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# ON GROUPS OF FINITE NORMAL RANK

**Summary**. In the article the definition of a normal rank of a group is introduced. A theorem on a normal rank of the nonabelian metabelian $p$-group with some restrictions is proved.

## O GRUPACH SKOŃCZONJ NORMALNEJ RANGI

**Streszczenie**. W pracy wprowadza się definicję normalnej rangi grupy oraz dowodzi się twierdzenia o skończoności normalnej rangi dla nieabelowej metabelowej $p$-grupy.

A number of authors studied the groups in which finiteness conditions were laid on some systems of their subgroups [1]. Earlier the author investigated the groups of finite $\mathcal{F}$-rank [2], where $\mathcal{F}$ was some system of nonabelian finitely generated subgroups of a group.

In this article an investigation of the groups of finite normal rank is begun.

**Definition 1.** *We shall say that a group $G$ has finite normal rank $r$, if $r$ is a minimal number with the property that for any finite set of elements $g_1, g_2, ..., g_n$ of a group $G$ there are the elements $h_1, h_2, ..., h_m$ of $G$ such that $m \leq r$ and*

$$< g_1, g_2, ..., g_n >^G = < h_1, h_2, ..., h_m >^G .$$

*In the case when there is no such a number $r$, the normal rank of a group $G$ is considered to be infinite.*

We shall use the notation $r_n(G)$ for the normal rank of a group $G$.

The principal result of this article is the theorem.

**Theorem 1.** *Let $G$ be a nonabelian $p$-group, where $p$ is a prime number. Let $A$ be a normal subgroup of $G$ which is an elementary abelian $p$-group. The quotient group $G/A$ is isomorphic to the quasicyclic $p$-group. If the subgroup $A$ can be generated as a normal subgroup by $n$ elements, i.e.*

$$A =< a_1, a_2, ..., a_n >^G,$$

*then the normal rank of a group $G$ is finite and $r_n(G) \leqslant n+1$.*

This result was announced in [3] earlier.

We shall need the following lemma to prove the theorem.

**Lemma 1.** *The normal rank of the wreath product of a group of prime order $p$ and quasicyclic $p$-group is equal to 2.*

**Proof.** Let $A$ be the basis of the wreath product $W$, $W =< a > wrP$, where $P$ is a quasicyclic $p$-group. We shall prove at first that for any $b_1, b_2, ..., b_n$ from $A$ there is such an element $b \in A$, for which

$$< b_1, b_2, ..., b_n >^G =< b >^G .$$

Since $P = \bigcup_{i=1}^{\infty} < g_i >, |g_i| = p^i$, then $W = \bigcup_{i=1}^{\infty} (< a > wr < g_i >)$, and the elements $b_1, b_2, ..., b_n$ are contained in a subgroup $V =< a > wr < g_i >$ for some number $i$. The upper central series of the subgroup $V$ is

$$E = Z_0 < Z_1 < ... < Z_{p^i - 1} < Z_{p^i} < V,$$

where $Z_{p^i}$ is the basis of the wreath product $V$, factors $Z_{k+1}/Z_k, k = 0, 1, ..., p^i - 1$ have the orders $p$, the factor $V/Z_{p^i}$ is cyclic group of order $p^i$ [4].

The subgroups $B_k =< b_k >^{<g_i>}, k = 1, 2, ..., n$ are normal in the group $V$, therefore the intersections $B_k \bigcap Z_j, j = 1, 2, ..., p^i$ are nontrivial. Since the factors $Z_{k+1}/Z_k, k = 0, 1, ..., p^i - 1$ are cyclic of prime order $p$ then the equalities $B_k \bigcap Z_j = Z_j, j = 0, 1, ..., t_k$, are valid, where $t_k \leq p^i$. From this it follows that $B_k = Z_{t_k}$, therefore for any $l, m \leq n$ one of the subgroups $B_l, B_m$ is embeded in another. Consequently the subgroups $B_k, k = 1, 2, ..., n$ form the series of embeded subgroups

$$B_{k_1} < B_{k_2} < ... < B_{k_n} = B,$$

where $B =< b_1, b_2, ..., b_n >^{g_i}$. Therefore $B =< b >^{<g_i>}$, where $b = b_{k_n}$. The equality follows

$$< b_1, b_2, ..., b_n >^G =< b >^G .$$

Now we shall prove that for any $c_1, c_2, ..., c_r \in W$, the subgroup $C =< c_1, c_2, ..., c_r >^G$ can be generated as a $G$-subgroup by no more than 2 elements. It is sufficient to consider the case $C_1 \not\leq A$, where $C_1 =< c_1, c_2, ..., c_r >$. Since $C_1A/A \simeq C_1/C_1 \bigcap A$, the subgroup $C_1$ is finite and the quotient group $C_1A/A$ is cyclic, then we can choose the elements $d_1, d_2, ..., d_s$ such that

$$C =< d_1, d_2, ..., d_{s-1}, d_s >^G$$

and $d_i \in A, i = 1, 2, ..., s-1, d_s \notin A$. As we proved, there is an element $d \in A$ for which

$$< d_1, d_2, ..., d_{s-1} >^G =< d >^G,$$

therefore $C =< d, d_s >^G$. Consequently the normal rank of the wreath product $W$ is no more than 2.

