ZESZYTY NAUKOWE POLITECHNIKI ŚLĄSKIEJ 1997

Seria: MATEMATYKA-FIZYKA z. 82 Nr kol. 1360

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THE NEUTRIX CONVOLUTION PRODUCT $(X_-^\lambda \ln^R X_-) \circledast (X_+^\mu \ln^S X_+)$

Summary. The existence of the neutrix convolution product $f \circledast g$ as well as the explicit formulae are proved for the distributions $f(x) = x_-^\lambda \ln^r x_-$ and $g(x) = x_+^\mu \ln^s x_+$, where $r, s \in \{0\} \cup \mathbf{N}$ and λ, μ are real numbers such that $\lambda, \mu \notin -\mathbf{N}$ and $\lambda + \mu \notin \mathbf{Z}$, and for some related pairs of distributions. The theorems of the paper generalize earlier results proved in the case $r = s = 0$ and $\lambda, \mu, \lambda + \mu \notin \mathbf{Z}$.

SPLOT $(X_-^\lambda \ln^R X_-) \circledast (X_+^\mu \ln^S X_+)$ W SENSIE NEUTRIKSU

Streszczenie. W pracy dowodzi się istnienia splotu $f \circledast g$ w sensie neutriksu i znajduje jego wartość dla dystrybucji $f(x) = x_-^\lambda \ln^r x_-$ i $g(x) = x_+^\mu \ln^s x_+$, gdzie $r, s \in \{0\} \cup \mathbf{N}$, a λ, μ są liczbami rzeczywistymi, takimi że $\lambda, \mu \notin -\mathbf{N}$ i $\lambda + \mu \notin \mathbf{Z}$, a także dla pewnych innych par dystrybucji. Twierdzenia podane w pracy uogólniają wcześniejsze wyniki otrzymane w przypadku, gdy $r = s = 0$ oraz $\lambda, \mu, \lambda + \mu \notin \mathbf{Z}$.

In the following we denote by $\mathbf{Z}$ the set of all integers, by $\mathbf{N}$ the set of all positive integers, by $-\mathbf{N}$ the set of all negative integers, by $\mathbf{N}_0$ the set of all nonnegative integers, by $-\mathbf{N}_0$ the set of all nonpositive integers and by $\mathbf{R}$ the set of all reals.

Moreover, we let $\mathcal{D}$ be the space of infinitely differentiable functions on $\mathbf{R}$ with compact support and $\mathcal{D}'$ be the space of distributions on $\mathbf{R}$, i.e. linear continuous functionals defined on $\mathcal{D}$ endowed with an appropriate topology (see e.g. [7]).

The research of the second named author was partially supported by the Royal Society, Great Britain

Definition 1. *Suppose that f and g are distributions in $\mathcal{D}'$ whose supports A and B satisfy the following condition of compatibility: for every compact set $K \subset \mathbf{R}$, the set $(K - A) \cap B$ is compact in $\mathbf{R}$. Then the convolution product $f * g$ in $\mathcal{D}'$ is defined by the formula:*

$$\langle (f * g)(x), \phi \rangle = \langle\, f(x)\,,\, \langle g(y), \phi(x + y)\rangle\,\rangle$$

for arbitrary ϕ in $\mathcal{D}$ (cf. [7]).

Obviously, the above definition embraces the two following particular cases of compatible supports:

(a) either A or B is bounded;

(b) A and B are bounded on the same side.

It follows from Definition 1 that if the convolution product $f * g$ exists, then also the convolution products $g * f$, $f * g'$, $f' * g$ and $f^{\vee} * g^{\vee}$ exist and

$$f * g = g * f, \tag{1}$$

$$(f * g)' = f * g' = f' * g, \tag{2}$$

$$(f \circledast g)^{\vee} = f^{\vee} \circledast g^{\vee}, \tag{3}$$

where $^{\vee}$ is the operation of replacing variable x by $-x$, defined formally as follows: $\phi^{\vee}(x) := \phi(-x)$ for $\phi \in \mathcal{D}, x \in \mathbf{R}$ and $\langle f^{\vee}, \phi \rangle := \langle f, \phi^{\vee} \rangle$ for $f \in \mathcal{D}', \phi \in \mathcal{D}$ (see [6]).

There exist in the literature various general, without any restrictions on the supports, definitions of the convolution product of distributions (cf. [6]), but for many pairs of distributions such convolution products do not exist.

In [2] the neutrix convolution product was defined so that it exists for a considerably larger class of pairs of distributions. In order to recall the definition of the neutrix convolution product we first of all let τ be a fixed function in $\mathcal{D}$ satisfying the following properties:

(i) $\tau(x) = \tau(-x)$,

(ii) $0 \le \tau(x) \le 1$,

(iii) $\tau(x) = 1$ for $|x| \le \frac{1}{2}$,

(iv) $\tau(x) = 0$ for $|x| \ge 1$.

Next we define the sequence $\{\tau_n\}$ of functions setting

$$\tau_n(x) = \begin{cases} 1 & \text{if } |x| \le n \\ \tau(n^n x - n^{n+1}) & \text{if } x > n \\ \tau(n^n x + n^{n+1}) & \text{if } x < -n, \end{cases}$$

for $n \in \mathbf{N}$.

