular to u^α in all indices. Thus

$$G^\alpha u^\beta - G^\beta u^\alpha = d_\tau (s^{\alpha\beta} + k^{\lambda\alpha\beta} \dot{u}_\lambda). \tag{47}$$

(43) and (47) are the equation of motion of the dipole-quadrupole particle. Putting

$$G^\alpha u_\alpha = - m_0, \tag{48}$$

we obtain from (47) the following expression for G^α

$$G^\alpha = m_0 u^\alpha + s^{\alpha\beta} \dot{u}_\beta + k^{\lambda\alpha\beta} \dot{u}_\lambda \dot{u}_\beta. \tag{49}$$

m_0 is a constant as $G_\alpha \dot{u}^\alpha = 0$ from (49).

Eliminating G^α between (43), (47) and (49), we get two non linear equations of the second and third order respectively

$$m_0 u^\beta + s^{\beta\lambda} \ddot{u}_\lambda + k^{\lambda\beta\mu} \ddot{u}_\mu \dot{u}_\lambda = 0, \tag{50}$$

$$\dot{s}^{\alpha\beta} + d_\tau (k^{\lambda\alpha\beta} \dot{u}_\lambda) = s^{\alpha\lambda} \dot{u}_\lambda u^\beta - s^{\beta\lambda} \dot{u}_\lambda u^\alpha + k^{\lambda\mu\alpha} \dot{u}_\lambda \dot{u}_\mu u^\beta - k^{\lambda\mu\beta} \dot{u}_\lambda \dot{u}_\mu u^\alpha,$$

as generalization of the equations found by Mathisson.

Mathisson (*1, 1947*) deduced from his variational principle the equations of motion of a particle possessing a dipole and a quadrupole moment, but he omitted in the decomposition of the tensor $q^{\lambda\mu\alpha\beta}$ the term $q^{\lambda\mu\alpha}$ which alone appears in the equations of motion. Therefore his equations of motion for a dipole-quadrupole particle (in the case of special relativity) are the same as for a dipole particle.

Two tensors in the decomposition (35) may be regarded as the 4-dimensional generalization of the 3-dimensional quadrupole moments. The first of them is $q^{\lambda\mu} = q^{\lambda\mu\alpha\beta} u_\alpha u_\beta$ and gives in nonrelativistic approximation

$$q^{ik} = q^{ik\,44} = \int \mu_0 x^i x^k \, dV \qquad q^{\alpha 4} = 0. \tag{51}$$

The second is $q^{\lambda\mu\alpha} = -q^{\lambda\mu\alpha\beta} u_\beta$; in nonrelativistic approximation

$$\begin{aligned} -q^{ikj} &= q^{ikj4} = \int \mu_0 x^i x^k v^i \, dV, \\ -q^{ik4} &= q^{ik44} = \int \mu_0 x^i x^k \, dV, \end{aligned} \tag{52}$$

all the other components vanishing.

The q^{ikj} appear in the equations of motion only in the following combinations

$$2 q^{i[kj]} = \int \mu_0 x^i (x^j v^k - x^k v^j) \, dV.$$

The motion of the particle is thus independent of the static quadrupole moments (51) as well as of the expressions of the form

$$\int \mu_0 x^i (x^j v^k + x^k v^j) \, dV.$$

5. Solution of the equations of motion of a free quadrupole particle

Let us now assume a particle possesing a mass m_0, a quadrupole moment and no dipole moment. From (43), (47) and (49) with $s^{\alpha\beta} = 0$ we have

$$\dot{G}^\beta = 0, \tag{53}$$

$$d_\tau (k^{\lambda\alpha\beta} \dot{u}_\lambda) = G^\alpha u^\beta - G^\beta u^\alpha, \tag{54}$$

$$G^\beta = m_0 u^\beta + k^{\lambda\alpha\beta} \dot{u}_\lambda \dot{u}_\alpha, \tag{55}$$

$$k^{\lambda\alpha\beta} u_\lambda = k^{\lambda\alpha\beta} u_\alpha = k^{\lambda\alpha\beta} u_\beta = 0. \tag{56}$$

As usual, we assume the vector G^β to be a time-like vector. Since it is constant we may choose a system of reference Σ_c in which $G^i = 0$, $G^4 = \text{const} \neq 0$, and hence

$$u^4 = \text{const} \tag{57}$$

as

$$m_0 = -G^\alpha u_\alpha, \tag{58}$$

in virtue of (55), [see (7, *1947*) p. 14]. All the following work will be done in this special system of coordinates. From (57) we conclude that

$$v^i v_i = \text{const}.$$

Three of the equations (55) take now the form

$$m_0 u^i + k^{jli} \dot{u}_j \dot{u}_l = 0.$$

Transforming from τ and u^i to t and v^i and putting

$$M = m_0 (\sqrt{1 - v^2})^3$$

we get:

$$M v^i + k^{jli} \frac{dv_j}{dt} \frac{dv_l}{dt} = 0. \tag{59}$$

Putting

$$l^{ik} = k^{mik}\frac{dv_m}{dt} \tag{60}$$

we get from (46), (54), (56) and (58)

$$l^{ik} = -l^{ki} = \text{const.}$$

Inserting (60) in (59) we obtain

$$Mv^i + l^{ik}\frac{dv_k}{dt} = 0,$$

equations of the same form as for a dipole particle of the first kind, their general solution was shown to be a uniform circular motion with angular velocity

$$\omega = \frac{Mc^2}{l}$$

and radius of the circle

$$r = \frac{l}{Mc}\frac{v}{c},$$

where l is the (3-dimensional) length of the axial vector (l^{23}, l^{31}, l^{12}) and v is the velocity of the particle on the circle.

I should like to express my thanks to Prof. Jan Weyssenhoff for suggesting this problem and many valuable discussions.

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THEORETICAL REMARKS ON CRANE AND HALPERN'S EXPERIMENTAL EVIDENCE FOR THE EXISTENCE OF THE NEUTRINO*

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(Received May 9, 1947)

The relation between the recoil energy of the nucleus in the process $Cl^{38} \xrightarrow{\beta} A^{38}$ and the number of droplets produced by this in a cloud chamber has been calculated on the assumption of Crane and Halpern that the number of droplets is approximately equal to the number of nitrogen and oxygen atoms dissociated in collision chains initiated by the recoil atom. The relation appears to be linear with two constants which have been evaluated.

The occurrence of several ions in clusters of droplets has been explained as caused by X-rays or Auger processes releasing the fraction of the total binding energy of the orbital electrons of the atom connected with the change of the nuclear charge during the beta decay.

Finally, statistical errors in Crane and Halpern's experiments have been discussed. The analysis leads to the conclusion that these experiments give a definitely positive answer in the question of the non-conservation of momentum in the «two-body» beta-decay.

Introduction

In 1944 A. P. Grinberg published in Russian an excellent survey *(1, 1944)* of the experimental evidence for the existence of a neutrino, written chiefly in connection with the important experiments of J. S. Allen *(2, 1942)* which he considered as a definitive discovery of this particle. In this report the former work on this topics by H. R. Crane and J. Halpern (abbreviated henceforth to: C&H) was also discussed and subjected to severe criticism *(3, 4, 1938; 5, 1939)*. Grinberg writes on their work in conclusion: «the distribution of their experimental points quite certainly cannot be regarded as probable». He is not isolated in his critical opinion of C.&H.'s method, although others views are not so extreme. Kan Chang Wang *(6, 1942)* writes: «...owing to the smallness of the

* Summary of the dissertation for the degree of M. A. (magister filozofii) presented before the Jagiellonian University, Kraków, January 20, 1946.

ionization effect of the recoil atom, it seems worth while to consider a different method of detecting it». Allen *(2, 1942)* states: «Since little is known about the energy range relation of very slow atoms in gases, the momenta of the recoil nuclei could not be measured with any accuracy. Although this method was somewhat refined in another experiment, no very definite evidence regarding the neutrino was discovered» [1]. Even C.&H. recognize *(5, 1939)* that: «The principal unknown in the method is, and always has been, the relation between the number of droplets and the energy of the recoil atom». Provoked by Grinberg's article I undertook an attempt to investigate this «principal unknown» in the method of C.&H. from a theoretical point of view. After Allen's work the analysis seems to be important not only as looking for a confirmation, but also since the C.&H.'s method can give theoretically more information than that of Allen, namely the distribution of angles between the directions of emission of the neutrino and the electron, which is of great importance for the Fermi theory of beta decay.

In the course of the investigation it became apparent to me that the mechanism of ion creation in C.&H.'s experiments is not quite clear even after their reply to the criticism of L. Wertenstein *(9, 1938)* on this subject. This problem will be discussed in the section «The origin of ions». It also seems essential to investigate in detail the statistical errors in the method of C.&H.

The recoil energy as a function of the number of droplets

To obtain theoretically this relation we assume like C.&H. *(5, 1939)* that in their experiments the formation of droplets was initiated in general by a dissociation of nitrogen or oxygen molecules in collisions caused by the recoil of the nucleus. We suppose for the sake of simplicity that every such dissociated atom originates a droplet and that the presence in the chamber of other gases besides air can be neglected in the theoretical treatment of collision processes. Both these assumptions seem to be plausible in the face of the experimental conditions and some results of C.&H.

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[1] E. J. Konopinski in his report on beta decay in the Rev. of Mod. Phys. *(7, 1943)* writes only: «More satisfyingly direct observation of the neutrino was undertaken by Crane and Halpern and Allen. These investigators attempted to observe the recoils of nuclei from neutrino emission. Allen's work seems the most nearly conclusive». Das Gupta and Ghosh *(8, 1946)* qualify the C.&H. experiments as «of limited accuracy», although they estimate their conclusions as «highly probable».

The recoil energy of a nucleus not exceeding c. 450 eV, we are mainly interested in the order of energy 100 eV for impinging molecules in collisions. Struck gas molecules can be considered as at rest by reason of the relative smallness of their thermal energies (10^{-2} eV). The time of such a collision is c. 10^{-14} sec whereas the periods of quantum processes in nitrogen or oxygen molecules which can be excited by collision to higher energy levels have orders: 10^{-16} sec for the (classical) rotation of the orbital electrons, 10^{-14} sec for the oscillation of the atoms in the molecule, and 10^{-12} sec for the rotation of the molecule as a whole. We see, after C. & H. *(5, 1939)*, that a resonance is only possible with the oscillatory energy level, which leads to the dissociation of the molecule, and that other excitations can be neglected. We can therefore dispense from an exact quantum discussion of collision problems, which would involve almost insuperable calculation difficulties.

The argon atom as well the homopolar molecules of nitrogen and oxygen can be considered in the first approximation as spheres. The familiar classical theory of inelastic collision of two spheres with absorption of the energy of dissociation shows that this absorption is only then possible when the angle ϑ between the direction of the impinging molecule and the line of centers of the molecules in the moment of impact does not exceed a certain maximum value i. e. when

$$\cos\vartheta \geqslant \sqrt{\frac{M+m}{m}\frac{D}{E}},$$

where M and m denote respectively the masses of the impinging and the struck molecules, E kinetic energy of the impinging molecule before the collision and D dissociation energy. We see that the maximum value of ϑ (or the minimum one of $\cos\vartheta$) depends upon E and this fact creates the «classical» dependence of the pissociation cross-section upon E, which must be taken into consideration even in our narrow range of energy (where the above mentioned «quantum» dependence of this cross-section upon E mav be neglected).

It should be stressed that the part of a dissociating agent is played not only by the recoil atom itself but also by all gas molecules, dissociated or not, which receive in collisions sufficient kinetic energy to dissociate in turn other modecules of air. Every recoil atom produces chains of collisions branching forth like a genealogical tree (or, e. g. a cosmic ray shower). These collision chains can be regarded as terminated when the kinetic energy of the colliding molecules drops to thermal level, but we are not interested beyond the point where further dissociation processes are not more possible. There are

10 types of collisions which may lead to dissociation: $A - N_2$, $A - O_2$, $N - N_2$, $N - O_2$, $O - N_2$, $O - O_2$, $N_2 - N_2$, $N_2 - O_2$, $O_2 - N_2$, $O_2 - O_2$. As calculation shows, the last four types of collisions may be neglected since — in the energy range considered — they produce no dissociation at all.

The following items were computed succesively:

(1) the dissociation cross-section for the above mentioned 10 types of collisions as a function of E,

(2) the average energies of all products of these collisions, also as functions of E, and

(3) on the basis of (1) and (2) the average shape of the «collision genealogical tree» for recoil atoms with energies[2]: 25, 100, 200, 300 and 400 eV. Finally, the numbers of dissociated atoms of nitrogen and oxygen were summed up for every «tree».

As dissociation energies were assumed: 9.5 eV for nitrogen, and 6.2 eV for oxygen.

Table I gives the results of these somewhat tedious and lengthy (over 100 pages) but straightforward calculations.