For proving the equality $r_n(W) = 2$ we numerate the elements of the subgroup $P$ as $h_1, h_2, ...$ and assume that $a^{h_i} = a_i$. According to the structure of the subgroup $W$ the subgroup $A_0 =< a_i a_j^{-1}, i, j = 1, 2, ... >$ is normal in $W$ and the quotient group $W/A_0$ is isomorphic to the direct product of a group of the prime order $p$ and a quasicyclic $p$-group. Since the normal rank of the quotient group $W/A_0$ is equal to 2 and $r_n(W/A_0) \leq r_n(W)$, where $r_n(W) \leq 2$, then we have the equality $r_n(W) = 2$. Lemma is proved.∎

**Proof of the theorem.** At first we shall prove that for any finite set of elements $b_1, b_2, ..., b_k$ of $A$ there are such elements $c_1, c_2, ..., c_t$ of $A$ that $t \leq n$ and

$$< b_1, b_2, ..., b_k >^G =< c_1, c_2, ..., c_t >^G .$$

We shall prove this statement by the induction on the number $l$ of elements $a_1, a_2, ..., a_l$, where

$$A =< a_1, a_2, ..., a_l >^G .$$

If $l = 1$ then $A =< a_1 >^G$, therefore the group $G$ is isomorphic to some quotient group of the wreath product of a group of prime order $p$ and a quasicyclic $p$-group. From the proof of the lemma it follows that there is an element $b \in A$ for which

$$< b_1, b_2, ..., b_k >^G =< b >^G .$$

Let our statement be valid for $l = n - 1$. Let $l = n$ and

$$B =< b_1, b_2, ..., b_k >^G, A_1 =< a_1, a_2, ..., a_{n-1} >^G .$$

If the subgroup $B$ is contained in $A_1$ then according to the inductive assumption there exist such elements $c_1, c_2, ..., c_t, t \leq n$ that $B =< c_1, c_2, ..., c_t >^G$ . Let now $B \not\leq A_1$. The quotient group $G/A_1$ is isomorphic to some quotient group of the wreath product of a group of prime order $p$ and a quasicyclic $p$-group. From this fact and the isomorphism

$$BA_1/A_1 \simeq B/B \bigcap A_1$$

it follows by the lemma that there is an element $b \in B$ for which

$$B/B\bigcap A_1 =< b(B\bigcap A_1) >^G .$$

Consequently for every $b_i, i = 1,2,...,k$, there are such integers $n_1, n_2, ..., n_{r_i}$ and elements $g_1, g_2, ..., g_{r_i}$ of $G$ that the equalities

$$b_i = (b^{n_1})^{g_1}(b^{n_2})^{g_2}...(b^{n_{r_i}})^{g_{r_i}}h_i,$$

are valid where $h_i \in B\bigcap A_1$. Since the element $b$ belongs to the subgroup $B$ then

$$B =< b, h_1, \ldots, h_k >^G,$$

therefore

$$B =< b >^G< h_1, h_2, ..., h_k >^G . \tag{1}$$

According to the inductive assumption and the inclusion

$$< h_1, h_2, ..., h_k >^G\leq A_1$$

there are such elements $d_1, d_2, ..., d_m$ of $A$, that $m \leq n - 1$ and

$$< h_1, h_2, ..., h_k >^G=< d_1, d_2, ..., d_m >^G .$$

From this equality and (1) it follows that $B =< b, d_1, ..., d_m >^G, m \leq n - 1$. Our statement is proved.

Let now $B =< b_1, b_2, ..., b_k >^G$, where even if one of the elements $b_i, i = 1, 2, ..., k$ does not belong to the subgroup $A$. Since the subgroup $D$ generated by the elements $b_1, b_2, ..., b_k$ is finite then the intersection $D\bigcap A$ is finite too. Therefore there are the elements $c_1, c_2, ..., c_j, j \leq n$, for which

$$< D\bigcap A >^G=< c_1, c_2, ..., c_j >^G .$$

Since the quotient group $G/A$ is locally cyclic and $DA/A \simeq D/D\bigcap A$, there is such an element $c_{j+1}$ of $D$ that

$$< D >^G=< c_1, c_2, ..., c_j, c_{j+1} >^G .$$

Consequently the equality $B =< c_1, c_2, ..., c_{j+1} >^G$ is valid where $j + 1 \leq n + 1$. The theorem is proved.∎

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## Abstract

In the article the definition of a normal rank of a group is introduced.

**Definition.** *We shall say that a group $G$ has finite normal rank $r$ if $r$ is a minimal number with the property that for any finite set of elements $g_1, g_2, ..., g_n$ of a group $G$ there are the elements $h_1, h_2, ..., h_m$ of $G$ such that $m \leq r$ and*

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**Theorem.** *Let $G$ be a nonabelian $p$-group where $p$ is a prime number. Let $A$ be a normal subgroup of $G$ which is an elementary abelian $p$-group. Quotient group $G/A$ is isomorphic to the quasicyclic $p$-group. If the subgroup $A$ can be generated as a normal subgroup by $n$ elements, i.e.*

$$A =< a_1, a_2, ..., a_n >^G,$$

*then the normal rank of a group $G$ is finite and $r_n(G) \leqslant n + 1$.*

The following lemma plays an important role in the proof of the theorem.

**Lemma.** *The normal rank of a wreath product of a group of prime order $p$ and a quasicyclic $p$-group is equal to 2.*