Throughout the paper, given a distribution f, by f_n we denote the distributions of the form

$$f_n := f\,\tau_n$$

for $n \in \mathbf{N}$.

The notion of a neutrix which allows the extension of limits of numerical sequences was introduced by van der Corput in [1] and is based on a suitably chosen set of negligible functions.

As in [2], we adopt in this paper the following definition of negligible functions:

Definition 2. *The set of negligible functions of the neutrix N with the domain $N' = \mathbf{N}$ and the range $N'' = \mathbf{R}$ consists of all finite linear sums of the functions*

$$n^\lambda \ln^{r-1} n, \; \ln^r n \quad (\lambda > 0,\; r \in \mathbf{N})$$

and all functions which converge to zero in the usual sense as n tends to infinity.

Recall now the definition of the neutrix convolution product given in [2].

Definition 3. *The neutrix convolution product $f \circledast g$ of two distributions f and g in $\mathcal{D}'$ is defined as the neutrix limit of the sequence $\{f_n * g\}$, provided that the limit $h \in \mathcal{D}'$ exists in the sense that*

$$\underset{n\to\infty}{\mathrm{N-lim}} \langle f_n * g, \phi\rangle = \langle h, \phi\rangle$$

for all ϕ in $\mathcal{D}$, where N is the neutrix described in Definition 2.

Note that in this definition the convolution product $f_n * g$ is meant in the sense of Definition 1 (the distributions f_n have bounded support since the support of τ_n is contained in the interval $[-n - n^{-n}, n + n^{-n}]$) and that the distribution h in Definition 3 is unique.

The following theorem was proved in [6] and shows that Definition 3 is an extension of Definition 1.

Theorem 1. *Let f and g be distributions with compatible supports. Then the neutrix convolution product $f \circledast g$ exists and*

$$f \circledast g = f * g.$$

The neutrix convolution product has the following important properties, analogous to the first of equations in (2) and to (3) (see [2] and [6]):

Theorem 2. *Let f and g be distributions in $\mathcal{D}'$ and suppose that the neutrix convolution product $f \circledast g$ exists. Then the neutrix convolution products $f \circledast g'$ and $f^{\vee} \circledast g^{\vee}$ exist and*

$$\begin{aligned} (f \circledast g)' &= f \circledast g'; \\ (f \circledast g)^{\vee} &= f^{\vee} \circledast g^{\vee}. \end{aligned}$$

Note however that equation (1) does not necessarily hold for the neutrix convolution product and that $(f \circledast g)'$ is not necessarily equal to $f' \circledast g$.

In [2] the following result was obtained:

Theorem 3. *The neutrix convolution product $x_-^\lambda \circledast x_+^s$ exists and*

$$x_-^\lambda \circledast x_+^s = (-1)^{s+1} B(\lambda+1, s+1)\, x_-^{\lambda+s+1} \tag{4}$$

for $\lambda \in (-1, \infty)$ and $s \in \mathbf{N}_0$, where B denotes the beta function.

Later, the following two theorems were proved in [3] and [4], respectively:

Theorem 4. *The neutrix convolution product $x_-^\lambda \circledast x_+^s$ exists and satisfies equation* (4) *for $\lambda \in (-\infty, -1] \setminus (-\mathbf{N})$ and $s \in \mathbf{N}_0$.*

Theorem 5. *The neutrix convolution product $x_-^s \circledast x_+^\lambda$ exists and*

$$x_-^s \circledast x_+^\lambda = (-1)^{s+1} B(\lambda+1, s+1)\, x_+^{\lambda+s+1}$$

for $\lambda \in \mathbf{R} \setminus \mathbf{Z}$ and $s \in \mathbf{N}_0$.

The next theorem was proved in [5].

Theorem 6. *The neutrix convolution product $x_-^\lambda \circledast x_+^\mu$ exists and*

$$x_-^\lambda \circledast x_+^\mu = B(-\lambda-\mu-1, \mu+1)\, x_-^{\lambda+\mu+1} + B(-\lambda-\mu-1, \lambda+1)\, x_+^{\lambda+\mu+1}, \tag{5}$$

for $\lambda, \mu \in \mathbf{R}$ such that $\lambda, \mu, \lambda+\mu \notin \mathbf{Z}$.

In the following, we are going to generalize the last theorem by proving the existence of the convolution products of the form $(x_-^\lambda \ln^r x_-) \circledast (x_+^\mu \ln^s x_+)$ for all $r, s \in \mathbf{N}_0$ and λ, μ such that $\lambda, \mu \notin -\mathbf{N}$ and $\lambda + \mu \notin \mathbf{Z}$. It appears that these convolution products may be expressed in a concise form as the respective distributional derivatives with respect to λ and μ of the right hand side of (5). For this aim we need some auxiliary results on the beta function.