Table I

Energy of A^{38} (eV) (E)	Average Number of Collisions in a «Tree»	Average Number of Dissociated Atoms			Average Energy Spent (E/N) (eV/diss. atom)
		of Nitrogen	of Oxygen	Total (N)	
1	2	3	4	5	6
25	1	0.3	0.2	0.5	50
100	10	4.0	1.6	5.6	18
200	23	9.3	3.6	12.9	15.5
300	41	14.4	5.8	20.2	14.8
400	47	20.8	7.8	28.6	14.0

The numbers of columns 3, 4 and 5 are presented graphically in Fig. 1, those of column 6 in Fig. 2. We see that the points in Fig. 1 lay on straight lines not through the origin of coordinates (there-

[2] The calculation for 25 eV was made chiefly for the purpose of checking. The results are in this case not so certain as for other value of E since then the time of collision is 10^{-13} sec and an excitation of the rotatory spectrum of the struck gas molecule it is possible in some degree. It is clear therefore that in reality the dissociation effect will be on an average still smaller than indicated in the table, and the energy spent per diss. atom larger. But since, in any case, the mean number of dissociated atoms is under 1, the loss in accuracy is of little importance for our problem.

fore, the points in Fig. 2, which indicate the ratio E/N, lay on a hyperbola). Thus we have obtained the solution of our problem in the form

$$E = aN + b,$$

where a and b are constants. Evaluating them from the graph we get

$$E = 13.3\ N + 22, \qquad (1)$$

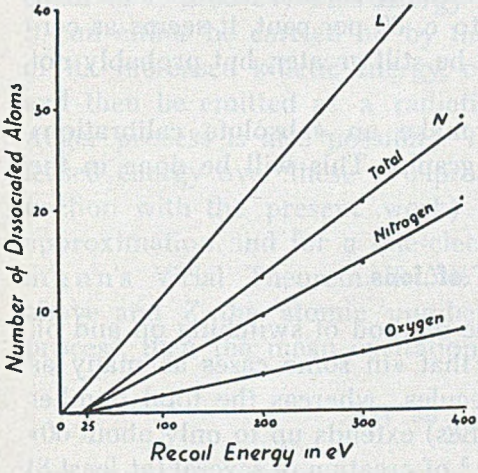

Fig. 1

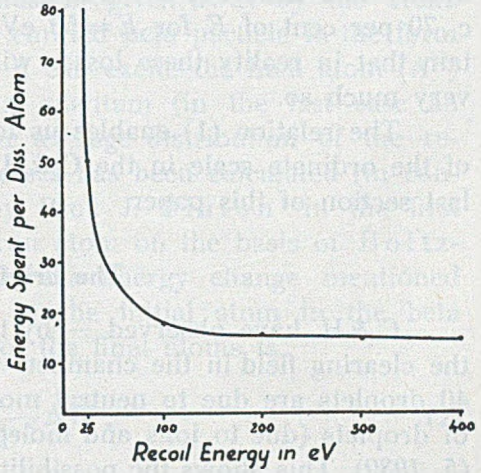

Fig. 2

where E is measured in eV. Hence, the energy spent per droplet is

$$\frac{E}{N} = 13.3 + \frac{22}{N} = 13.3 + \frac{292}{E - 22}.$$

This ratio is a function of E (or of N), but for large values of E it approaches the constant 13.3 eV/droplet. In the interval 300—500 eV the ratio is nearly 14 eV/droplet.

C. & H. assumed as a working hypothesis the linear proportionality of the relation. This includes the supposition that the free term $b = 0$. We see that for E of order 10^2 eV (or N of order 10^1) this approximation is not so bad in spite of Grinberg's criticism.

It is clear to-day — due to L. Wertenstein — that the first estimation of a by C. & H., namely 15 eV/droplet (*3, 1938*), was based on a false assumption. Nevertheless, we see that this value is very near ours, much nearer than that which was admitted hy C. & H. in their second paper (*5, 1938*), c. 8 eV/droplet, although in the last case the principle of determination seems quite correct. The question of this discrepancy will be discussed below in the section «Statistical errors».

The line L in Fig. 1 represents the number of dissociated atoms of N and O under the assumption that all the energy of the recoil is spent in dissociation and no fraction of it remains for elastic collisions and other losses. The mean value of D for air is c. 8.8 eV and thus the function to compute is equal directly to $E/8.8$ (E in eV). The differences of the ordinates of the lines L and T corresponding to the same abscissas represent the values of energy transformed ultimately into heat. For $E = 400$ eV the energy loss so computed equals c. 70 per cent of E, for $E = 50$ eV to c. 80 per cent. It seems as certain that in reality these losses will be still greater, but probably not very much so.

The relation (1) enables us to make an «absolute calibration» of the ordinate scale in the C.&H. graphs. This will be done in the last section of this paper.

The origin of ions

C.&H. have observed — by the method of switching on and off the clearing field in the chamber — that «in some cases as many as 40 droplets are due to neutral molecules, whereas the total number of droplets (due to ions and molecules) extends up to only about 60» *(5, 1939)*. This shows the possibility [3] of creation of several (at least 8) ion pairs by one recoil atom, in spite of the recognised argument of Wertenstein that this atom can produce only one ion pair [4].

To remove this difficulty I put here forward the idea that ions are produced as a result of the orbital radiation connected with the nuclear decay. By nuclear disintegration the charge of the nucleus is changed and, hence, the total binding energy of the electrons with the nucleus of the atom must be changed too. The electrons must pass in some way from the normal state of the original atom to the normal state of the final one. This phenomenon, so far as I know, has not yet been investigated, probably because of its weakness in comparison with the nuclear radiation.

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[3] From the shape of the curves on Fig. 3 and Fig. 4 (see below) we can infer that the difference is not caused by casual fluctuations in either set of observations.

[4] Grinberg *(1, 1944)* supposes that ions can result from Auger processes of gas atoms excited in collisions. But in the Auger processes electrons of the inner atom shells must be excited and in the case in question the energy required for so doing is not at our disposal (the binding energy of inner electrons of O or N is of the order 10^3—10^4 eV whereas the maximum energy of the recoil atom is c. 400 eV).

The approximate formula of E. A. Milne and A. B. Baker (see *10, 1933*, where further references are given) for the total binding energy W (in eV) of electrons in an atom with atomic number Z, namely

$$W = -20.8\, Z^{7/3},$$

leads in the case of the process $Cl^{38} \xrightarrow{\beta} A^{38}$ to a value of the change ΔW equal to c. 2200 eV. This energy must be given away by the atom. It can either be carried off by the emitted beta particle in the form of its increased kinetic energy, or it can excite the final atom (A^{38}) and then be emitted as a radiation quantum (in the last case an Auger process is also possible). The average distribution of the released energy over these two processes has been calculated (in connection with the present work) by Prof. J. Blaton[5] in the first approximation and for a one-electron atom on the basis of Boltzmann's Virial Theorem. If ΔW is the energy change mentioned above and Z the atomic number of the initial atom in the beta process, then the mean excitation of the final atoms is

$$\Delta W' = \frac{1}{2Z+1}\Delta W\,, \qquad (2)$$

whereas the emitted beta particle gets on the average the energy

$$\Delta W'' = \frac{2Z}{2Z+1}\Delta W\,.$$

For an atom with 17 electrons, such as that of chlorine, Blaton's result can give only a very rough evaluation (when we choose in a suitable way a «screen number» to diminish the value of Z in the formula (2)). Probably, for large atoms some methods of approximation (of Thomas-Fermi or of Hartree) with the aid of machine integration might give a second approximation for the problem. But, since relatively large statistical fluctuations (see next section) lay in the very nature of the experiment discussed, it is not worth while making a precise evaluation of $\Delta W'$, and for our purpose a half-quantitative estimation on the basis of Blaton's formula suffices. We diminish Z by the average of the screen numbers of all electrons of the chlorine atom, namely by c. 5, and from (2) we obtain for $\Delta W'$ c. 1×10^2 eV. It is the mean for a large number of disintegration processes.

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[5] These calculations were kindly given to me in manuscript.

Table II

Limits of Spectral Series of Argon

Series	Energy		Wave length (Å)	For normal air	
	(Ry)	(eV)		number of ion pairs produced	mean range (mm)
1	2	3	4	5	6
K	237	3200	3.84	106	5×10^{1}
L	18	244	50.6	8	2×10^{-2}
M	0.6	8	1545	—	—

Table II gives the data of possible excitations of the X-ray spectrum of an argon atom and of the effects of respective radiations in normal air (the figures of column 2 are interpolated from (*11, 1926*), the others are calculated on their basis). The ionization of air by an X-ray photon is composed in general of one photo-effect (with absorption of the ionization energy amounting to c. 16.7 eV in the average, i. e. 16.7 eV/ion pair) and many ionizations by secondary electrons with average energy spent of 30 eV/ion pair. Therefore we may adopt rather the latter value for the calculation of the figures in column 5.

We see that in our phenomenon an excitation of the K-series is impossible (since the total ΔW is only 2200 eV), but the following series can be excited. The excitation of the L-series can produce 8 ion pairs in air. It seems that agreement of this number with the experimental result of C.&H. mentioned at the beginning of this section is not accidental and that it supports our hypothesis. The range in air of the respective X-ray photon (2×10^{-2} mm) lies quite within the limits of the dimensions of the clusters of droplets observed by C.&H. (radii of 1—2 mm after diffusion of the ions).

As mentioned above, on the average over a great number of beta decays the excitation energy of the final atom is c. 100 eV and after releasing it gives probably in air about 3 ion pairs or 6 droplets. This figure we take as the first approximation of the average number of ions produced by the phenomenon discussed, which we may call e. g., the *nucleo-electronic* effect. It is obvious that this effect has no connection with the recoil of the nucleus after disintegration and that the observed number of droplets in a cluster should be reduced by the obtained figure, i. e. 6, to get the value of N which is connected with the recoil energy.

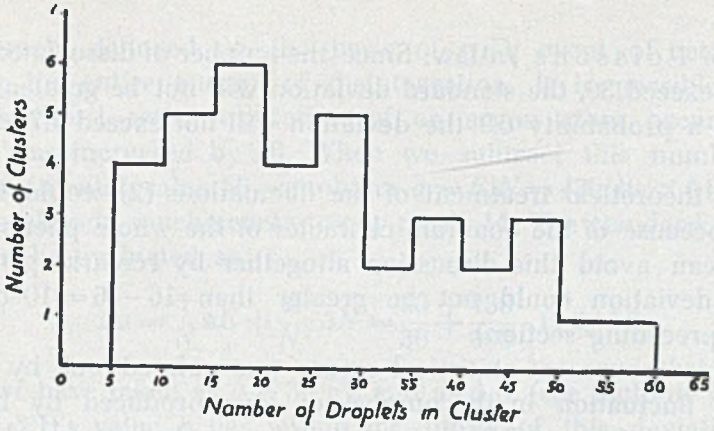

Fig. 3

It we compare the statistics of the droplet numbers in the clusters observed by C. & H. containing ions with that for clusters without ions (Fig. 3 and Fig. 4), we see that the displacement of the maxima of the respective curves corresponds to about 5 droplets. This fact seems also to support our aforesaid result. It must, however, be stressed that Fig. 4 is probably distorted by the existence of the lower energy component in the chlorine beta-ray spectrum and this circumstance considerably diminishes the strength of the last evidence.

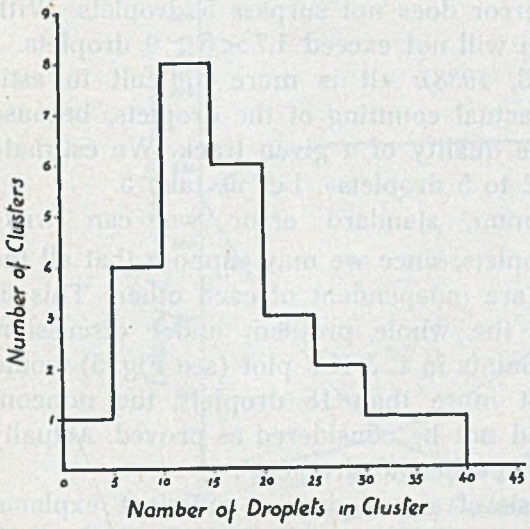

Fig. 4

Statistical errors

The preceding considerations show the existence of two sources of possible deviations in the result of the experiment:

(1) fluctuations in the number of dissociated nitrogen and oxygen atoms, and

(2) fluctuations in the number of ion pairs produced by the nucleo-electronic effect.

The first deviations are of a purely «classical» type known from kinetic theories (they are caused by fluctuations in number and types of collisions) and can therefore well be estimated with

the aid of Poisson's $\sqrt{n}$-law. Since the number of dissociated atoms does not exceed 30, the standard deviation will not be greater than 6, and with a probability 0.9 the deviation will not exceed $1.7 \times 6 \approx 10$ droplets.

The theoretical treatment of the fluctuations (2) would be very difficult because of the quantum character of the whole phenomenon, but we can avoid this discussion altogether by remarking that the relevant deviation could not be greater than $16 - 6 = 10$ droplets (see the preceding section).

A third source of statistical errors was pointed out by C. & H. *(3, 1938)*: fluctuation in the number of ions produced by beta-ray electrons per 1 cm of their path within a sphere over which the droplets due to the nucleus extend. This error is equal to about $\sqrt{\rho R}$, where ρ denotes the linear density of droplets along the electron track and R the radius of the sphere. From C. & H.'s graph *(3, 1938)* we see that this standard error does not surpass 5 droplets. With a probability 0.9 the deviation will not exceed $1.7 \times 5 \approx 9$ droplets.

Finally, C. & H. write *(3, 1938)*: «It is more difficult to estimate the uncertainty in the actual counting of the droplets, because this depends largely upon the quality of a given track. We estimate that this error ranges from 2 to 5 droplets». Let us take 5.

As a resulting maximum standard error we can write $\sqrt{10^2 + 10^2 + 9^2 + 5^2} \approx 18$ droplets, since we may suppose that all the sources of errors mentioned are independent of each other. This figure seems very essential to the whole problem under discussion. Indeed, if the observational points in C. & H.'s plot (see Fig. 5) would deviate from curve 2 by not more than 18 droplets, the nonconservation of momentum could not be considered as proved. Actually in many cases this deviation reaches 40 droplets [6].

