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PARAMETERS ESTIMATES FOR LINEAR STOCHASTIC EQUATION WITH CONTROL

Summary. The strong consistency of maximum likelihood estimates of unknown parameter of a continuous time linear stochastic equation, is shown in this paper. The situation was considered, when the controled and uncontrolled processes have been observed. The unknown parameter appears in the linear transformations of the state and the control. The parameter estimates are used, to define the adaptive control.

ESTYMACJA PARAMETRÓW DLA JEDNOWYMIAROWEGO RÓWNANIA ZE STEROWANIEM

Streszczenie. W pracy wykazano silną zgodność estymatora największego prawdopodobieństwa dla współczynników jednowymiarowego liniowego równania stochastycznego ze sterowaniem. Rozważamy sytuację, gdy obserwacji podlega proces sterowany i pewien proces nie sterowany. Estymowanymi parametrami są liniowa transformacja stanu oraz sterowania. Uzyskane wyniki zastosowano do zadania sterowania adaptacyjnego.

1. Introduction

There exist many papers about the parameter estimates and adaptive control in linear continuous differential equations, see [1]-[3], also [5]-[7] and [11]. The assumptions, which are done in those papers are different from this, which we made. For example in papers [5] and [7] unknown parameters occur in linear transformations of the state and some specified asymptotic behavior of determinant is required. In our paper we succeeded to omit this assumption in one-dimensional case. Some results in the case, when unknown parameters apper in the linear state and control transformation, were obtained also in paper [6], but the system taken under consideration had particular form and in one-dimensional case it reduces to less general in relation to this that we consider.

On the other hand, a restriction on the control, in papers [1]-[3], [8] and [11], makes it impossible for us, to use these results to consider a cost functional.

In this paper we consider the stochastic system of differential equations:

$$dx_t = (ax_t + bu_t)dt + dW_t^{(1)},\ x_0 = 0, \tag{1}$$

$$dy_t = ax_t dt + dW_t^{(2)},\ y_0 = 0,$$

where x_0, y_t, y_0, y_t, $u_t \in R$ and $W = (W_t^{(1)}, W_t^{(2)})$, $t \geq 0$ is a Wienner process. It is assumed, that the unknown parameters $a \in (\alpha_1, \alpha_2)$, $b \in (\beta_1, \beta_2)$ and α_1, $\beta_1 > 0$.
At any time $t \geq 0$ based on the measurements x_s, y_s, $0 \leq s \leq t$ we hope to estimate the unknown parameters a and b and to design the adaptive control with the purpose of minimizing the cost functional:

$$C = \lim_{t\to\infty} \frac{1}{t} C_t, \tag{2}$$

where

$$C_t = \int_0^t (qx_s^2 + ru_s^2)ds, \tag{3}$$

and $q > 0$, $r > 0$.

2. Parameter estimates and adaptive control law

In this section we first define the parameter estimates $\hat{a}(t)$ for a and $\hat{b}(t)$ for b and then define the adaptive control u_t on the basis of the observations (x_s, y_s), $0 \leq s \leq t$ and the estimates $\hat{a}(t)$ and $\hat{b}(t)$ obtained as well.

The likelihood function is obtained from the mutual absolute continuity of the probability measure of the process (x_t, y_t), $t \geq 0$ and the Wiener measure for $W(t)$, $t \geq 0$ (see [7]) as

$$L_t(a, b) = \exp\left[\int_0^t \langle Az_s + Bu_s, dz_s\rangle - \frac{1}{2}\int_0^t \|Az_s + Bu_s\|^2 ds\right]. \tag{4}$$

where $z_t = \begin{bmatrix} x_t \\ y_t \end{bmatrix}$, $A = \begin{bmatrix} a & 0 \\ a & 0 \end{bmatrix}$, $B = \begin{bmatrix} b \\ 0 \end{bmatrix}$ and $\langle\cdot,\cdot\rangle$ is the standard inner product in R^2. The maximum likelihood estimates $\hat{a}(t)$ of a and $\hat{b}(t)$ of b are obtained by maximizing $L_t(a, b)$. To maximizing $L_t(a, b)$ it is necessary (and in fact sufficient) that:

$$\frac{d\ln L_t(a,b)}{db} = 0,\ \frac{d\ln L_t(a,b)}{da} = 0, \tag{5}$$

then

$$\hat{a}(t)=\frac{\int_0^t x_s dx_s \int_0^t u_s^2 ds+\int_0^t x_s dy_s \int_0^t u_s^2 ds-\int_0^t u_s dx_s \int_0^t x_s u_s ds}{2\int_0^t x_s^2 ds \int_0^t u_s^2 ds-\left(\int_0^t x_s u_s ds\right)^2}, \tag{6}$$

$$\hat{b}(t)=\frac{2\int_0^t u_s dx_s \int_0^t x_s^2 ds+\int_0^t x_s dx_s \int_0^t x_s u_s ds-\int_0^t x_s dy_s \int_0^t x_s u_s ds}{2\int_0^t x_s^2 ds \int_0^t u_s^2 ds-\left(\int_0^t x_s u_s ds\right)^2}. \tag{7}$$