It was proved in [9] that

$$B_{r,s}(\lambda,\mu) := D_\lambda^r D_\mu^s B(\lambda,\mu) = \mathrm{N}\!-\!\lim_{n\to\infty} \int_{1/n}^{1-1/n} t^{\lambda-1} \ln^r t\,(1-t)^{\mu-1} \ln^s(1-t)\,dt$$

for $r, s \in \mathbf{N}_0$ and $\lambda, \mu \notin -\mathbf{N}_0$, where

$$D_\lambda^r := \frac{\partial^r}{\partial\lambda^r}, \quad D_\mu^s := \frac{\partial^s}{\partial\mu^s}.$$

In particular, if $\mu > 0$ and $\lambda \notin -\mathbf{N}_0$, the above expression can be replaced by

$$B_{r,s}(\lambda,\mu) = \mathrm{N}\!-\!\lim_{n\to\infty} \int_{1/n}^{1} t^{\lambda-1} \ln^r t\,(1-t)^{\mu-1} \ln^s(1-t)\,dt. \tag{6}$$

In the lemma below (α_n) is the sequence of positive numbers tending to 0, given by one of the formulae:

(a) $\alpha_n = 1/n$ for $n \in \mathbf{N}$;

(b) $\alpha_n = x/n$ for $n \in \mathbf{N}$;

(c) $\alpha_n = x/(x+n)$ for $n \in \mathbf{N}$.

Lemma. *If $x > 0$, then*

$$B_{r,s}(\lambda,\mu) = \mathrm{N}\!-\!\lim_{n\to\infty} \int_{\alpha_n}^{1} t^{\lambda-1} \ln^r t\,(1-t)^{\mu-1} \ln^s(1-t)\,dt \tag{7}$$

for $r, s \in \mathbf{N}_0$, $\mu \in (0,\infty)$ and $\lambda \in \mathbf{R} \setminus (-\mathbf{N}_0)$, where (α_n) is any of the three numerical sequences given by formulas $(a) - (c)$ above.

Proof. Choose a positive integer p such that $p + \lambda > 0$ and let $\sum_{i=0}^{p-1} a_i t^i$ be the sum of the first p terms in the Taylor expansion of $(1-t)^{\mu-1} \ln^s(1-t)$.

We shall prove first that

$$\mathrm{N}\!-\!\lim_{n\to\infty} \int_{\alpha_n}^{1} t^{\lambda+i-1} \ln^r t\,dt = \frac{(-1)^r r!}{(\lambda+i)^{r+1}} \tag{8}$$

for $r \in \mathbf{N}_0$ in all cases $(a) - (c)$, defined above.

Since $\lambda + i \neq 0$, we have

$$\int_\alpha^1 t^{\lambda+i-1}\,dt = \frac{1-\alpha^{\lambda+i}}{\lambda+i} \tag{9}$$

and further, if $r \in \mathbf{N}$,

$$\int_\alpha^1 t^{\lambda+i-1}\ln^r t\,dt = -\frac{\alpha^{\lambda+i}\ln^{r+1}\alpha}{\lambda+i} - \frac{r}{\lambda+i}\int_\alpha^1 t^{\lambda+i-1}\ln^{r-1} t\,dt. \tag{10}$$

Replacing α in (9) and (10) by $\alpha_n = x/(x+n)$, we have

$$\int_{\alpha_n}^1 t^{\lambda+i-1}\,dt = \frac{1-n^{-\lambda-i}x^{\lambda+i}(1+x/n)^{-\lambda-i}}{\lambda+i}$$

and

$$\int_{\alpha_n}^1 t^{\lambda+i-1}\ln^r t\,dt = -\frac{1}{(\lambda+i)}\left[\alpha_n^{\lambda+i}(\ln x - \ln(x+n))^r + r\int_{\alpha_n}^1 t^{\lambda+i-1}\ln^{r-1} t\,dt\right],$$

since λ is not an integer. It follows by induction that

$$\mathrm{N}\!-\!\lim_{n\to\infty}\int_{x/(x+n)}^1 t^{\lambda+i-1}\ln^{r+1} t\,dt = \frac{(-1)^r r!}{(\lambda+i)^{r+1}}$$

for $r \in \mathbf{N}$, i.e. (8) follows in case (c).

In a similar way, one can check that (8) holds true in cases (a) and (b).

Since $p + \lambda > 0$, the integral

$$I := \int_0^1 t^{\lambda-1}\ln^r t\left[(1-t)^{\mu-1}\ln^s(1-t) - \sum_{i=0}^{p-1} a_i t^i\right]dt$$

exists. Hence, in view of (6) and (8),

$$\begin{aligned}
B_{r,s}(\lambda,\mu) &= \mathrm{N}\!-\!\lim_{n\to\infty}\int_{1/n}^1 t^{\lambda-1}\ln^r t\,(1-t)^{\mu-1}\ln^s(1-t)\,dt \\
&= I + \sum_{i=0}^{p-1} a_i\left[\mathrm{N}\!-\!\lim_{n\to\infty}\int_{1/n}^1 t^{\lambda+i-1}\ln^r t\,dt\right] \\
&= I + \sum_{i=0}^{p-1} a_i\left[\mathrm{N}\!-\!\lim_{n\to\infty}\int_{\alpha_n}^1 t^{\lambda+i-1}\ln^r t\,dt\right] \\
&= \mathrm{N}\!-\!\lim_{n\to\infty}\int_{\alpha_n}^1 t^{\lambda-1}\ln^r t\,(1-t)^{\mu-1}\ln^s(1-t)\,dt
\end{aligned}$$

for the sequences (α_n) of the form (b) and (c).