It seems that our analysis of errors gives a sufficient explanation for the discrepancy between the value of a obtained theoretically by us and that determined experimentally by C. & H. The latter

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[6] To exclude the possibility of an accidental addition of errors of the types named, we can multiply the deviation of 18 droplets once again by 1.7, and obtain so the number of c. 30 droplets. The actual deviation will not exceed this value with a probability of 0.9 at least. 30 is still smaller than 40. It must be pointed out, however, that so large a deviation occurs in very exceptional cases only, and that the mean fluctuation (not the maximum possible mean fluctuation) can be estimated as

$$\sqrt{6^2 + 8^2 + 4^2 + 3^2} = c.11 \quad \text{for } N = 30 \text{ droplets, and as}$$

$$\sqrt{2^2 + 6^2 + 4^2 + 2^2} = c.8 \quad \text{for } N = 5 \text{ droplets.}$$

was namely deduced on the basis of one event of beta particle carrying the entire energy of disintegration. It is possible that in this case an L-series excitation of an argon atom occurred, and thus N was increased by 16. When we subtract this number from the experimental value 56 we obtain $a = E/N = 430/40 \approx 11$ and this figure is already much nearer to our result 14. The standard deviation of a can be evaluated as

$$\Delta a = \frac{1}{N}\Delta E + \frac{E}{N^2}\Delta N = \frac{30}{30} + \frac{430}{900} \cdot 11 \approx 6.5,$$

where we have taken as ΔE 30 eV and as ΔN (see footnote [6]) 11 droplets. C. & H's value 8 lies within the limits of this deviation from our result 14.

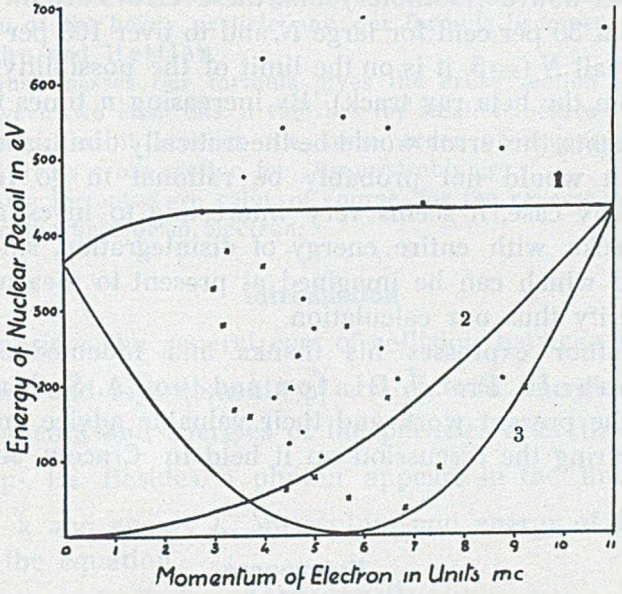

Fig. 5

Conclusions

In Fig. 5 are presented the experimental results of C. & H. as interpreted by our analysis. The numbers of droplets given by C. & H. *(3, 1938), (5, 1939)* have been diminished by 6 (the mean result of the nucleo-electronic effect) and from the values so obtained the relative recoil energies E have been calculated by the aid of formula (1).

We see that only 7 points lie above the upper curve and that of these all are contained within the limits of a deviation of 18 droplets (260 eV) from this curve.

We thus come to the conclusion that, if our preceding considerations are correct, C. & H's deductions from their experiments of the non-conservation of momentum in the «two-body» beta disintegration seem to be well founded in spite of criticisms mentioned at the beginning of this paper.

However, the above quantitatively determined possibility of large statistical fluctuations in the experiments shows that the drawing of conclusions about the statistics of angles between the directions of emission of the neutrino and the electron from so small a number of experimental points (35) is very uncertain. In this respect C. & H. are wholly correct when they write *(5, 1939):* «It is not safe to attach much significance to the results obtained on this aspect of the problem, because of the possibility of rather large experimental errors». We have seen above (footnote [6]) that these errors are on an average equal to about 30 per cent for large N, and to over 100 per cent (150% or so) for small N ($=5$, it is on the limit of the possibility distinguish a cluster from the beta ray track). By increasing n times the number of measurements, this error would be theoretically diminished $\sqrt{n}$ times. In practice it would not probably be rational to go further than $n = 16$. In any case, it seems very interesting to investigate further the beta tracks with entire energy of disintegration, since it is the only method which can be imagined at present to measure the ratio a and to verify thus our calculation.

The author expresses his thanks and indebtedness to Prof. J. Weyssenhoff, Prof. J. Blaton and Prof. A. Sołtan for their interest in the present work and their valuable advice and opinions expressed during the discussion on it held in Cracow, June 6, 1945.

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RADIATIVE COLLISIONS BETWEEN TWO ELECTRONS

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(Received Mai 15, 1947)

The differential cross section is calculated for a radiative collision between two particles. The assumptions are: Both particle obey the Dirac equation and their mutual interaction is static.

In the limiting case when the mass of one of the particles tends to infinity we get the well known formula of Bethe and Heitler *(1, 1934)*. However, in the case of great energies it is not allowed to neglect the change of momentum of the heavy particle and our formula becomes different from that of Bethe and Heitler.

For equal masses our formula gives the cross section for a radiative collision between two electrons. It vanishes for small velocities of the particles. For large velocities it reaches the same order of magnitude as that of Bethe-Heitler's cross section for „Bremsstrahlung".

The same formulae are valid, of course, for the reverse process of pair production in the field of an electron.

Introduction

We consider the general case of collision between two particles of masses m_1 and m_2, momenta $\vec{p}_1^0$ and $\vec{p}_2^0$, and energies E_1^0 and E_2^0. Let the momenta and energies of the particles after the collision be $\vec{p}_1$, E_1 and $\vec{p}_2$, E_2. Besides, a photon appears in the final state with momentum $\vec{k}$ and energy k. Momentum and energy of a particle are related by the equation

$$E^2 = p^2 + \mu^2. \tag{1}$$

For p we choose energy units, μ is the rest energy of the particle. Both particles obey the Dirac equation.

Cross section

Considering the interaction H between particle and radiation and V between two particles as small we can calculate the cross section by the usual perturbation method *(2, 1944)*. Since the matrix elements of the direct transition between the initial and the final state vanish, the process must occur through some intermediate states.

There are four possibilities:

1) Firstly m_1 emits $\vec{k}$ and gets the momentum $\vec{p}'_1$ and then it interacts with m_2 and gets the final momentum $\vec{p}_1$, while m_2 is going over to the final state with the momentum $\vec{p}_2$. The matrix elements corresponding to these transitions vanish unless momentum is conserved. Thus

$$\vec{p}_1^{\,0} = \vec{p}'_1 + \vec{k}\,, \qquad \vec{p}'_1 + \vec{p}_2^{\,0} = \vec{p}_1 + \vec{p}_2\,. \tag{2}$$

2) m_1 firstly interacts with m_2 and then emits $\vec{k}$. The corresponding conservation laws are

$$\vec{p}_1^{\,0} + \vec{p}_2^{\,0} = \vec{p}''_1 + \vec{p}_2\,, \qquad \vec{p}'_1 = \vec{p}_1 + \vec{k}\,. \tag{3}$$

3) and 4) The roles of the particles m_1 and m_2 are interchanged. The conservation laws are

$$\vec{p}_2^{\,0} = \vec{p}'''_2 + \vec{k}\,, \qquad \vec{p}'''_2 + \vec{p}_1^{\,0} = \vec{p}_2 + \vec{p}_1\,; \tag{4}$$

$$\vec{p}_2^{\,0} + \vec{p}_1^{\,0} = \vec{p}_2^{\,IV} + \vec{p}_1\,, \qquad \vec{p}_2^{\,IV} = \vec{p}_2 + \vec{k}\,. \tag{5}$$

Elimination of the momentum of the particle in the intermediate state from the above pairs of equations (2, 3, 4, 5) shows that momentum is conserved in the transitions between the initial and the final states

$$\vec{p}_1^{\,0} + \vec{p}_2^{\,0} = \vec{p}_1 + \vec{p}_2 + \vec{k}\,. \tag{6}$$

If we neglect spin interaction between the particles the matrix elements corresponding to the transitions (2, 3, 4, 5) are (2, *1949*)

$$\left.\begin{aligned} H_{AF} &= -\,ehc\sqrt{2\pi/k}\,(u_1^{0*}\,\alpha_1\,u'_1) \\ V_{IF} &= \frac{4\pi\hbar^2c^2e^2}{|\vec{p}'_1 - \vec{p}_1|^2}(u'^{*}_1 u_1)(u_2^{0*} u^2) \\ H_{IIF} &= -\,ehc\sqrt{2\pi/k}\,(u''^{*}_1\,\alpha_1\,u_1) \\ V_{AII} &= \frac{4\pi\hbar^2c^2e^2}{|\vec{p}_1^{\,0} - \vec{p}''_1|^2}(u_1^{0*} u''_1)(u_2^{0*} u_2) \end{aligned}\right| \quad \begin{aligned} &\text{and similar matrix elements} \\ &\text{for the other two possibilities} \\ &\text{with the roles of the particles} \\ &1 \text{ and } 2 \text{ interchanged,} \end{aligned} \tag{7}$$

where the u's are spin functions of free particles obeying the Dirac equation, $\vec{\alpha}$, β are the Dirac matrices and α is the component of $\vec{\alpha}$ in the direction of polarization. The u's are normalised as follows

$$\sum_{s=1}^{4} |u(s)|^2 = 1\,. \tag{8}$$

Hence, the matrix element H_{AF} of the transition from the initial to the final state is

$$H_{AF} = -4\pi(\hbar c e)^3\sqrt{2\pi/k}\left\{\frac{(u_2^{0*}u_2)}{|\vec{p}_2-\vec{p}_2^{\,0}|^2}\sum\left[\frac{(u_1^{0*}\alpha_1 u_1')(u_1'^{*}u_1)}{E_A-E_I}+\frac{(u_1^{0*}u_1'')(u_1''^{*}\alpha_1 u_1)}{E_A-E_{II}}\right]\right.$$
$$\left.+\frac{(u_1^{0*}u_1)}{|\vec{p}_1-\vec{p}_1^{\,0}|^2}\sum\left[\frac{(u_2^{0*}\alpha_2 u_2''')(u_2'''^{*}u_2)}{E_A-E_{III}}+\frac{(u_2^{0*}u_2^{IV})(u_2^{IV*}\alpha_2 u_2)}{E_A-E_{IV}}\right]\right\}. \tag{9}$$

The summation concerns both signs of the energy and both positions of the spin possible in the intermediate states. A simple calculation yields

$$H_{AF} = 4\pi(\hbar ce)^3\sqrt{2\pi/k}\cdot\frac{1}{2k}\left\{\frac{(u_2^{0*}u_2)}{|\vec{p}_2-\vec{p}_2^{\,0}|^2}\left[\frac{(u_1^{0*}M_1u_1)}{E_1^0-p_1^0\cos\Theta_1^0}-\frac{(u_1^{0*}N_1u_1)}{E_1-p_1\cos\Theta_1}\right]\right.$$
$$\left.+\frac{(u_1^{0*}u_1)}{|\vec{p}_1-\vec{p}_2^{\,0}|^2}\left[\frac{(u_2^{0*}M_2u_2)}{E_2^0-p_2^0\cos\Theta_2^0}-\frac{(u_2^{0*}N_2u_2)}{E_2-p_2\cos\Theta_2}\right]\right\}, \tag{10}$$

where

$$\begin{aligned}
M_1 &= 2(\vec{p}_1^{\,0}\vec{n})-k(\vec{\alpha}_1\vec{n})+(\vec{\alpha}_1\vec{k})(\vec{\alpha}_1\vec{n}),\\
N_1 &= 2(\vec{p}_1\vec{n})+k(\vec{\alpha}_1\vec{n})+(\vec{\alpha}_1\vec{k})(\vec{\alpha}_1\vec{n}),\\
M_2 &= 2(\vec{p}_2^{\,0}\vec{n})-k(\vec{\alpha}_2\vec{n})+(\vec{\alpha}_2\vec{k})(\vec{\alpha}_2\vec{n}),\\
N_2 &= 2(\vec{p}_2\vec{n})+k(\vec{\alpha}_2\vec{n})+(\vec{\alpha}_2\vec{k})(\vec{\alpha}_2\vec{n}),
\end{aligned} \tag{11}$$

$$\begin{aligned}
\Theta_1 &= \sphericalangle(\vec{p}_1\vec{k}), \qquad & \Theta_1^0 &= \sphericalangle(\vec{p}_1^{\,0}\vec{k}),\\
\Theta_2 &= \sphericalangle(\vec{p}_2\vec{k}), \qquad & \Theta_2^0 &= \sphericalangle(\vec{p}_2^{\,0}\vec{k}),
\end{aligned} \tag{12}$$

and $\vec{n}$ is the unit vector in the direction of polarization.