Because

$$dx_t=(ax_t+bu_t)dt+dW_t^{(1)},$$

$$dy_t=ax_t dt+dW_t^{(2)},$$

by rewriting (6) and (7), we obtain:

$$\hat{a}(t)-a=\frac{\int_0^t x_s dW_s^{(1)} \int_0^t u_s^2 ds+\int_0^t x_s dW_s^{(2)} \int_0^t u_s^2 ds-\int_0^t u_s dW_s^{(1)} \int_0^t x_s u_s ds}{2\int_0^t x_s^2 ds \int_0^t u_s^2 ds-\left(\int_0^t x_s u_s ds\right)^2}, \tag{8}$$

$$\hat{b}(t)-b=\frac{2\int_0^t u_s dW_s^{(1)} \int_0^t x_s^2 ds-\int_0^t x_s dW_s^{(1)} \int_0^t x_s u_s ds-\int_0^t x_s dW_s^{(2)} \int_0^t x_s u_s ds}{2\int_0^t x_s^2 ds \int_0^t u_s^2 ds-\left(\int_0^t x_s u_s ds\right)^2}. \tag{9}$$

We now proceed to define the adaptive control. It is well known (see, for example, [4]), that the feedback control $u_t=kx_t$, where $k=-r^{-1}bp$, and $p=\frac{a+\sqrt{a^2+qr^{-1}b^2}}{b^2r^{-1}}$, makes the cost functional (2) minimal and equal to $C=p$.

Howewer, this optimal control can not be used here since a and b, and hence p are unknown. It is natural to replace (see [5], [7]) a and b by their estimates $\hat{a}(t)$ and $\hat{b}(t)$. Define p_t and k_t as follows

$$p_t=\frac{\hat{a}(t)+\sqrt{(\hat{a}(t))^2+qr^{-1}(\hat{b}(t))^2}}{(\hat{b}(t))^2 r^{-1}}, \tag{10}$$

$$k_t=-r^{-1}\hat{b}(t)p_t. \tag{11}$$

We now proceed to define the adaptive control. Take any $k_0\neq 0$ as the initial value for the adaptive feedback gain. The time axis is partitioned by stopping times τ_n, $n=0,1,2,\dots$ and the adaptive control is defined to be

$$u_t=\begin{cases} k_{\tau_n}x_t & \text{if } t \text{ belongs to some } [\tau_n,\tau_{n+1}) \\ k_{\tau_{n-1}}x_t & \text{if } t \text{ belongs to some } [\tau_n,\tau_{n+1}) \end{cases} \tag{12}$$

The stopping times are given as follows:

$$\tau_0 = 0 \tag{13}$$

$$\tau_n = \max\left\{\inf\left\{t : \int_0^t x_s^2 ds \geq n\right\}, \inf\left\{t : \int_0^t u_s^2 ds \geq n\right\}\right\}.$$

3. Strong consistency of parameter estimates and optimality of adaptive control

We first prove a lemma.

Lemma 1. *For any n, τ_n is finite a. s.*

Proof. Assume that for a set $\Lambda \subset \Omega$ with $P(\Lambda) > 0$, and for $n \in N$, for every $\omega \in \Lambda$ holds:

$$\tau_n(\omega) = \infty,$$

then by definition of τ_n, we have:

$$\int_0^\infty x_s^2(\omega)ds < 0 \quad \text{or} \quad \int_0^\infty u_s^2(\omega)ds < 0.$$

If $\int_0^\infty u_s^2(\omega)ds < 0$ then by definition of the adaptive control we have, that $|k_{tau_k}| \geq \min\{|k_{\tau_1}|, \ldots, |k_{\tau_n}|\} > 0$, for $k = 1, 2, \ldots, n$ (the feedback gain takes only finite numbers of values: $k_{\tau_0}, k_{\tau_1}, \ldots, k_{\tau_n}$), so

$$\int_0^\infty x_s^2(\omega)ds < 0. \tag{14}$$

A contradiction will be established. Itô's formula implies that:

$$x_{t\wedge\tau_n}^2 - x_0^2 = 2\int_0^{t\wedge\tau_n} (ax_s + bu_s)x_s dsd + 2\int_0^{t\wedge\tau_n} x_s dW_s + t \wedge \tau_n. \tag{15}$$

The first integral in (15) can be bounded by using the Schwarz inequality and the boundedness on k_t, $t \geq 0$ and the second integral is a square integrable martingale by the definition of stopping times. Letting $t \to \infty$ it follows, that for $\omega \in \{\tau_n = \infty\}$ we have $\lim_{t\to\infty} x_t^2(w) = \infty$. However this contradicts the assumption (14).∎

Theorem 1. *Under the adaptive control defined by (12) and (13) the family of maximum likelihood estimates $\hat{a}(t)$ is strongly consistent, that is*

$$P_{a,b}(\lim_{t\to\infty} \hat{a}(t) = a) = 1.$$

Proof. To verify this Theorem the following result [10] is important.