Consequently, equation (7) follows. □

Theorem 7. *The neutrix convolution product* $(x_-^\lambda \ln^r x_-) \circledast (x_+^\mu \ln^s x_+)$ *exists and*

$$\begin{aligned}&(x_-^\lambda \ln^r x_-) \circledast (x_+^\mu \ln^s x_+)\\ &= D_\lambda^r D_\mu^s [B(-\lambda-\mu-1, \mu+1)\, x_-^{\lambda+\mu+1} + B(-\lambda-\mu-1, \lambda+1)\, x_+^{\lambda+\mu+1}]\end{aligned} \tag{11}$$

for $r, s \in \mathbf{N}_0$ *and* $\lambda, \mu \in \mathbf{R} \setminus (-\mathbf{N})$ *such that* $\lambda + \mu \notin \mathbf{Z}$.

Proof. The proof will consist of three parts, depending on the values of λ and μ. In all three parts we assume that $r, s \in \mathbf{N}_0$.

Part I: $\lambda, \mu > -1$; $\lambda + \mu \notin \{-1\} \cup \mathbf{N}_0$.

First notice that in our case $x_-^\lambda \ln^r x_-$ and $x_+^\mu \ln^s x_+$ are locally integrable functions. Put

$$(x_-^\lambda \ln^r x_-)_n := (x_-^\lambda \ln^r x_-)\, \tau_n(x).$$

Then the convolution product $(x_-^\lambda \ln^r x_-)_n * (x_+^\mu \ln^s x_+)$ exists both in the sense of Definition 1 and in the classical sense and we have

$$\begin{aligned}&(x_-^\lambda \ln^r x_-)_n * (x_+^\mu \ln^s x_+)\\ &= \int_{-\infty}^{\infty} y_-^\lambda \ln^r y_- \tau_n(y)(x-y)_+^\mu \ln^s (x-y)_+ \, dy = I_1^n + I_2^n,\end{aligned} \tag{12}$$

where

$$I_1^n := \int_{-n}^{0} (-y)^\lambda \ln^r(-y)\; (x-y)_+^\mu \ln^s (x-y)_+ \, dy; \tag{13}$$

$$I_2^n := \int_{-n-n^{-n}}^{-n} (-y)^\lambda \ln^r(-y) \tau_n(y)(x-y)_+^\mu \ln^s (x-y)_+ \, dy. \tag{14}$$

In the case $x < 0$, we substitute in the integral in (13) $y = xt^{-1}$ and obtain

$$\begin{aligned}I_1^n &= \int_{-n}^{x} (-y)^\lambda \ln^r(-y)(x-y)^\mu \ln^s(x-y)\, dy\\ &= (-x)^{\lambda+\mu+1} \int_{-x/n}^{1} t^{-\lambda-\mu-2} [\ln(-x) - \ln t]^r (1-t)^\mu [\ln(-x) + \ln(1-t) - \ln t]^s \, dt\\ &= \sum_{i=0}^{r} \sum_{k=0}^{s} \sum_{j=0}^{k} c_{i,j,k}\, I_{i,j,k}^n\, (-x)^{\lambda+\mu+1} \ln^{r+s-i-k}(-x),\end{aligned} \tag{15}$$

where

$$c_{i,j,k} := (-1)^{i+j} \binom{r}{i} \binom{s}{k} \binom{k}{j}; \quad I_{i,j,k}^n := \int_{-x/n}^{1} t^{-\lambda-\mu-2} \ln^{i+j} t\; (1-t)^\mu \ln^{k-j}(1-t)\, dt$$

for the respective integers i, j, k.

Since $-x > 0$, it follows from the lemma that

$$\mathrm{N}\text{-}\lim_{n\to\infty} I^n_{i,j,k} = B_{i+j,k-j}(-\lambda-\mu-1,\mu+1) \tag{16}$$

for the respective i,j,k.

On the other hand, using the formula $x_+^\lambda \ln^p x_+ = D^p_\lambda x^\lambda_+$ for $p\in\mathbf{N}$ (see [8]) and the equations:

$$\begin{aligned}
&D_\lambda B_{l,m}(-\lambda-\mu-1,\mu+1) = -B_{l+1,m}(-\lambda-\mu-1,\mu+1);\\
&D_\mu B_{l,m}(-\lambda-\mu-1,\mu+1) =\\
&\qquad = -B_{l+1,m}(-\lambda-\mu-1,\mu+1) + B_{l,m+1}(-\lambda-\mu-1,\mu+1),
\end{aligned}$$

valid for any $l,m\in\mathbf{N}$, one can prove by induction with respect to $r+s$ that

$$\begin{aligned}
&D^r_\lambda D^s_\mu[B(-\lambda-\mu-1,\mu+1)\,x_-^{\lambda+\mu+1}] =\\
&= \sum_{i=0}^{r}\sum_{k=0}^{s}\sum_{j=0}^{k} c_{i,j,k}\, B_{i+j,k-j}(-\lambda-\mu-1,\mu+1)\; x_-^{\lambda+\mu+1}\ln^{r+s-i-k} x_-.
\end{aligned} \tag{17}$$

Combining (15), (16) and (17), we get

$$\mathrm{N}\text{-}\lim_{n\to\infty} I^n_1 = D^r_\lambda D^s_\mu[B(-\lambda-\mu-1,\mu+1)\,x_-^{\lambda+\mu+1}] \tag{18}$$

in the case $x<0$.