Let us consider the case where one of the particles, say m_2, is initially at rest ($\vec{p}_2^{\,0}=0$). In such a system the cross section is

$$d\Phi = \frac{2\pi}{\hbar}\cdot\frac{E_1^0}{cp_1^0}|H_{AF}|^2\rho_F, \tag{13}$$

where $\frac{cp_1^0}{E_1^0}$ represents the velocity of the incident particle and ρ_F is the density of states in the final state. As we have three particles in the final state, ρ_F will be of the form

$$\rho_F = \rho_{E_1}dE_1\,\rho_k\left(\frac{\partial k}{\partial E_F}\right) = \frac{p_1E_1k^2dE_1}{(2\pi\hbar c)^6}\cdot\frac{E_2}{E_2-p_2\cos\Theta_2}\,d\Omega_1\,d\Omega_k, \tag{14}$$

with

$$E_F = E_1+E_2+k.$$

The square of the modulus of H_{AF} has still to be summed over the spin directions of the final state, averaged over the spin directions of the initial state, and summed over the polarisations of $\vec{k}$. After a simple but rather tedious calculation, putting everywhere $\vec{p}_2^{\,0}=0$, we get

$$
\begin{aligned}
d\Phi = \frac{e^4}{8\pi^2}\left(\frac{e^2}{hc}\right)\cdot\frac{p_1}{p_1^0}\cdot\frac{1}{k}\Bigg|\left(1+\frac{\mu_2}{E_2}\right)\frac{1}{p_2^4}\Bigg[\frac{4E_1(E_1^0-k)+(E_2-\mu_2)^2-p_2^{\,2}}{(E_1^0-p_1^0\cos\Theta_1^0)^2}\,p_1^{0\,2}\sin^2\Theta_1^0 \\
+\frac{4E_1^0(E_1+k)+(E_2-\mu_2)^2-p_2^{\,2}}{(E_1-p_1\cos\Theta_1)^2}\,p_1^2\sin^2\Theta_1 \\
-2\,\frac{4E_1^0E_1+(E_1^0-E_1)^2+k^2-p_2^{\,2}}{(E_1^0-p_1^0\cos\Theta_1^0)(E_1-p_1\cos\Theta_1)}\,p_1^0p_1\sin\Theta_1^0\sin\Theta_1\cos\varphi_1 \\
+2k^2\,\frac{p_1^{0\,2}\sin^2\Theta_1^0+p_1^2\sin^2\Theta_1}{(E_1^0-p_1^0\cos\Theta_1)(E_1-p_1\cos\Theta_1)}\Bigg] \\
+\frac{E_1^0E_1+(\vec{p}_1^{\,0}\vec{p}_1)+\mu_1^2}{|\vec{p}_2+\vec{k}|^4}\Bigg[\frac{4\mu_2(E_2+k)+(E_1^0-E_1)^2-(\vec{p}_1^{\,0}-\vec{p}_1)^2}{\mu_2E_2(E_2-p_2\cos\Theta_2)^2}\,p_2^2\sin^2\Theta_2 \\
+2k^2\,\frac{p_2^2\sin^2\Theta_2}{\mu_2^2E_2(E_2-p_2\cos\Theta_2)}\Bigg]+\frac{1}{p_2^2|\vec{p}_2+\vec{k}|^2}\left(\frac{2E_2+2\mu_2+k}{E_2(E_2-p_2\cos\Theta_2)}\right. \\
\left.+\frac{k}{\mu_2E_2}\right)\Bigg[\Bigg(\frac{2E_1^0E_1+2(\vec{p}_1^{\,0}\vec{p}_1)+2\mu_1^2+E_1^0k+(\vec{p}_1^{\,0}\vec{k})}{E_1-p_1\cos\Theta_1}+k\Bigg)\,p_1p_2\sin\Theta_1\sin\Theta_2\cos\varphi_1^2 \\
-\Bigg(\frac{2E_1^0E_1+2(\vec{p}_1^{\,0}\vec{p}_1)+2\mu_1^2-E_1k-(\vec{p}_1\vec{k})}{E_1^0-p_1^0\cos\Theta_1^0}-k\Bigg)p_1^0p_2\sin\Theta_1^0\sin\Theta_2\cos\varphi_{1,0}^2\Bigg]\Bigg|\frac{E_2\,dE_1\,d\Omega_1\,d\Omega_k}{E_2-p_2\cos\Theta_2},
\end{aligned}
\tag{15}
$$

where φ_1, φ_1^2 and $\varphi_{1,0}^2$ are the angles between the planes $\vec{p}_1^{\,0}\times\vec{k}$ and $\vec{p}_1\times\vec{k}$, $\vec{p}_1\times\vec{k}$ and $\vec{p}_2\times\vec{k}$, $\vec{p}_1^{\,0}\times\vec{k}$ and $\vec{p}_2\times\vec{k}$ respectively. $d\Omega_1$ and $d\Omega_k$ are the elements of the solid angles about the directions $\vec{p}_1$ and $\vec{k}$:

$$
\begin{gathered}
\varphi_1=\sphericalangle(\vec{p}_1^{\,0}\times\vec{k},\ \vec{p}_1\times\vec{k}),\quad \varphi_1^2=\sphericalangle(\vec{p}_1\times\vec{k},\ \vec{p}_2\times\vec{k}),\quad \varphi_{1,0}^2=\sphericalangle(\vec{p}_1^{\,0}\times\vec{k},\ \vec{p}_2\times\vec{k}), \\
d\Omega_1=\sin\Theta_1\,d\Theta_1\,d\varphi_1,\qquad d\Omega_k=\sin\Theta_1^0\,d\Theta_1^0\,d\varphi_1.
\end{gathered}
\tag{16}
$$

Discussion

1) For $\mu_2\to\infty$ we should obtain the formula of Bethe and Heitler *(1, 1934)*. Indeed $\mu_2\to\infty$ leads to $E_2\to\infty$ and thus (15) becomes

$$d\Phi = \left(\frac{e^4}{4\pi^2}\right)\left(\frac{e^2}{hc}\right)\frac{p_1}{p_1^0}\cdot\frac{dE_1}{k}\cdot\frac{d\Omega_1\, d\Omega_k}{p_2^4}\Bigg|\frac{4E_1^2-p_2^2}{(E_1^0-p_1^0\cos\Theta_1^0)^2}p_1^{0\,2}\sin^2\Theta_1^0$$

$$+\frac{4E_1^{0\,2}-p_2^2}{(E_1-p_1\cos\Theta_1)^2}p_1^2\sin^2\Theta_1-2\frac{4E_1^0E_1+2k^2-p_2^2}{(E_1^0-p_1^0\cos\Theta_1^0)(E_1-p_1\cos\Theta_1)}p_1^0p_1\sin\Theta_1^0\sin\Theta_1\cos\varphi_1 \tag{17}$$

$$+2k^2\frac{p_1^{0\,2}\sin^2\Theta_1^0+p_1^2\sin^2\Theta_1}{(E_1^0-p_1^0\cos\Theta_1^0)(E_1-p_1\cos\Theta_1)}\Bigg|,$$

which is exactly the formula of Bethe and Heitler.

2) (15) becomes different from (17) if the mass of the second particle is large but finite ($\mu_2 = Mc^2$, M = mass of the proton). This difference becomes important when the momentum $\vec{p}_2$ taken over by the proton is of the same order of magnitude as the rest energy of the proton: $p_2 \geqslant Mc^2 \cong 1000$ MeV. Suppose all three particles have energies of the order of magnitude of Mc^2. Then, only the terms in the first bracket in (15) are of importance for the cross section, the terms in the second bracket being smaller by the factor

$$\left(\frac{E_1-p_1\cos\Theta_1}{E_2-p_2\cos\Theta_2}\right)^2\simeq\left(\frac{\mu_1}{\mu_2}\right)^2\left(\frac{\mu_1}{E_1}\right)^2,$$

and the terms in the third bracket by the factor $\frac{\mu_1}{\mu_2}\cdot\frac{\mu_1}{E_1}$. Thus we get

$$d\Phi=\frac{e^4}{8\pi^2}\left(\frac{e^2}{hc}\right)\cdot\frac{p_1}{p_1^0}\frac{dE_1}{k}\cdot\frac{E_2\,d\Omega_1\,d\Omega_k}{E_2-p_2\cos\Theta_2}\left(1+\frac{\mu_2}{E_2}\right)\frac{1}{p_2^4}\Bigg|\frac{4E_1(E_1^0-k)+(E_2-\mu_2)^2-p_2^2}{(E_1^0-p_1^0\cos\Theta_1^0)^2}p_1^{0\,2}\sin^2\Theta_1^0$$

$$+\frac{4E_1^0(E_1+k)+(E_2-\mu_2)^2-p_2^2}{(E_1-p_1\cos\Theta_1)^2}p_1^2\sin^2\Theta_1 \tag{18}$$

$$-2\frac{4E_1^0E_1+(E_1^0-E_1)^2+k^2-p_2^2}{(E_1^0-p_1^0\cos\Theta_1^0)(E_1-p_1\cos\Theta_1)}p_1^0p_1\sin\Theta_1^0\sin\Theta_1\cos\varphi_1+2k^2\frac{p_1^{0\,2}\sin^2\Theta_1^0+p_1^2\sin^2\Theta_1}{(E_1^0-p_1^0\cos\Theta_1^0)(E_1-p_1\cos\Theta_1)}\Bigg|$$

as the cross section for the «Bremsstrahlung» applying to the case of extremely great energies instead of Bethe-Heitler's cross section.

3) For $\mu_1 = \mu_2 = \mu$ we must consider two further possibilities:

a) Small velocities: In this case

$$p_1 << \mu, \qquad E_1 \cong \mu, \tag{19}$$

$$k = E_1^0 - E_1 + \mu - E_2 \cong \frac{p_1^{0\,2}-p_1^2}{2\mu} - \frac{p_2^{\,2}}{2\mu} << p_1. \tag{19'}$$

The subscript i means that (19) is correct for any of the energies E_1^0, E_1, E_2 or momenta $\vec{p}_1^{\,0}$, $\vec{p}_1$, $\vec{p}_2$. With (19), (15) becomes

$$d\Phi = \frac{e^4}{\pi^2}\left(\frac{e^2}{hc}\right)\frac{p_1}{p_1^0}\frac{dE_1}{k}\,d\Omega_1\,d\Omega_k\,\frac{[2(\vec{p}_2\vec{k})+k^2]^2}{p_2^4(\vec{p}_2+\vec{k})^4}(p_1^{0\,2}\sin^2\Theta_1 \tag{20}$$

$$+ p_1^2\sin^2\Theta_1 - 2p_1^0 p_1 \sin\Theta_1^0\sin\Theta_1\cos\varphi_1).$$

Our cross section for radiative collision between two slow electrons is smaller then Heitler's cross section (2, *1944, p.* 166) for «Bremsstrahlung» of slow electrons by the factor $\left(2\frac{k}{p_2}\cos\Theta_2\right)^2$. This factor is $<<$ 1 because of (19′).

b) Great velocities: We can neglect μ against E and p. E and p are then nearly equal. Formula (15) becomes

$$d\Phi = \frac{e^4}{8\pi^2}\left(\frac{e^2}{hc}\right)\frac{p_1}{p_1^0}\cdot\frac{dE_1}{k}\cdot\frac{E_2\,d\Omega\,d\Omega_k}{E_2-p_2\cos\Theta_2}\left|\frac{1}{p_2^4}\left[\frac{4E_1(E_1^0-k)p_1^{0\,2}\sin^2\Theta_1^0}{(E_1^0-p_1^0\cos\Theta_1^0)^2}+\frac{4E_1^0(E_1+k)p_1^2\sin^2\Theta_1}{(E_1-p_1\cos\Theta)^2}\right.\right.$$

$$-4\frac{E_1^0(E_1+k)+E_1(E_0^1-k)}{(E_1^0-p_1^0\cos\Theta_1^0)(E_1-p_1\cos\Theta_1)}p_1^0 p_1\sin\Theta_1^0\sin\Theta_1\cos\varphi_1$$

$$\left.+2k^2\frac{p_1^{0\,2}\sin^2\Theta_1^0+p_1^2\sin^2\Theta_1}{(E_1^0-p_1^0\cos\Theta_1)(E_1-p_1\cos\Theta_1)}\right]+2\frac{E_1^0E_1+(\vec{p}_0^{\,1}\vec{p}_1)}{|\vec{p}_2+\vec{k}|^4}\left[\frac{2\mu_2(E_2+k)-E_1^0E_1+\vec{p}_1^{\,0}\vec{p})}{\mu_2E_2(E_2-p_2\cos\Theta_2)}\right.$$

$$\left.+\frac{k^2}{\mu_2^2E_2(E_2-p_2\cos\Theta_2)}\right]p_2^2\sin^2\Theta_2+\frac{1}{p_2^2|\vec{p}_2+\vec{k}|^2}\left(\frac{2E_2+k}{E_2(E_2-p_2\cos\Theta_2)}\right. \tag{21}$$

$$+\left.\frac{k}{\mu_2E_2}\right)\left[\left(\frac{2E_1^0E_1+2(\vec{p}_1^{\,0}\vec{p}_1)+E_1^0k+\vec{p}_1^{\,0}\vec{k})}{E_1-p_1\cos\Theta_1}+k\right)p_1p_2\sin\Theta_1\sin\Theta_2\cos\varphi_1^2\right.$$

$$\left.\left.-\left(\frac{2E_1^0E_1+2(\vec{p}_1^{\,0}\vec{p}_1)-E_1k-(\vec{p}_1\vec{k})}{E_1^0-p_1^0\cos\Theta_1^0}-k\right)p_1^0p_2\sin\Theta_1^0\sin\Theta_2\cos\varphi_{1,0}^2\right]\right|.$$

It is impossible to integrate formula (21) elementary. The approximate methods seem to change very much the character of the formula. We will try therefore to draw some conclusions about the total cross section from the differential cross section without integration.