Lemma 2. *If (W_t, F_t), $t \geq 0$ is a Wiener process on the probability space (Ω, F, P) and the process (f_t) is F_t-adapted for $t \geq 0$, such that:*

$$\int_0^t f_s^2 ds < \infty \quad a.s \text{ for } \ 0 \leq t < \infty,$$

and

$$\int_0^\infty f_s^2 ds = \infty \quad a.s,$$

then the process (z_s, G_s), where $z_s = \int_0^{\tau_s} f_t dW_t$, $G_s = F_{\tau_s}$ and $\tau_s = \inf\left\{t : \int_0^t f_U^2 du > s\right\}$, is a Wiener process and $\lim_{t\to\infty} \frac{\int_0^t f_s dW_s}{\int_0^t f_s^2 ds} = 0$ a.s.

Because $\int_0^t x_s^2 ds \int_0^t u_s^2 ds - \left(\int_0^t x_s u_s ds\right)^2 \geq 0$, then from (8), we have

$$|\hat{a}(t) - a| \leq \left|\frac{\int_0^t x_s dW_s^{(1)}}{\int_0^t x_s^2 ds}\right| + \left|\frac{\int_0^t x_s dW_s^{(2)}}{\int_0^t x_s^2 ds}\right| + \left|\frac{\int_0^t u_s dW_s^{(1)}}{\int_0^t u_s^2 ds}\right| \cdot \left|\frac{\int_0^t x_s u_s ds}{\int_0^t x_s^2 ds}\right|. \tag{16}$$

It follows from condition, that $a \in (\alpha_1, \alpha_2)$, $b \in (\beta_1, \beta_2)$ where $\alpha_1, \beta_1 > 0$ and (10)-(13), that

$$\left|\frac{\int_0^t x_s u_s ds}{\int_0^t x_s^2 ds}\right| = \left|\frac{\int_0^t k_s x_s^2 ds}{\int_0^t x_s^2 ds}\right| \leq \sup_{0 \leq s < \infty} |k_s| = c < \infty, \tag{17}$$

where $c_2 > 0$. Using the inequality (17) to inequality (16), we have:

$$|\hat{a}(t) - a| \leq \left|\frac{\int_0^t x_s dW_s^{(1)}}{\int_0^t x_s^2 ds}\right| + \left|\frac{\int_0^t x_s dW_s^{(2)}}{\int_0^t x_s^2 ds}\right| + \left|\frac{\int_0^t u_s dW_s^{(1)}}{\int_0^t u_s^2 ds}\right| \cdot c. \tag{18}$$

From (18) and Lemma 2 we obtain:

$$\lim_{t\to\infty} \hat{a}(t) = a \quad \text{a.s.}$$

This completes the proof.■

Similary, as Theorem 1, we can show the next Theorem about the estimate for b.

Theorem 2. *Under the adaptive control defined by (12) and (13) the family of maximum likelihood estimates $\hat{b}(t)$ is strongly consistent, that is*

$$P_{a,b}(\lim_{t\to\infty} \hat{b}(t) = b) = 1.$$

Theorems 1 and 2 have established the convergence of $\hat{a}(t)$ to a and $\hat{b}(t)$ to b as $t \to \infty$, so

$$\lim_{k_t} = k, \tag{19}$$

where $k = -r^{-1}bp$, $p = \frac{a+\sqrt{a^2+qr^{-1}b^2}}{b^2r^{-1}}$ and k_t is given by $k_t = k_{\tau_n}$ if t belongs to some $[\tau_n, \tau_{n+1})$. In the paper [7] the following theorem, was proved:

Theorem 3. *If* $\lim_{t\to\infty} k_t = \frac{a+\sqrt{a^2+qr^{-1}b^2}}{b}$ *then control police* $u_t = k_t x_t$ *is optimal for system (1) with cost functional (2) i.e* $\lim_{t\to\infty} \frac{C_t}{t} = p$.

From Theorems 1, 2 and 3, we obtain the next main theorem of our paper.

Theorem 4. *The control police given by (12) and (13) is optimal for system (1) with cost funcional (2) i.e* $\lim_{t\to\infty} \frac{C_t}{t} = p$.

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Streszczenie

W pracy rozważamy jednowymiarowy liniowy układ sterowania stochastycznego na nieskończonym przedziale czasowym z kwadratowym funkcjonałem kosztów. Rozpatrujemy adaptacyjne zadanie sterowania optymalnego. Polega ono na znalezieniu sterowania minimalizującego wartość funkcjonału kosztów, przy założeniu że współczynniki rozpatrywanego równania są nieznane. Rozwiązanie tego zadania podzielono na dwie części. Częścią pierwszą jest identyfikacja obiektu sterowania na podstawie obserwacji realizacji. Identyfikację przeprowadza się metodą największej wiarygodności. Dodatkowo dla

uzyskania silnej zgodności estymatorów założono, że dysponujemy dodatkową informacją w postaci obserwacji procesu niesterowanego. Częścią drugą jest konstrukcja optymalnej strategii sterowania. Strategię tę konstruujemy w oparciu o uzyskaną wcześniej ocenę największego prawdopodobieństwa, zastępując prawdziwe wartości parametrów ich ocenami. Wykazano, że tak skonstruowane oceny są silnie zgodne, a strategia sterowania jest optymalna.