If $x>0$, we use the substitution $y = x\,(1-t^{-1})$ in the integral in (13) and then

$$\begin{aligned}
I^n_1 &= \int_{-n}^{0}(-y)^\lambda \ln^r(-y)(x-y)^\mu \ln^s(x-y)\,dy\\
&= x^{\lambda+\mu+1}\int_{x/(x+n)}^{1} t^{-\lambda-\mu-2}[\ln x + \ln(1-t) - \ln t]^r(1-t)^\lambda[\ln x - \ln t]^s\,dt\\
&= \sum_{i=0}^{s}\sum_{k=0}^{r}\sum_{j=0}^{k} c_{i,j,k}\, J^n_{i,j,k}\; x^{\lambda+\mu+1}\ln^{r+s-i-k} x
\end{aligned} \tag{19}$$

where

$$J^n_{i,j,k} := \int_{x/(x+n)}^{1} t^{-\lambda-\mu-2}\ln^{i+j} t\;(1-t)^\lambda \ln^{k-j}(1-t)\,dt.$$

By the lemma,

$$\mathrm{N}\text{-}\lim_{n\to\infty} J^n_{i,j,k} = B_{i+j,k-j}(-\lambda-\mu-1,\lambda+1) \tag{20}$$

for the respective integers i,j,k.

On the other hand, replacing $-x$ by x and interchanging λ and μ as well as r and s in (17), we get

$$\begin{aligned}
&D^r_\lambda D^s_\mu[B(-\lambda-\mu-1,\lambda+1)\,x_-^{\lambda+\mu+1}] =\\
&= \sum_{i=0}^{s}\sum_{k=0}^{r}\sum_{j=0}^{k} c_{i,j,k}\, B_{i+j,k-j}(-\lambda-\mu-1,\lambda+1)\, x_+^{\lambda+\mu+1}\ln^{r+s-i-k} x_+.
\end{aligned} \tag{21}$$

Combining (19), (20) and (21), we obtain

$$\operatorname*{N-lim}_{n\to\infty} I_1^n = D_\lambda^r D_\mu^s [B(-\lambda-\mu-1, \lambda+1) x_+^{\lambda+\mu+1}] \tag{22}$$

in the case $x > 0$.

Further, it is easily seen that

$$I_2^n = O(n^{-n+\lambda+\mu} \ln^{r+s} n),$$

so

$$\operatorname*{N-lim}_{n\to\infty} I_2^n = 0. \tag{23}$$

It now follows from equations (12), (18), (22) and (23) that

$$\begin{aligned} &\operatorname*{N-lim}_{n\to\infty} (x_-^\lambda \ln^r x_-)_n * (x_+^\mu \ln^s x_+) = \\ &= D_\lambda^r D_\mu^s [B(-\lambda-\mu-1, \mu+1) x_-^{\lambda+\mu+1} + B(-\lambda-\mu-1, \lambda+1) x_+^{\lambda+\mu+1}], \end{aligned}$$

which completes the proof of Part I.

Part II: $\lambda > -1$; $\mu \notin -\mathbf{N}$; $\lambda + \mu \notin \mathbf{Z}$.

Denote $\mathbf{N}_0^- := \emptyset$, $\mathbf{N}_i^- := \{-1, -2, \dots, -i\}$ and

$$X_i := \{(\lambda, \mu) : \lambda > -1,\ \mu > -i,\ \mu \notin \mathbf{N}_{i-1}^-,\ \lambda + \mu \notin \mathbf{N}_i^- \cup \mathbf{N}_0, \}.$$

for $i \in \mathbf{N}$.

To prove the assertion of the theorem under the conditions of this part we will show the existence of $(x_-^\lambda \ln^r x_-) \circledast (x_+^\mu \ln^s x_+)$ and equation (11) for arbitrary $i \in \mathbf{N}$ and $(\lambda, \mu) \in X_i$ by induction with respect to i.