Because of $p\cong E$ the cross section (21) is large only for small angles Θ_1^0 and Θ_1. All the momenta of the final state will therefore be contained in a small solid angle about the direction of $\vec{p}_1^{\,0}$.

Further we can say, that the magnitude of (21) is of the same order as that of Bethe-Heitler's cross section for «Bremsstrahlung».

Indeed, the largest terms in the cross section for «Bremsstrahlung» are of the order $1/\mu^2$ multiplied by a certain factor. After integration of (21) we would obtain terms proportional to $1/f(E_i)$ with the same proportionality factor. Here $f(E_i)$ means a homogeneous function of the second order of the E's which is not equal to μ^2. This follows from a difference in the conservation energy equations. In the case of «Bremsstrahlung» we have $E_1^0 = E_1 + k$ and in our case we have $E_1^0 + \mu = E_1 + E_2 + k$. One factor has not yet been considered. The ratio of the density functions for both cases is $E_2/(E_2 - p_2 \cos\Theta_2) \cong 2\frac{E_2^2}{\mu_2^2}$. As $f(E_i)$ is of the same order of magnitude as E_2^2 the ratio of the two cross sections will be of the order of magnitude of unity. This is true for nuclei with the charge e. For heavy nuclei with the charge Ze the ratio of both cross section will, of course, be $1/Z^2$. The same result is obtained by Heitler *(2, 1944)* by means of the Williams-Weizsäcker method. However, the application of this method for two particles with equal masses is not fully justified.

We have not considered the exchange forces between the two electrons. This, however, would not affect the order of magnitude of the process either in case a) nor in b).

All our considerations are, for sufficiently large energies ($k > 4mc^2$) *(3, 1933)*, valid also in the case of the reverse process of pair production in the field of an electron *(4, 1945)*. This follows from the fact that in this process we have to deal with matrix elements conjugate complex to those of the primary process. We shall not write down the corresponding formulae, as the adaption of the formula for the process of pair production runs in exactly the same way as the adaptation of the «Bremsstrahlung» formula to the pair production in the field of a nucleus *(2, 1944)*.

To adapt formula (15) for experimental test the spin interaction and the exchange forces should be taken into account. This complicates the calculations very much but do not make them impossible, at least in case of the differential cross section. The total cross section can allways be evaluated numerically.

Such a formula could be tested experimentally if the intensity of the emitted radiation in the process of scattering of high energy electrons by free electrons would prove measurable.

I should like to thank Prof. Cz. Białobrzeski and Prof. W. Rubinowicz for many helpfull discussions on this work.

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ON SIMULTANEOUS INTERACTION OF SEVERAL FIELDS AND THE SELF-ENERGY PROBLEM

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(Received November 3, 1948)

The problem of simultaneous interaction of several fields is studied within the framework of the Heisenberg-Pauli formalism. For a suitable mixture of fields of the well known types the divergent parts of the self-energy terms may be eliminated. The main interest is devoted to the problem of vacuum charge fluctuations. It is found that the photon self-energy vanishes if a suitable mixture of charged spinor and scalar fields is assumed. The calculations are performed to the second order of approximation in e^2 only.

I. General Introduction

The well known inconsistencies encountered in the theory of quantized fields, e. g. the divergent expressions for the eigenvalues of certain observables (e. g. self-energy) and the non-existence of a solution of the time-dependent Schrödinger equation, may arise from the following facts: (1) the formalism of field quantization is erroneous, (2) the very idea of „field" is wrong, (3) the field models, hitherto discussed, are not suitable or, at least, incomplete.

All the attempts to modify either the formalism itself or the conception of field haye been fruitless as yet, and it seems not very much probable that any progress may be achieved in this direction without introducing a fundamentally new idea. As this idea is still lacking, it seems natural to investigate the third and simplest possibility. Thus, the question is whether it is possible to obtain a consistent theory without a fundamental change in the conception of the field (as a set of quantities $\varphi_i(xyzt)$ obeying some linear partial differential equations) and without any premature departure from the Heisenberg-Pauli formalism of field quantization, but simply by a suitable improvement of the model of the field.

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Recently Bohr[1] expressed the opinion that «it may be an extravagant wish to hope to have electron theory stand alone. It might be possible to obtain the cut-off from phenomena involving mesons, neutrons etc...». In agreement with this point of view, we suppose that the field models usually considered were incomplete in so far as the interactions between the various fields were taken separately into consideration instead of being investigated simultaneously. Of course, we do not know, as yet, all the types of fields occuring in nature, neither do we know the types of interactions between them. But still, using the already well known types of fields (spinor, scalar, vector, pseudoscalar etc...). and assuming various interactions between them, we may investigate whether the cut-off is actually possible.

As far as I know, the first (to some extent successful) attempt to proceed on these lines was done by Pais *(1, 1947)*, who introduced a coupling of a spinor field with a neutral meson field. The (logarithmically divergent) self-energy of the electron due to the interaction with a scalar field is negative and may compensate the self-energy produced by the interaction with the electromagnetic field. By a suitable choice of the interaction constant $f^2 = 2e^2$ the logarithmically divergent parts of both self-energies cancel. The deviations of the results of this theory from the predictions of the usual theory will be the smaller the larger is the mass constant of the meson field.

A straightforward extension of the procedure of Pais is needed in case of a charged meson field interacting with the electromagnetic field. In this case the self-energy of the charged particles is divergent quadratically. Thus, the introduction of an interaction with a scalar neutral field with a suitable interaction constant f is not sufficient. This would remove only the divergence of the highest order, while there would remain still a term divergent logarithmically. But, in this case we may introduce two further fields, one of a vector type, and one of a scalar type. Thus, we have three coupling constants at our disposal, which permit us to get rid of the logarithmic part of the self-energy too. This procedure removes the infinite self-energies of charged particles of any type, at least in the lowest order of the perturbation calculus in e^2 where self-energies occur.

The physical meaning of this procedure is obvious: in order to obtain a consistent theory the (charged) particles must be subject

[1] Pocono Conference of Physics, 30 March — 1 April 1948. Cited after unofficial notes by J. A. Wheeler.

to other forces of a non-electromagnetic nature. This was already anticipated in the classical theory, where the charged particles are not stable unless we introduce some cohesive forces of a non-electromagnetic nature. Indeed, no model of a charged particle may be consistently achieved in a frame of purely electromagnetic linear theory.

As well known, there is another difficulty which has no counterpart in the classical theory: In the theory of quantized fields neutral particles coupled with a charged field possess a (divergent) self-energy produced by the charge fluctuations of the vacuum. These vacuum polarization effects are equivalent to ascribing a proper mass to the neutral particles. This situation is particularly unpleasant in case of photons whose rest mass must be exactly zero. By obtaining in the course of the calculation a result different from zero (finite or infinite) for the rest mass of the photon, we encounter a paradox, since the formalism is known to be gauge invariant, and there is no way of putting in a non-zero value for the photon mass. The reason for this conspicuous inconsistency is obviously as follows: The proof of the gauge invariance of the formalism is based on the assumption that a solution of the equations exists. This is not true. The Schrödinger equation in case of the electron field interacting with the electromagnetic field has no solution and, consequently, the Maxwell equations in their quantized form are not valid, either.

Responsible for the non-solubility of the field equations is the operator of charge and current j^{μ}_{el} (as the Maxwell equations possess a solution in the interaction-free case). Hence, we may hope that a suitable modification of this vector operator may improve the situation. The only straightforward possibility is to convert the electromagnetic charge and current operator into an operator for several fields:

$$j^{\mu} = j^{\mu}_1 + j^{\mu}_2 + \ldots,$$

where $j^{\mu}_n, n = 1, 2 \ldots$ are the usual operators for spinor, scalar, vector, ... fields with arbitrary mass constants m_n and coupling constants e_n.

If we succeed in combining several charged fields in such a way that the self-energy of the photon interacting with the common charge j^{μ} becomes exactly zero, then we shall possess also some necessary conditions for the existence of the solution of the Schrödinger equations. In the next sections we shall show that this is actually possible for a mixture of spinor and (charged) scalar fields.

II. Charge Fluctuations of a Spinor Field

The Schrödinger equation for the electromagnetic field interacting with a spinor field is

$$i\frac{\partial \Phi}{\partial t} = \left(\overline{H}_0 + \overline{H}'\right)\Phi, \tag{1}$$

where H_0 is the Hamilton operator for the interaction-free case, H' is the energy density of interaction, $\overline{H} = \int d^3x H$,

$$H' = -s_\mu A_\mu, \tag{2}$$

where A_μ is the four-vector potential of the electromagnetic field, and s_μ is the four-vector of charge and current density

$$s_\mu = -\frac{ie}{2}\left(\bar{\psi}\gamma_\mu\psi - \psi\tilde{\gamma}_\mu\bar{\psi}\right), \tag{3}$$

where $\bar{\psi} = \psi^*\gamma_4$ and $\tilde{\gamma}^{(\mu)}_{\alpha\beta} = \gamma^{(\mu)}_{\beta\alpha}$. This form of the current four-vector is symmetric in positons and negatons.

Following Tomonaga (*2, 1947*) and Schwinger (*3, 1948*), we transform the Schrödinger equation to the «interaction representation» by putting

$$\Phi \to e^{-i\overline{H}_0 t}\Phi, \qquad \overline{H}' \to e^{-i\overline{H}_0 t}\overline{H}' e^{i\overline{H}_0 t}.$$

This is a mixed Heisenberg-Schrödinger representation in which both the Schrödinger functional Φ and the field quantities ψ_σ, A_μ are time-dependent. The Schrödinger equation becomes

$$i\frac{\partial \Phi}{\partial t} = \overline{H}'\Phi \tag{4}$$

while the field quantities obey the same commutation relations and field equations as in the interaction-free case:

$$\Box A_\mu = 0, \qquad [A_\mu(x),\ A_\nu(x')] = \frac{1}{i}\delta_{\mu\nu} D(x-x'), \tag{5a}$$

$$\left(\gamma_\mu\frac{\partial}{\partial x_\mu} + \varkappa\right)\psi = 0, \quad [\psi_\alpha(x), \bar{\psi}_\beta(x')]_+ = iS_{\alpha\beta}(x-x') = i\left(\gamma^{(\mu)}_{\alpha\beta}\frac{\partial}{\partial x_\mu} - \delta_{\alpha\beta}\varkappa\right)\Delta(x-x'). \tag{5b}$$

Here

$$\Delta = \frac{1}{(2\pi)^3}\int \frac{e^{i\bar{k}\bar{x}}\sin\sqrt{k^2+\varkappa^2}\,t}{\sqrt{k^2+\varkappa^2}}\,d^3k, \qquad D = \frac{1}{(2\pi)^3}\int \frac{e^{i\bar{k}\bar{x}}\sin k\,t}{k}\,d^3k \tag{6}$$

are the invariant Jordan-Pauli delta-functions. It is often convenient to split these functions into positive and negative frequency parts, e. g.

$$D^+ = D_1 - iD, \qquad D^- = D_1 + iD, \tag{7}$$

where

$$D_1 = \frac{1}{(2\pi)^3}\int d^3k \frac{e^{ik\bar{x}} \cos kt}{k}. \tag{8}$$

The field quantities may be also split into positive and negative energy parts, e. g.

$$\psi = \psi^+ + \psi^-. \tag{9}$$

The commutation relations for these parts are:

$$\begin{aligned} [\psi_\alpha^+(x), \bar{\psi}_\beta^+(x')]_+ &= -\frac{1}{2} S_{\alpha\beta}^+(x-x'), \\ [\psi_\alpha^-(x), \bar{\psi}_\beta^-(x')]_+ &= \frac{1}{2} S_{\alpha\beta}^-(x-x'), \\ [\psi_\alpha^+(x), \bar{\psi}_\beta^-(x')]_+ &= 0. \end{aligned} \tag{10}$$

The $S_{\alpha\beta}^+$, $S_{\alpha\beta}^-$ functions are, of course, positive and negative frequency parts of $S_{\alpha\beta}$ defined by (5b). The field equations and the commutation relations are obviously Lorentz invariant in this representation. A generalization of (4) consisting in replacing the planes $t = \text{const}$ by arbitrary space-like surfaces converts also the Schrödinger equation into a Lorentz-invariant form discovered by Tomonaga. But we shall not make use of this generalization in the course of our calculations.