For $i = 1$ our assertion is valid, in view of Part I. Fix $k \in \mathbf{N}$ and suppose that the assertion holds true for $i = k$. Now take $(\lambda, \mu) \in X_{k+1}$ and put $\bar\mu := \mu + 1$. Since $(\lambda, \bar\mu) \in X_k$ and $\bar\mu \neq 0$, the neutrix convolution product $(x_-^\lambda \ln^r x_-) \circledast (x_+^{\bar\mu} \ln^s x_+)$ exists for arbitrary $r, s \in \mathbf{N}_0$. Moreover, by Theorem 2 and induction hypothesis, the neutrix convolution product $(x_-^\lambda \ln^r x_-) \circledast (x_+^{\bar\mu} \ln^s x_+)'$ exists and

$$\begin{aligned} &(x_-^\lambda \ln^r x_-) \circledast (x_+^{\bar\mu} \ln^s x_+)' = (x_-^\lambda \ln^r x_-) \circledast [\bar\mu x_+^\mu \ln^s x_+ + s x_+^\mu \ln^{s-1} x_+] = \\ &= [(x_-^\lambda \ln^r x_-) \circledast (x_+^{\bar\mu} \ln^s x_+)]' = D_\lambda^r D_\mu^s [(\lambda+\mu+2) f_{\lambda,\mu}(x)], \end{aligned} \tag{24}$$

for $r, s \in \mathbf{N}_0$, where

$$f_{\lambda,\mu}(x) := B(\lambda-\mu-2, \mu+2)\, x_-^{\lambda+\mu+1} + B(\lambda-\mu-2, \lambda+1)\, x_+^{\lambda+\mu+1}.$$

Notice that

$$-(\lambda+\mu+2) B(-\lambda-\mu-2, \mu+2) = \bar\mu B(-\lambda-\mu-1, \mu+1); \tag{25}$$

$$(\lambda+\mu+2) B(-\lambda-\mu-2, \lambda+1) = \bar\mu B(-\lambda-\mu-1, \lambda+1), \tag{26}$$

due to the known property of the gamma function and its relation to the beta function. It follows from (25) and (26) that

$$(\lambda+\mu+2)f_{\lambda,\mu}(x)=\bar{\mu}g_{\lambda,\mu}(x), \tag{27}$$

where

$$g_{\lambda,\mu}(x):=B(-\lambda-\mu-1,\mu+1)\,x_-^{\lambda+\mu+1}+B(-\lambda-\mu-1,\lambda+1)\,x_+^{\lambda+\mu+1}.$$

By (27),

$$\begin{aligned}D_\lambda^r D_\mu^s[(\lambda+\mu+2)f_{\lambda,\mu}(x)] &= D_\lambda^r D_\mu^{s-1} D_\mu^1[(\mu+1)\,g_{\lambda,\mu}(x)] =\\ &= D_\lambda^r D_\mu^{s-1}[(\mu+1)D_\mu^1\,g_{\lambda,\mu}(x)]+D_\lambda^r D_\mu^{s-1}g_{\lambda,\mu}(x) =\\ &= D_\lambda^r D_\mu^{s-2}[(\mu+1)D_\mu^2\,g_{\lambda,\mu}(x)]+2D_\lambda^r D_\mu^{s-1}g_{\lambda,\mu}(x).\end{aligned}$$

and, by induction,

$$\begin{aligned}&D_\lambda^r D_\mu^s[(\lambda+\mu+2)f_{\lambda,\mu}(x)] =\\ &= D_\lambda^r[(\mu+1)D_\mu^s\,g_{\lambda,\mu}(x)]+sD_\lambda^r D_\mu^{s-1}\,g_{\lambda,\mu}(x).\end{aligned}$$

Consequently, by (24),

$$\begin{aligned}&(x_-^\lambda \ln^r x_-) \circledast [\bar{\mu}x_+^\mu \ln^s x_+ + sx_+^\mu \ln^{s-1} x_+]\\ &= \bar{\mu}D_\lambda^r D_\mu^s\,g_{\lambda,\mu}(x)+sD_\lambda^r D_\mu^{s-1}g_{\lambda,\mu}(x)\end{aligned} \tag{28}$$

for $r,s\in\mathbf{N}_0$.

Taking $s=0$, we see that $(x_-^\lambda \ln^r x_-) \circledast x_+^\mu$ exists and, since $\bar{\mu}\neq 0$,

$$(x_-^\lambda \ln^r x_-) \circledast x_+^\mu = D_\lambda^r\,g_{\lambda,\mu}(x),$$

i.e. (11) holds for $s=0$.

Assume that $(x_-^\lambda \ln^r x_-) \circledast (x_+^\mu \ln^{s-1} x_+)$ exists and

$$(x_-^\lambda \ln^r x_-) \circledast (x_+^\mu \ln^{s-1} x_+) = D_\lambda^r D_\mu^{s-1}g_{\lambda,\mu}(x) \tag{29}$$

for some $s\in\mathbf{N}_0$. Since the neutrix convolution products in (28) and (29) exist, it follows that also $(x_-^\lambda \ln^r x_-) \circledast (x_+^\mu \ln^s x_+)$ exists and

$$\begin{aligned}&\bar{\mu}(x_-^\lambda \ln^r x_-) \circledast (x_+^\mu \ln^s x_+) =\\ &= (x_-^\lambda \ln^r x_-) \circledast [\bar{\mu}x_+^\mu \ln^s x_+ + sx_+^\mu \ln^{s-1} x_+] - s(x_-^\lambda \ln^r x_-) \circledast (x_+^\mu \ln^{s-1} x_+) =\\ &= D_\lambda^r D_\mu^s[(\lambda+\mu+2)f_{\lambda,\mu}(x)] - sD_\lambda^r D_\mu^{s-1}g_{\lambda,\mu}(x) = \bar{\mu}D_\lambda^r D_\mu^s\,g_{\lambda,\mu}(x).\end{aligned}$$

Since $\bar{\mu}\neq 0$, it follows by induction with respect to s that our assertion holds for $(\lambda,\mu)\in X_{k+1}$ and this completes the proof of Part II.

Part III: the general case.

Denote

$$Y_i := \{(\lambda,\mu):\ \lambda > -i,\ \lambda \notin \mathbf{N}^-_{i-1},\ \mu \notin -\mathbf{N};\ \lambda+\mu \notin \mathbf{Z}\}.$$

We have to prove the assertion of the theorem for $(\lambda,\mu) \in Y_i$ and all $i \in \mathbf{N}$.

Evidently, the assertion is true for $(\lambda,\mu) \in Y_1$, due to Part II. Assume that the assertion holds for all pairs in Y_k, $k \in \mathbf{N}$ and let $(\lambda,\mu) \in Y_{k+1}$. Clearly, $(\bar\lambda,\mu) \in X_k$ and $\bar\lambda \neq 0$, where $\bar\lambda := \lambda + 1$.

Since the convolution product $(x_-^{\lambda+1}\ln^r x_-)_n * (x_+^\mu \ln^s x_+)$ exists in the sense of Definition 1 for $n \in \mathbf{N}$, equations (2) can be used. Given an arbitrary $\phi \in \mathcal{D}$ (let the support of ϕ be contained in the interval $[a,b]$), we have

$$\begin{aligned}\langle[(x_-^{\bar\lambda}\ln^r x_-)_n * (x_+^\mu \ln^s x_+)]', \phi(x)\rangle &= -\langle(x_-^{\bar\lambda}\ln^r x_-)_n * (x_+^\mu \ln^s x_+), \phi'(x)\rangle = \\ &= \langle\{(\bar\lambda)(x_-^{\lambda}\ln^r x_- + r(x_-^{\lambda}\ln^{r-1} x_-]_n * (x_+^\mu \ln^s x_+), \phi(x)\rangle + \\ &+\langle[x_-^{\bar\lambda}\ln^r x_- \tau_n'(x)] * (x_+^\mu \ln^s x_+), \phi(x)\rangle. \end{aligned} \tag{30}$$

The support of $\tau_n'(x)$ is contained in the interval $[-n-n^{-n}, -n]$. Therefore, for $n > -a$,

$$\langle(x_-^{\bar\lambda}\ln^r x_- \tau_n'(x)) * (x_+^\mu \ln^s x_+), \phi(x)\rangle = \int_a^b \phi(x)\, I_n(x)\, dx, \tag{31}$$

where

$$I_n(x) := \int_{-n-n^{-n}}^{-n} (-y)^{\bar\lambda}\ln^r(-y)\tau_n'(y)(x-y)^\mu \ln^s(x-y)\, dy\, dx,$$

with the functions $(-y)^{\bar\lambda}\ln^r(-y)$ and $(x-y)^\mu\ln^s(x-y)$ integrable on the domain of integration. Integration by parts yields

$$I_n(x) = h_n(x) + \int_{-n-n^{-n}}^{-n} [(-y)^{\bar\lambda}\ln^r(-y)(x-y)^\mu \ln^s(x-y)]'\tau_n(y)\, dy, \tag{32}$$

where

$$h_n(x) := n^{\bar\lambda}\ln^r n\ (x+n)^\mu \ln^s(x+n).$$

Choosing a positive integer p greater than $\lambda+\mu$, we can put

$$h_n(x) = n^{\lambda+\mu+1}\sum_{i=0}^{p-1}\sum_{j=0}^{s} \frac{a_{ij}x^i \ln^j n}{n^i} + O(n^{\lambda+\mu+1-p}\ln^{r+s} n).$$

Since $\lambda+\mu$ is not an integer, we conclude that

$$\mathrm{N}\!-\!\lim_{n\to\infty} h_n(x) = 0. \tag{33}$$

It is easily seen that

$$\lim_{n\to\infty}\int_{-n-n^{-n}}^{-n} [(-y)^{\bar\lambda}\ln^r(-y)\,(x-y)^\mu \ln^s(x-y)]'\tau_n(y)\, dy = 0 \tag{34}$$

and thus, by (31), (32), (33) and (34),

$$\mathrm{N-}\lim_{n\to\infty}\langle (x_-^{\bar\lambda}\ln^r x_-\,\tau_n'(x)) * (x_+^\mu \ln^s x_+), \phi(x)\rangle = 0. \tag{35}$$

Now, using (30), (35), the induction hypothesis and the fact that $\bar\lambda \neq 0$, it can be proved by induction with respect to r that the assertion of the theorem is true for $(\lambda,\mu) \in Y_{k+1}$, in much the same way as in Part II (the role of μ in Part II is now played by λ).