In order to calculate the self-energies, we introduce a unitary transformation e^S, $\Phi \to e^S \Phi$, and obtain

$$i\frac{\partial \Phi}{\partial t} = \left\{ -i e^{-S} \frac{\partial}{\partial t} e^S + e^{-S} \bar{H}' e^S \right\} \Phi. \tag{11}$$

This yields, to the second order of approximation,

$$i\frac{\partial \Phi}{\partial t} = \left\{ -i\dot{S} - \frac{i}{2}[\dot{S}, S] + \bar{H}' + [\bar{H}', S] \right\} \Phi. \tag{12}$$

The condition for vanishing of the first order term is

$$i\dot{S} = \bar{H}' = -\int s_\nu A_\nu \, d^3x, \tag{13}$$

and the eq. (12) becomes

$$i\frac{\partial}{\partial t}\Phi = \frac{1}{2}[\bar{H}, S]\Phi. \tag{14}$$

From (13) we obtain

$$S = i\int^{t} dx_0 \int s_\nu(x) A_\nu(x) d^3x \tag{15a}$$

with an arbitrary lower limit of integration. A very convenient form of the S matrix, introduced by Schwinger, is half the sum of two such integrals with the lower limits of integration + and — infinity:

$$S = \frac{i}{2}\left(\int_{-\infty}^{t} dt + \int_{+\infty}^{t} dt\right)\int s_\nu(x) A_\nu(x) d^3x. \tag{15b}$$

Now we may write

$$[H', S] = -i\int_{\pm\infty}^{t} dx'_0 \int d^3x' [s_\nu(x) A_\nu(x),\ s_\mu(x') A_\mu(x')], \tag{16}$$

where $\int_{\pm\infty}^{t} dx'_0$ is used as short for half the sum of the time integrals (15b). The commutator in this integral yields

$$[s_\nu A_\nu, s'_\mu A'_\mu] = [s_\nu, s'_\mu] A_\nu A'_\mu + s'_\mu s_\nu \frac{1}{i}\delta_{\mu\nu} D(x-x'). \tag{17}$$

The second term in (17) is independent of the electromagnetic potentials A_μ, so that the self-energy of photons, in which we are interested, is contained in the first term only. From (14), (16) and (17) this self-energy term is

$$\langle H\rangle_{\text{fluct}} = -\frac{i}{2}\int_{-\infty}^{+\infty} d^3x' \int_{\pm\infty}^{t} dx'_0 \langle [s_\mu(x), s_\nu(x')]\rangle_{\text{vac}} A_\mu(x) A_\nu(x'). \tag{18}$$

In this formula are contained the effects of fluctuations of the vacuum charge. In order to obtain the self-energy of the actually existing photons, we should subtract from $A_\mu A'_\nu$ the expectation value of the same operator in the state without photons

$$A_\mu A'_\nu \to A_\mu A'_\nu - \langle A_\mu A'_\nu\rangle_{\text{vac}}$$

However, we do not need to do this, but may study the effects of the charge fluctuations more generally. From the commutation relations (5 b)

$$[s_\mu(x), s_\nu(x')] = -ie^2(\bar{\psi}(x)\gamma_\mu S(x-x')\gamma_\nu\psi(x') - \bar{\psi}(x')\gamma_\nu S(x'-x)\gamma_\mu\psi(x)) \tag{19}$$

the expectation value for the vacuum is

$$\langle [s_\mu, s'_\nu]\rangle_{\text{vac}} = 4ie^2\left|\frac{\partial\Delta(x-x')}{\partial x_\mu}\cdot\frac{\partial\Delta_1(x-x')}{\partial x_\nu} + \frac{\partial\Delta(x-x')}{\partial x_\nu}\cdot\frac{\partial\Delta_1(x-x')}{\partial x_\mu}\right.$$
$$\left. - \delta_{\mu\nu}\left(\frac{\partial\Delta(x-x')}{\partial x_\lambda}\cdot\frac{\partial\Delta_1(x-x')}{\partial x_\lambda} + \varkappa^2\Delta(x-x')\Delta_1(x-x')\right)\right|. \tag{20}$$

In order to derive this expression, we make use of the following properties of the field quantities

$$\langle\bar{\psi}_\alpha \psi'_\beta\rangle_{\text{vac}} = \langle\bar{\psi}^+_\alpha \psi'^+_\beta\rangle_{\text{vac}} + \langle\bar{\psi}^-_\alpha \psi'^-_\beta\rangle_{\text{vac}},$$

since the mixed terms $\psi^+\psi^-$ have no diagonal elements. The term $\langle\bar{\psi}^+_\alpha \psi'^+_\beta\rangle_{\text{vac}}$ vanishes, since $\psi'^+\Phi_{\text{vac}} = 0$. (It should be remembered that the operator ψ^+ annihilates negatons). Hence

$$\langle\bar{\psi}_\alpha \psi'_\beta\rangle_{\text{vac}} = \langle\bar{\psi}^-_\alpha \psi'^-_\beta\rangle_{\text{vac}} = \langle\bar{\psi}^-_\alpha \psi'^-_\beta + \psi'^-_\beta \bar{\psi}^-_\alpha\rangle_{\text{vac}};$$

here the term $\psi'^-_\beta \bar{\psi}^-_\alpha$ has been added which gives $\bar{\psi}^-\Phi_{\text{vac}} = 0$ ($\bar{\psi}^-$ annihilates positons). Thus

$$\langle\bar{\psi}_\alpha \psi'_\beta\rangle_{\text{vac}} = \langle[\bar{\psi}^-_\alpha, \psi'^-_\beta]\rangle_{\text{vac}} = \frac{1}{2}S^-_{\beta\alpha}(x'-x) = \frac{1}{2}\left(\gamma^{(\lambda)}_{\beta\alpha}\frac{\partial}{\partial x_\lambda} - \varkappa\delta_{\beta\alpha}\right)\Delta^-(x'-x). \quad (21)$$

The formula $$\Delta^+(x-x') = \Delta^-(x'-x)$$

together with the expressions for the traces of the products of the Dirac matrices enables the derivation of the expression (20) from (19) and (21) by a comparatively long but elementary calculation. From (18) and (20) we obtain

$$\langle H\rangle_{\text{fluct}} = 2e^2 A_\mu(x)\int_{+\infty}^{-\infty} d^3\xi \int_{\pm\infty}^{0} d\xi_0 \left\{\frac{\partial\Delta}{\partial\xi_\mu}\frac{\partial\Delta_1}{\partial\xi_\nu} + \frac{\partial\Delta}{\partial\xi_\nu}\frac{\partial\Delta_1}{\partial\xi_\mu} - \delta_{\mu\nu}\left(\frac{\partial\Delta}{\partial\xi_\lambda}\frac{\partial\Delta_1}{\partial\xi_\lambda} + \varkappa^2\Delta\Delta_1\right)\right\} A_\nu(x-\xi),$$

where $\xi_\mu = x_\mu - x'_\mu$. By introducing under the integral the quantity $\frac{\xi_0}{|\xi_0|}$ we get

$$\langle H\rangle_{\text{fluct}} = e^2 A_\mu(x)\int d^4\xi \left\{\frac{\partial\Delta_S}{\partial\xi_\mu}\frac{\partial\Delta_1}{\partial\xi_\nu} + \frac{\partial\Delta_S}{\partial\xi_\nu}\frac{\partial\Delta_1}{\partial\xi_\mu} - \delta_{\mu\nu}\left(\frac{\partial\Delta_S}{\partial\xi_\lambda}\frac{\partial\Delta_1}{\partial\xi_\lambda} + \varkappa^2\Delta_S\Delta_1\right)\right\} A_\nu(x-\xi), \quad (22)$$

where $\Delta_S = \frac{\xi^0}{|\xi_0|}\Delta(\xi)$. Now $A_\nu(x-\xi)$ may be Fourier analysed

$$A_\nu(x-\xi) = \sum_k a^{(\nu)}_k e^{ik_\mu(x_\mu-\xi_\mu)} + \text{compl. conj.}$$

$$\langle H\rangle_{\text{fluct}} = e^2 A_\mu(x)\sum_k a^{(\nu)}_k e^{ik_\mu x_\mu}\int d^4\xi \left\{\frac{\partial\Delta_S}{\partial\xi_\mu}\cdot\frac{\partial\Delta_1}{\partial\xi_\nu} + \frac{\partial\Delta_S}{\partial\xi_\nu}\cdot\frac{\partial\Delta_1}{\partial\xi_\mu}\right.$$
$$\left. - \delta_{\mu\nu}\left(\frac{\partial\Delta_S}{\partial\xi_\lambda}\frac{\partial\Delta_1}{\partial\Delta_\lambda} + \varkappa^2\Delta_S\Delta_1\right)\right\} e^{-ik_\mu\xi_\mu} + \text{compl. conj.}$$

Following an argument of Schwinger we may say: — The integral represents formally a tensor $T_{\mu\nu}$ dependent on k. The only tensor with this property is

$$T_{\mu\nu} = A\cdot k_\mu k_\nu + B\,\delta_{\mu\nu},$$

where A and B are scalars. The first term gives zero from the Lorentz condition[2] $\sum_k a_k^{(\nu)} \cdot k_\nu \Phi = 0$, so that only the second one gives a contribution[3] to $\langle H \rangle_{\text{fluct}}$. Thus in order to calculate the self-energy term, we have to take

$$\langle H \rangle_{\text{fluct}} = e^2 A_\mu \sum_k a_k^{(\nu)} e^{ikx} \cdot B \delta_{\mu\nu} + \text{compl. conj.}, \tag{23}$$

or

$$\langle H \rangle_{\text{fluct}} = e^2 A_\mu(x) A_\mu(x) \cdot B \tag{24}$$

(B is real). From the last expression we see that the self-energy term is formally representable as a change of the mass constant Δm

$$\langle H \rangle_{\text{fluct}} = e^2 B A_\mu^2(x) = \pm (\Delta m)^2 A_\mu^2(x). \tag{25}$$

Now, we have to calculate the invariant quantity B

$$\begin{aligned} B &= \frac{1}{4} \operatorname{Tr} T_{\mu\mu} = \frac{1}{4} \int d^4\xi \left\{ 2 \frac{\partial \Delta_S}{\partial \xi_\mu} \frac{\partial \Delta_1}{\partial \xi_\mu} - 4 \left(\frac{\partial \Delta_S}{\partial \xi_\lambda} \frac{\partial \Delta_1}{\partial \xi_\lambda} + \varkappa^2 \Delta_S \Delta_1 \right) \right\} e^{-ik_\mu \xi_\mu} \\ &= - \int d^4\xi \left(\frac{1}{2} \frac{\partial \Delta_S}{\partial \xi_\lambda} \frac{\partial \Delta_1}{\partial \xi_\lambda} + \varkappa^2 \Delta_S \Delta_1 \right) e^{-ik_\mu \xi_\mu}. \end{aligned} \tag{26}$$

In order to compute this integral, we make use of the following representations of the delta functions

$$\Delta_S(\xi) = \frac{2}{(2\pi)^4} \int d^4q \, \frac{e^{iq_\mu \xi_\mu}}{q_\nu^2 + \varkappa^2}, \qquad \Delta_1 = \frac{1}{(2\pi)^3} \int d^4p \, \delta(p_\nu^2 + \varkappa^2) e^{ip_\mu \xi_\mu}. \tag{27}$$

In the integral representing Δ_S the principal value is to be taken

$$B = - \frac{2}{(2\pi)^7} \int d^4\xi \, d^4q \, d^4p \left(- \frac{q_\lambda p_\lambda}{2} + \varkappa^2 \right) \frac{\delta(p_\nu^2 + \varkappa^2)}{q_\nu^2 + \varkappa^2} \cdot e^{i(p_\mu + q_\mu - k_\mu)\xi_\mu}.$$

The integration over ξ yields four delta-functions, which enables the integration over q_μ, so that we get

$$B = \frac{2}{(2\pi)^3} \int d^4p \left(\frac{q_\lambda p_\lambda}{2} - \varkappa^2 \right) \frac{\delta(p_\nu^2 + \varkappa^2)}{q_\nu^2 + \varkappa^2},$$

where $q_\mu = k_\mu - p_\mu$. Due to the occurrence of $\delta(p_\nu^2 + \varkappa^2)$ under the integral, we may also easily integrate over dp_0. First split the in-

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[2] The Lorentz condition in the interaction representation is not $\frac{\partial A_\mu}{\partial x_\mu} \Phi = 0$, but the deviation from the usual form is proportional to higher powers of e and gives rise to higher order effects only.

[3] The existence of the term $B\delta_{\mu\nu}$ was pointed out by Wentzel.

tegral $\int_{-\infty}^{\infty} dp_0$ into $\int_{-\infty}^{0} dp_0 + \int_{0}^{\infty} dp_0$, then transform p_μ into $-p_\mu$ in the first of these integrals. This yields

$$B = \frac{2}{(2\pi)^3}\int_{-\infty}^{+\infty} d^3p \int_0^{\infty} dp_0 \left(\frac{-\frac{k_\mu p_\mu}{2} - \frac{p_\mu^2}{2} - \varkappa^2}{p_\mu^2 + 2p_\mu k_\mu + \varkappa^2} + \frac{\frac{k_\mu p_\mu}{2} - \frac{p_\mu^2}{2} - \varkappa^2}{p_\mu^2 - 2p_\mu k_\mu + \varkappa^2} \right) \delta(p_\nu^2 + \varkappa^2).$$

Now introduce a new variable $p_0^2 = z$, and obtain finally

$$B = -\frac{1}{2(2\pi)^3}\int \frac{d^3p}{\sqrt{p^2+\varkappa^2}} \tag{28}$$

or

$$\langle H \rangle_{\text{fluct}} = -\frac{e^2}{2(2\pi)^3}\int \frac{d^3p}{\sqrt{p^2+\varkappa^2}} \cdot A_\mu^2(x). \tag{29}$$

This integral is formally Lorentz-invariant but diverges quadratically.

III. Vacuum Charge Fluctuations of a Scalar Field

R. Jost[4] has calculated the «photon self-energy» in case of the interaction with a charged scalar meson-field. In this case the interaction energy density consists of two terms proportional to e and e^2, respectively

$$H' = H'_1 + H'_2 = -ieA_\nu\left(\frac{\partial\varphi^*}{\partial x_\nu}\varphi - \frac{\partial\varphi}{\partial x_\nu}\varphi^*\right) + e^2\varphi^*\varphi A_i A_i, \tag{30}$$

$\nu = 1,2,3,4$, $i = 1,2,3$, φ means the scalar meson field quantity. Denote the charge and current density in the interaction-free case by

$$\sigma_\mu = ie\left(\frac{\partial\varphi^*}{\partial x_\mu}\varphi - \frac{\partial\varphi}{\partial x_\mu}\varphi^*\right). \tag{31}$$

Performing a unitary transformation e^S the Schrödinger equation becomes to the second order of approximation

$$i\frac{\partial}{\partial t}\Phi = \left|\frac{1}{2}[\bar{H}_1, S] + \bar{H}_2\right|\Phi, \tag{32}$$

where

$$S = i\int_{\pm\infty}^{t} dx_0 \int \sigma_\nu A_\nu \, dx^3, \tag{33}$$

$$[H_1, S] = -i\int_{\pm\infty}^{t} dx'_0 \int d^3x' \, [\sigma_\nu A_\nu, \sigma'_\mu A'_\mu], \tag{34}$$

where

$$[\sigma_\nu A_\nu, \sigma'_\mu A'_\mu] = [\sigma_\nu, \sigma'_\mu] A_\nu A'_\mu + \sigma'_\mu \sigma_\nu D(x - x'). \tag{35}$$

[4] Unpublished.

The photon self-energy is contained in the first term and, of course, in H_2. After some calculation, (34) with the first term of (35) yield

$$[H_1, S] = -A_0 A_0 \varphi^* \varphi + \frac{1}{2} A_\nu \int d^4x' A_\mu(x') \Big| 2\varphi^* \frac{\partial \varphi'}{\partial x'_\mu} \cdot \frac{\partial \Delta}{\partial x_\nu} + \frac{\partial \varphi^*}{\partial x_\nu} \varphi' \frac{\partial \Delta}{\partial x'_\mu} - \frac{\partial \varphi^*}{\partial x_\nu} \frac{\partial \varphi'}{\partial x'_\mu} \Delta + \text{compl. conj.} \Big| . \tag{36}$$

The term $-A_0^2 \varphi^* \varphi$ has been obtained by a partial integration. This term, together with H'_2, yields an invariant, so that we obtain finally

$$H'' = A_\mu A_\mu \varphi^* \varphi + \frac{1}{2} A_\nu \int d^4x' A_\mu(x') \Big[2\varphi^* \frac{\partial \varphi'}{\partial x'_\mu} \cdot \frac{\partial \Delta}{\partial x_\nu} + \frac{\partial \varphi^*}{\partial x_\nu} \varphi' \frac{\partial \Delta}{\partial x'_\mu} - \frac{\partial \varphi^*}{\partial x_\nu} \frac{\partial \varphi'}{\partial x'_\mu} \Delta + \text{compl. conj.} \Big] \tag{37}$$

for this part of the interaction energy density which is responsible for the self-energy of photons. Take the expectation value for a state without mesons

$$\langle \varphi^* \varphi \rangle_{\text{vac}} = \frac{1}{2} \Delta^+(0), \tag{38}$$

where $$\Delta^+(x - x') = \frac{1}{2(2\pi)^3} \int \frac{d^3p}{\sqrt{m^2 + p^2}} e^{ip_\mu (x'_\mu - x_\mu)}.$$

m = mass constant for the meson field. Introducing the notation $\xi = x - x'$ and replacing the time integral

$$\int_{\mp\infty}^{t} dt' \equiv \frac{1}{2} \Big(\int_{-\infty}^{t} dt' + \int_{+\infty}^{t} dt' \Big) \quad \text{by} \quad \frac{1}{2} \int_{-\infty}^{+\infty} d\xi_0 \frac{\xi_0}{|\xi_0|}$$

we get (39)

$$\langle H'' \rangle_{\text{vac}} = \frac{e^2}{2} A_\nu A_\nu \Delta^+(0) + \frac{e^2}{4} A_\nu \int d^4\xi \frac{\xi_0}{|\xi_0|} A_\mu(x - \xi) \Big[-2 \frac{\partial \Delta_1}{\partial \xi_\mu} \cdot \frac{\partial \Delta}{\partial \xi_\nu} - \frac{\partial \Delta_1}{\partial \xi_\nu} \cdot \frac{\partial \Delta}{\partial \xi_\mu} + \frac{\partial^2 \Delta_1}{\partial \xi_\nu \partial \xi_\mu} \Delta \Big].$$

By a calculation quite analogous to that in the foregoing section we obtain finally the result

$$\langle H \rangle_{\text{vac}} = \frac{1}{4} \frac{e^2}{(2\pi)^3} \int \frac{d^3p}{\sqrt{m^2 + p^2}} A_\mu^2(x). \tag{40}$$

IV. A Compensation of Charge Fluctuations

We notice the following results of the two preceding sections: (1) In either case the mass factor Δm is divergent quadra-

tically [5]. (2) The signs of the self-energy terms are opposite. These two facts enable the compensation of the charge fluctuation effects and, consequently, a complete elimination of the self-energy of the photons. In order to achieve this result, we assume a suitable mixture of spinor and scalar charged fields. We take, e. g., besides the electron field with charge $e_0 = e$ and mass $m_0 = \varkappa$, another spinor field with charge e_1 and mass m_1, and two scalar fields whose charges and masses be denoted by e_2, e_3, m_2, m_3. [6] Introducing the following notation

$$-\frac{e_0^2}{e^2} = -1 = x_0\,, \quad -\frac{e_1^2}{e^2} = x_1\,, \quad \frac{1}{2}\frac{e_2^2}{e^2} = x_2\,, \quad \frac{1}{2}\frac{e_3^2}{e^2} = x_3\,,$$

the self-energy density of the electromagnetic field interacting with all these fields is

$$\frac{e^2}{2(2\pi)^3}\int d^3p \left(\sum_{i=0}^{3}\frac{x_i}{\sqrt{p^2+m_i^2}}\right)A_\mu^2(x). \tag{41}$$

The above integral vanishes if the following three conditions are satisfied

$$\sum_{i=0}^{3} x_i = 0\,, \qquad \sum_{i=0}^{3} x_i m_i^2 = 0\,, \qquad \sum_{i=0}^{3} x_i m_i^2 \ln m_i^2 = 0. \tag{42}$$

The first condition destroys the quadratically divergent terms, the second one cancels the logarithmically divergent parts, while the third one makes the remaining (convergent) part zero. (42) form a set of three linear inhomogeneous equations for the unknown x_1, x_2, x_3, which surely possesses a solution, since the masses m_i ($i=1,2,3$) may be always chosen so as to make the determinant different from zero. It is also possible to choose the masses m_i ($i = 1,2,3$) so large that some deviations from the usual electron theory would manifest

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[5] In case of the interaction with a spinor field Heisenberg (Zeits. f. Phys. **90**, 209, 1934) obtained a logarithmically divergent photon self-energy. The discrepancy between Heisenberg's and our results is due to the fact that Heisenberg arbitrarely substracts a part of the effect. Wentzel calculated the photon self energy using Schwinger's representation of the Δ functions and obtained a finite (but different from zero) value for the rest mass of the photon. This indicates that Schwinger's representation is not equivalent with the usual representation of the Δ-function. This is howerer another theory.

[6] It is also possible to put all the charges e_i equal to the elementary charge e, but in this case we have to introduce a larger number of auxiliary fields. (Note added in proof).

themselves in the extremely relativistic effects only. We may even let $m_1 \to \infty$, so that the additional fields would play, finally, only an auxiliary role in the calculations without appearing explicitely in the experiment. The equations (42) form a set of necessary conditions for the Schrödinger equation of a mixed field to possess a solution. Their physical meaning is the following: — they ensure the compensation of the charge fluctuations of the vacuum. Of course, the equations (42) are not yet sufficient conditions for the existence of a solution of the Schrödinger equation. First of all, we must add further fields and conditions in order to obtain finite self-energies of charged particles (sec. I). Generally, we must look for such a mixture of charged and neutral fields for which all the self-energies would become convergent (or disappear) by mutual compensation. This seems certainly possible to the second order of approximation. However, it is not certain whether the conditions obtained from the second order effects will automatically ensure also the convergence for higher order terms.

A c k n o w l e d g e m e n t. I wish to express my deep gratitude to Dr. R. J o s t for his kind advice and for several valuable discussions.

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ON THE EXISTENCE OF AN ELECTRIC FIELD IN SUPERCONDUCTORS

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(Received Mai 21, 1948)

By a generalization of Ohm's law and its application to the electric current in superconductors the fundamental laws of electrodynamics of superconductivity are deduced by purely formal analogy with the equations of motion of a free charge in an electromagnetic field. For stationary currents the condition $\hat{E}+\frac{1}{nec}\hat{i}\times\hat{H}=0$ asserts that there exist in superconductors an electric field acting upon the superconduction electrons but producing no Joule's heat, as its activity $\hat{i}\cdot\hat{E}=0$. By a train of thought starting from Hamilton's principle the electric field is found to derive from an electric polarization inside the superconductors. The appearence of an electric polarization may be traced to a spin coupling of the electrons due to the existence of exchange forces. A calculation of the number of electrons worked out for a simplified model of a superconductor leads to a fairly good agreement with experiment.

Up to this time a certain misconception is often to be found in the presentation of the electrodynamics of superconductivity. Let us call $\hat{E}$, $\hat{H}$ the intensity of the electromagnetic field, $\hat{i}$ the density of the current, m and e respectively the mass and charge of an electron, and ne the density of the electric charge. Usually one assumes a generalized Ohm's law in the form $\hat{E}=\rho\hat{i}+\lambda\dot{\hat{i}}$, where $\lambda=\frac{m}{ne^2}$ and ρ is the specific resistance. Here, a term expressing the influence of the magnetic field $\hat{H}$, viz. $\frac{1}{nec}\hat{i}\times\hat{H}$, has been omitted. This is justfied, of course, in the case of strong electric fields, but not for fields of the order of magnitude of the electric fields, which may be expected in supperconductors. Hence, the generalized Ohm's law has to be assumed as follows

$$\hat{E}+\frac{1}{nec}\hat{i}\times\hat{H}=\rho\hat{i}+\lambda\dot{\hat{i}}. \tag{1}$$

In case of non-stationary currents in superconductors we may put, according to experiment, $\rho = 0$ and (1) takes the form

$$\dot{\hat{E}} + \frac{1}{nec}\hat{i} \times \hat{H} = \lambda \dot{\hat{i}}, \tag{2}$$

which shows a striking analogy with the equation of motion of a free charge in an electromagnetic field. All the results of Hamilton's dynamics of such a particle may be applied in a purely formal manner to an element of current in a superconductor. The velocity of the electric charge satisfies the equation $\operatorname{rot} \hat{v} = -\frac{e}{mc}\hat{H}$, an analogous relation must hold for an element of current, viz. $\operatorname{rot} \frac{\hat{i}}{ne} = -\frac{e}{mc}\hat{H}$ or

$$c\lambda \operatorname{rot} \hat{i} = -\hat{H}, \tag{3}$$

which is nothing else as the well-known first fundamental equation of London's theory *(1, 1935; 2, 1936)*.

Taking into account that

$$\frac{d\hat{i}}{dt} = \frac{\partial \hat{i}}{\partial t} + \left(\frac{\hat{i}}{ne} \cdot \nabla\right)\hat{i} = \frac{\partial \hat{i}}{\partial t} - \frac{1}{2ne}\operatorname{grad} i^2 + \frac{1}{ne}\hat{i} \times \operatorname{rot} \hat{i},$$

equation (2) may be given the form

$$\dot{\hat{E}} + \frac{\lambda}{2ne}\operatorname{grad} i^2 - \lambda \frac{\partial \hat{i}}{\partial t} = \frac{1}{nec}\hat{i} \times (\hat{H} + c\lambda \operatorname{rot} \hat{i})$$

and hence, due to (3), we have

$$\dot{\hat{E}} + \frac{\lambda}{2ne}\operatorname{grad}\ i^2 - \lambda \frac{\partial \hat{i}}{\partial t} = 0, \tag{4}$$

the second fundamental equation of the theory of superconductivity with an additional term introduced some time ago by F. Bopp *(3, 1937)*.

In a superconductor carrying a steady current $\rho = 0$, $\dot{i} = 0$ and (2) goes over in

$$\dot{\hat{E}} + \frac{1}{nec}\hat{i} \times \hat{H} = 0. \tag{5}$$

At first sight this relation might appear trivial, but it shows that in a superconductor carrying a steady current there exists a non-vanishing electric field which, being perpendicular to $\hat{i}$ and $\hat{H}$, can neither do mechanical work nor generate Joule's heat.