By induction, the assertion in the general case follows. □

Corollary. *The neutrix convolution product* $(x_+^\lambda \ln^r x_+) \circledast (x_-^\mu \ln^s x_-)$ *exists and*

$$\begin{aligned}&(x_+^\lambda \ln^r x_+) \circledast (x_-^\mu \ln^s x_-) =\\ &= D_\lambda^r D_\mu^s [B(-\lambda-\mu-1,\mu+1)\,x_+^{\lambda+\mu+1} + B(-\lambda-\mu-1,\lambda+1)\,x_-^{\lambda+\mu+1}]\end{aligned}$$

for $r,s \in \mathbf{N}_0$ *and* $\lambda,\mu \in \mathbf{R}\setminus(-\mathbf{N})$ *such that* $\lambda+\mu \notin \mathbf{Z}$.

Proof. The assertion of the corollary follows immediately by the second part of Theorem 2, i.e. on replacing x by $-x$ in equation (11). □

The distributions $|x|^\lambda \ln^r|x|$ and $\operatorname{sgn} x.|x|^\lambda \ln^r|x|$ are defined by

$$|x|^\lambda \ln^r|x| = x_+^\lambda \ln^r x_+ + x_-^\lambda \ln^r x_-, \quad \operatorname{sgn} x.|x|^\lambda \ln^r|x| = x_+^\lambda \ln^r x_+ - x_-^\lambda \ln^r x_-.$$

We finally note that since the convolution products $(x_+^\lambda \ln^r x_+) * (x_+^\mu \ln^s x_+)$ and $(x_-^\lambda \ln^r x_-) * (x_-^\mu \ln^s x_-)$ exist by Definition 1 and since the neutrix convolution product is clearly distributive with respect to addition, it follows that further neutrix convolution products such as

$$(x_-^\lambda \ln^r x_-) \circledast (|x|^\mu \ln^s|x|), \quad (x_+^\lambda \ln^r x_+) \circledast (|x|^\mu \ln^s|x|),$$

$$(x_-^\lambda \ln^r x_-) \circledast (\operatorname{sgn} x.|x|^\mu \ln^s|x|), \quad (|x|^\lambda \ln^r|x|) \circledast (x_-^\mu \ln^s x_-)$$

exist for $r,s \in \mathbf{N}_0$, $\lambda,\mu \in \mathbf{R}\setminus(-\mathbf{N})$ with $\lambda+\mu \notin Z$.

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Streszczenie

Splot $f * g$ dystrybucji Schwartza $f, g \in \mathcal{D}'$ definiuje się wzorem

$$\langle f * g, \phi \rangle = \langle f\tau_n * g, \phi \rangle,$$

postulując, by granica istniała w przestrzeni $\mathcal{D}'$ dla dowolnych funkcji $\phi \in \mathcal{D}$ oraz ciągów (τ_n) elementów przestrzeni $\mathcal{D}$ aproksymujących funkcję 1 i należących do określonej klasy.

Splot dystrybucji f i g, oznaczany symbolem $f \circledast g$, rozumiany jest w tej pracy w ogólniejszym sensie, bowiem zakłada się, że granica po prawej stronie powyższej równości istnieje w sensie neutriksu (van der Corput [1]), wyznaczonego przez przestrzeń liniową funkcji zaniedbywalnych $f\colon \mathbf{N} \to \mathbf{R}$, generowaną przez wszystkie funkcje f zbieżne do 0 oraz funkcje f postaci:

$$n^\lambda \ln^{r-1} n, \ \ln^r n \quad (\lambda > 0, \ r \in \mathbf{N}).$$

Dowodzi się, że splot $f \circledast g$ istnieje w sensie neutriksu dla dystrybucji $f(x) = x_-^\lambda \ln^r x_-$ i $g(x) = x_+^\mu \ln^s x_+$ oraz zachodzi równość:

$$f \circledast g = D_\lambda^r D_\mu^s [B(-\lambda - \mu - 1, \mu + 1)\, x_-^{\lambda+\mu+1} + B(-\lambda - \mu - 1, \lambda + 1)\, x_+^{\lambda+\mu+1}]$$

dla $r, s \in \{0\} \cup \mathbf{N}$ oraz $\lambda, \mu \in \mathbf{R} \setminus (-\mathbf{N})$, takich że $\lambda + \mu \notin \mathbf{Z}$, gdzie

$$D_\lambda^r = \frac{\partial^r}{\partial \lambda^r}, \quad D_\mu^s = \frac{\partial^s}{\partial \mu^s},$$

a B oznacza funkcję beta Eulera.

Stąd wynika także istnienie następujących splotów:

$$(x_+^\lambda \ln^r x_+) \circledast (x_-^\mu \ln^s x_-); \quad (x_-^\lambda \ln^r x_-) \circledast (|x|^\mu \ln^s |x|)$$
$$(x_+^\lambda \ln^r x_+) \circledast (|x|^\mu \ln^s |x|); \quad (x_-^\lambda \ln^r x_-) \circledast (\operatorname{sgn} x . |x|^\mu \ln^s |x|);$$
$$(|x|^\lambda \ln^r |x|) \circledast (x_-^\mu \ln^s x_-)$$

przy tych samych warunkach na r, s, λ, μ, co poprzednio.