Returning to the above mentioned analogy between the behaviour of an element of current and the motion of a free charge in

an electromagnetic field, we may write Hamilton's principle for a unit volume-element of current as follows

$$\delta\int_{t_1}^{t_2} L\,dt = 0,$$

where $$L = T - U = \frac{1}{2}\lambda \hat{i}^2 - \left(neV - \frac{1}{c}\hat{A}\cdot\hat{i}\right)$$

and $$\hat{i} = ne\hat{v}, \quad \hat{E} = -\operatorname{grad} V, \quad \hat{H} = \operatorname{rot}\hat{A} \quad n = n(x,y,z).$$

In the sequel, as we shall limit ourselves to the consideration of steady electric current, we shall assume the kinetic energy $T = \frac{1}{2}\lambda \hat{i}^2$ to be a constant. Then, $\delta\int_{t_1}^{t_2} T\,dt = 0$, and from Hamilton's principle we get

$$\delta\int_{t_1}^{t_2} U\,dt = 0, \tag{6}$$

a condition which is equivalent to the Euler-Lagrange equations

$$ne\hat{E} + \frac{1}{c}\hat{i}\times\hat{H} - \left(eV - \frac{1}{nc}\hat{A}\cdot\hat{i}\right)\operatorname{grad} n = 0$$

Consequently, taking into account (5), we get $n = \text{const.}$ and come thus to the conclusion that the superconduction electrons are evenly distributed throughout the superconductor, in disaccord with the assertion of Bopp *(3, 1937)* drawn from his hydrodynamical theory of superconductivity. The distribution in space of the positive ions being also homogeneous, there is in a superconductor no «free electric charge» which might act as a source of electromagnetic induction vector. Hence, $\operatorname{div}\hat{D} = \operatorname{div}(\hat{E} + 4\pi\hat{P}) = 0$ and $\operatorname{div}\hat{E} = -4\pi\operatorname{div}\hat{P}$. On the other side, in the case of steady currents, (4) yields $\operatorname{div}\hat{E} = -\frac{\lambda}{2ne}\Delta\hat{i}^2 \neq 0$, and thus the electric intensity inside a superconductor is seen to be connected with the electric polarization. Besides, due to Maxwell's equation $\operatorname{rot}\hat{H} = \frac{4\pi}{c}\hat{i}$, the magnetic field is the proper field of the superconducted current. The question as to the moment of the appearence of superconducitivity can be brought back to the following question: At what moment while the temperature is being steadily reduced, an internal electric field may spontaneously appear? J. J. Thomson *(4, 1915)* was the first to try to explain the phenomenon of superconducitivity by spontaneous electric polarization. Though his dipole theory of conductivity does not comply with modern views, it seems that the underlying idea of seeking a solution of the problem by analogy with ferromagnetism contains a germ of truth. Indeed, the similarity of behaviour

of ferromagnetics and superconductors in the vicinity of their transition temperatures (abrupt changes of electric and magnetic properties, similar anomalies of specific heats) leads to suggest an essential parenthood of the mechanisms of both phenomena. It is very probable that in superconductors as well as in ferromagnetics there exists an «internal field» — due to exchange forces — put to light by the coupling of the spins of the electrons. P. A. M. Dirac *(5, 1929)* showed that the effective coupling of the spins of any two electrons in a crystal give rise to a potential energy $U_{ij} = -2J_{ij}\hat{s}_i \cdot \hat{s}_j$, where J_{ij} is the exchange integral depending on the states occupied by the given electrons, and the spin vector $\hat{s}_i$ is expressed in units of h. A necessary condition for the appearance of ferromagnetism is $J_{ij} > 0$, but more often $J_{ij} < 0$, in which case the spins tend to become antiparallel. The magnetic suceptibilities of some substances as MnO, MnS, Cr_2O_3 CrSb etc., called «antiferromagnetics» (see J. H. van Vleck, *6, 1945)* pass through a steep maximum with rising temperature. The properties of these antiferromagnetics may be explained by assuming that their crystals consist of two «sub-lattices» A and B, the A-atoms having as neighbours only B-atoms and vice versa. In case the exchange integral is negative, the mutual energy of two atoms is minimum when their spins are antiparallel. For the energy of the whole crystal to be a minimum the spins of all the A-atoms must be parallel among themselves but antiparallel to the spins of all the B-atoms. With rising temperature, the spontaneous internal field maintaining this orderly state grows weaker and weaker, until it vanishes altogether at the Curie point. Reasoning by analogy, we may imagine the electron spins in the «microdomains» of the superconductor tending to take antiparallel positions and forming thus an orderly state resembling two interwoven sublattices with antiparallel spins. We may think of the conduction electrons as forming one of these sublattices, the second one being constituted of the positive ions. The orderly arrangement of the electrons in their sublattice will bring about orderliness in the orientation of the ion dipoles — and an electric polarization of the medium will follow. For such a state to be in equilibrium the electric field due to polarization must be compensated according to the following condition of minimum potential energy

$$\hat{E} + \frac{1}{nec}\hat{i} \times \hat{H} = 0.$$

The existence of a spontaneous electric current perpendicular to the electric field may be regarded as a consequence of the above condition. Thus, the electric field acting upon the superconduction elec-

trons is always compensated by an «induced electric field» $\frac{1}{nec}\dot{i}\times\dot{H}$, and there is, therefore, in a superconductor no macroscopic electrostatic field, in contradistinction from a polarized dielectric, as was experimentally verified by H. London *(11, 1936)*.

The model of a superconductor sketched above renders it possible to calculate the number of superconduction electrons as a function of temperature, in analogy to the method of E. Stoner *(7, 1934)* used by him for ferromagnetism. Let us assume for simplicity that each atom of the superconductor has only two outward electrons (in addition to closed inward shells); one of these electrons is «quasi-free» and belongs rather to the «electron-fluid», the other one is bound to the rest of the atom. In a time average let the given ion be surrounded by neighbouring electrons of the electron-fluid. The mutual energy of a pair: ion-neighbouring electron depends, as in every two-electron problem, on the mutual orientation of electron spins; when the exchange integral $J<0$, the energy is greater or smaller by $|J|$ according to the spins being respectively parallel or antiparallel. Let z be the number of electrons surrounding the given ion, x the fraction of these electrons in antiparallel position to the ion spin and similarly y for parallel positions. Then, the increase of the energy (put equal to naught for $x=y$) is

$$\Delta u = -|J|(x-y).$$

We assume the number n of superconduction electrons in 1 cm^3 to be proportional to the excess of the number of antiparallel ion-electron pairs over the number of parallel pairs, $n=c(x-y)$, as this excess characterize the degree of orderliness as well as the internal electric polarization. In fact, in our model of a superconductor the electric polarization is due to the orderliness of the spin orientation of the ions and the measure of this orderliness is given by the excess of the antiparallel spin pairs (ion electron — neighbouring electron of the electron-fluid) over the parallel ones, as in a normal crystal there are as many parallel as antiparallel ion dipoles. Thus, only this excess gives rise to internal polarization and to superconductivity. In an ideal case, when all the spin pairs in the neighbourhood of a given ion are antiparallel their number is z; in this case $n_{max}=cz$, and hence $\frac{n}{n_{max}}=\frac{x-y}{z}=\frac{x-y}{x+y}$. Let the direction of the given ion be the singular direction of the ion sub-lattice. To each ion oriented in this direction or the opposite one (we assume these two

possibilities only) belongs the energy $\frac{1}{2}\Delta u = -\frac{1}{2}|J|(x-y)$ or $-\frac{1}{2}\Delta u$ respectively, as Δu refers to z ion pairs and each pair contains one ion electron and one neighbouring electron of the electron-fluid.

As the number of parallel ion dipoles is equal to the number of antiparallel spin-pairs we have, thanks to Boltzmann's law,

$$\frac{n}{n_{max}} = \frac{x-y}{x+y} = \frac{e^{-\Delta u/2kT} - e^{\Delta u/2kT}}{e^{-\Delta u/2kT} + e^{\Delta u/2kT}} = \text{th}\left(\frac{z|J|}{2kT}\frac{n}{n_{max}}\right)$$

or in parameter form

$$n/n_{max} = \text{th } a, \qquad T/\Theta = \frac{\text{th } a}{a}, \tag{7}$$

where $\Theta = z|J|/2k$ denotes a transition temperature analogous to the Curie point of ferromagnetics.

The number of superconduction electrons may be found from measurements of the so called penetration depth of a magnetic field into the superconductor $\delta = \sqrt{\frac{\lambda c^2}{4\pi}}$, where $\lambda = \frac{m}{ne^2}$.

In Fig. 1 we see, besides the graph of $n/n_{max} = f(T/\Theta)$ according to equation (7), the experimental curve plotted by F. London *(8, 1945)*

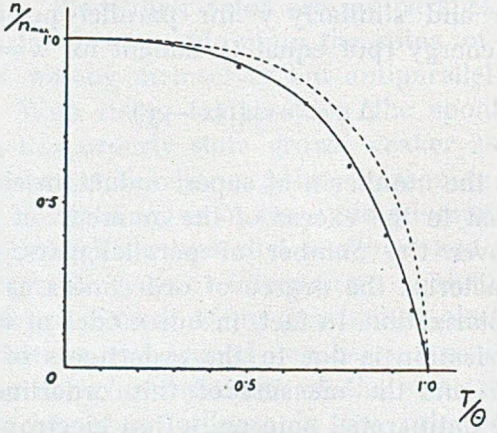

Fig. 1. Relative number of superconduction electrons as a function of reduced temperature. Solid curve after F. London *(8, 1945)*

from measurements of the susceptibility of colloidal mercury in a superconducting state [1]. These measurements were carried through by D. Shoenberg *(9, 1940)* and there results coincide with measurements on thin mercury-layers *(10, 1939)*.

[1] The experimental curve has been taken from the publication of F. London and the theoretical curve drawn to the same scale.

The striking similarity of both curves seems to indicate that we have come across a very promising way of elucidation the nature of superconductivity.

To sum up, we may emphasize the fact that the laws of electrodynamics point to the existence of an electric field in the interior of superconductors in stationary states, this field acting upon the superconduction electrons inside the superconductor. From the macroscopic point of view, this field does not manifest itself as it is just compesated by the „induced" electric field produced by spontaneous microcurrents. The origin of the internal electric field may be connected with the directional coupling of the ions and electrons under the influence of exchange forces, which explains the existence of a certain analogy between superconductivity and ferromagnetism.

In conclusion the writer takes the opportunity of thanking Prof. S. Szczeniowski, Prof. J. Weyssenhoff and Dr. B. Średniawa for their helpful advice and Prof. Niewodniczański for stimulating discussions and his continued interest in the subject of this paper.

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A GEOMETRICAL INTERPRETATION OF THE PHASE-DIFFERENCE ANGLE AND ITS APPLICATION TO A. C. PHASE MEASUREMENTS BY MEANS OF THE OSCILLOGRAPH

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(Received November 11, 1948)

It is well known that the trajectory of a point performing simultaneously two harmonic motions of the same frequency perpendicular to each other is an ellipse (first figure of Lissajous). It may be easily shown (see *e. g.* M. Born, *Optik*, 1933, p. 22) that between the amplitudes a_1, a_2, the phase difference δ of the two motions and the lengths a and b of the semi-axes of the ellipse the following relations exist

$$a_1^2 + a_2^2 = a^2 + b^2, \tag{1}$$

$$a_1 a_2 \sin \delta = a b. \tag{2}$$

According to the first and second theorems of Apollonius, well known in analitical geometry, these relations are valid for any two conjugated diameters of an ellipse and the angle δ between them. It may be proved almost directly that the conditions (1) and (2) are also sufficient for the real numbers a_1, a_2 and δ to represent in the ellipse with semi-axes a and b the lengths of two conjugated diameters and the angle between them.

On the basis of the above statement the angle δ can be constructed geometrically as follows (see Fig. 1). Two circles with radii

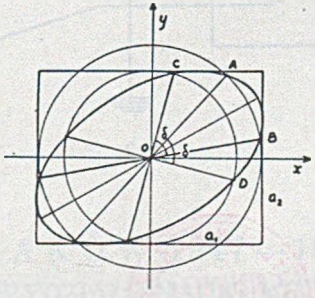

Fig. 1.

a_1 and a_2 are drawn from the centre 0 of the ellipse. Joining 0 with the points of intersection of the circles with the ellipse, we obtain two pairs of conjugated diameters. The angle δ is indicated in the figure.

Oscillographs give directly exact diagrams of resultant oscillation curves and our construction might be therefore applied to their interpretation. For instance, the measurement of the phase difference between tension and current by means of a cathode-ray oscillograph (e. g. with two electrostatic deflectors P_1, P_2) can be performed in principle as shown in Fig. 2. By suitable regulation of the amplitudes

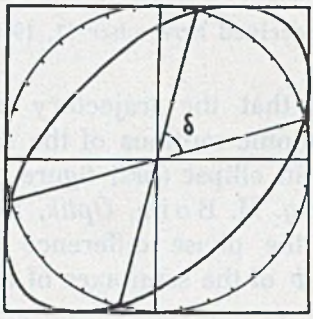

Fig. 2.

of the deflections caused by P_1 and P_2 both amplitudes a_2 and a_1 may be equalized. The common diameters of the ellipse and the circle given by their points of intersection determine the angle δ. By providing a suitable circular scale on the screen (see Fig. 3) the angle δ might be directly read from the scale.

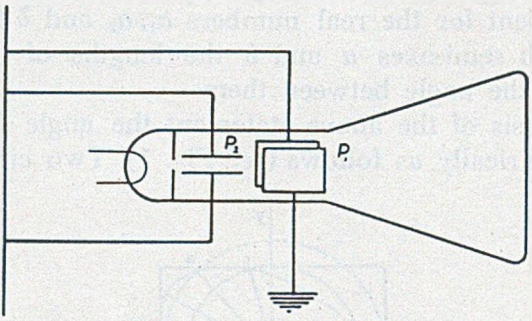

Fig. 3.

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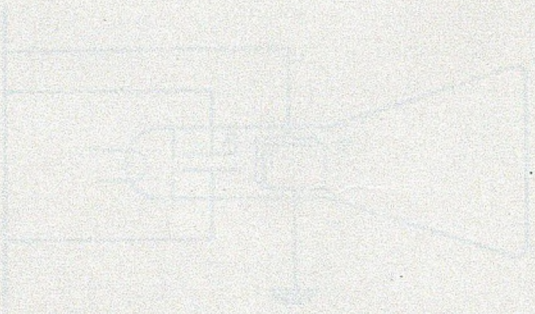

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