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ON PERIODIC DISTRIBUTIONS AND THE DIRICHLET PROBLEM ON THE UNIT DISC

Summary. A short description of the theory of periodic distributions is presented together with its applications to the Dirichlet problem for the Laplace equation on the unit disc with distributional data.

DYSTRYBUCJE OKRESOWE I PROBLEM DIRICHLETA DLA KOŁA JEDNOSTKOWEGO

Streszczenie. W pracy została zwięzle przedstawiona teoria dystrybucji okresowych oraz rozwiążanie problemu Dirichleta dla równania Laplace'a na kole jednostkowym z dystrybucyjnymi warunkami brzegowymi.

1. Let P denote the set of complex 2π -periodic smooth functions defined on \mathbb{R} . Similarly the symbol $L^2_{2\pi}$ denotes the space of 2π -periodic locally square integrable functions on \mathbb{R} . As usual, we shall denote by

$$(\varphi, \psi) := \frac{1}{2\pi} \int_{-\pi}^{\pi} \varphi(t) \overline{\psi(t)} dt$$

the inner product of φ and ψ in $L^2_{2\pi}$. The symbol $\|\varphi\| = (\varphi, \varphi)^{\frac{1}{2}}$ will mean the norm of φ in $L^2_{2\pi}$. The space $L^2_{2\pi}$ is complete under the norm $\|\cdot\|$.

We start with presenting of certain properties of the operator $M := -\frac{d^2}{dt^2} + 1$. Integration by parts shows that for every φ and ψ in P we have

$$(M^k \varphi, \psi) = (\varphi, M^k \psi), \quad k = 0, 1, 2, \dots, \tag{1}$$

$$(\varphi, \varphi) \leq (M\varphi, \varphi). \tag{2}$$

If we take P as the domain of M then M is a symmetric bounded from below linear operator in $L^2_{2\pi}$. One can show that

$$(M^k \varphi, \varphi) \leq (M^{k+1} \varphi, \varphi) \quad \text{for } \varphi \in P \text{ and } k = 0, 1, 2, \dots \quad (3)$$

Put $(\varphi, \psi)_k := (M^k \varphi, \psi)$ for $\varphi, \psi \in P$. Taking $\|\varphi\|_k := (\varphi, \varphi)_k^{\frac{1}{2}}$, it is easily seen that

$$\|\varphi\| = \|\varphi\|_0 \leq \|\varphi\|_1 \leq \dots \leq \|\varphi\|_k \leq \dots \quad \text{for } \varphi \in P. \quad (4)$$

It is known that these norms are compatible ([4], p.14) in the following sense: if a sequence $\{\varphi_v\}$ of elements φ_v belonging to P converges to zero with respect to the norm $\|\cdot\|_m$ and is a Cauchy sequence in the norm $\|\cdot\|_n$ ($m \leq n$) then it is also converges to zero in the norm $\|\cdot\|_n$ ([2], p.13). Since the norms $\|\cdot\|_k$, $k = 0, 1, 2, \dots$ are compatible and the relations (4) hold, therefore the completion P_k of P under the norm $\|\cdot\|_k$ may be regarded as a subspace of $L^2_{2\pi}$ ([2], p.18). It is easy to check that φ is in P_k if and only if the distributional derivative $D^\alpha \varphi \in L^2_{2\pi}$ for $\alpha = 0, \dots, k$. The norm $\|\cdot\|_k$ and the bilinear form $(\cdot, \cdot)_k$ can be extended by continuity on P_k and $P_k \times P_k$ respectively, such that

$$(\varphi, \psi)_k = (\psi, \varphi)_k \quad \text{for } \varphi, \psi \in P_k, \quad (5)$$

$$(\varphi, \varphi)_k \leq (\varphi, \varphi)_{k+1} \quad \text{for } \varphi \in P_{k+1}, \quad (6)$$

$$(\varphi, \psi)_k = (M^k \varphi, \psi) \quad \text{for } \varphi \in P \text{ and } \psi \in P_k. \quad (7)$$

The vector space P_k under the bilinear form $(\cdot, \cdot)_k$ is a Hilbert space. The vector space P will be equipped with the projective topology generated by the norms $\|\cdot\|_k$, $k = 0, 1, 2, \dots$ (countably Hilbert space, [3], p.57). The dual space of P will be denoted by P' .

Definition 1. A linear continuous form defined on P is said to be a periodic distribution.

Each periodic distribution Λ can be extended by continuity on some space P_k . This extension of Λ may be regarded as an element of P_{-k} , where P_{-k} is the dual space of P_k . A function ψ belonging to $L^2_{2\pi}$ is identified with the linear form (ψ, \cdot) .

It is known that the functions $e_v(t) = e^{ivt}$, $v \in Z$ constitute an orthonormal complete system in $L^2_{2\pi}$. Now we shall construct orthonormal systems in the spaces P_k , $k = 1, 2, \dots$. In order to we put $e_{vk}(t) = (v^2 + 1)^{-\frac{1}{2}} e_v(t)$.

Theorem 1. *The functions e_{vk} , $v \in Z$ constitute an orthonormal complete system in P_k , $k = 0, 1, \dots$*

Proof. From (7) it follows that functions e_{vk} constitute an orthonormal system in P_k with respect to the bilinear form $(\cdot, \cdot)_k$. It remains to prove that the system $\{e_{vk}\}$ is complete in P_k .

Let $\varphi \in P_k$, then

$$\varphi = \sum_{v \in Z} (\varphi, e_v) e_v \quad \text{in } L^2_{2\pi}.$$

The series $\sum_{v \in Z} (\varphi, e_{vk}) e_{vk}$ may be regarded as the Fourier series of φ under the orthonormal system $\{e_{vk}\}$. We know that $\sum_{v \in Z} |(\varphi, e_{vk})_k|^2 \leq \|\varphi\|_k^2$. In view of the completeness of P_k with respect to the norm $\|\cdot\|_k$ we infer that there exists ψ in P_k such that $\psi = \sum_{v \in Z} (\varphi, e_{vk})_k e_{vk}$ in P_k . Because of (7) we have

$$(\varphi, e_{vk})_k = (v^2 + 1)^{\frac{k}{2}} (\varphi, e_v). \quad (8)$$

Hence we get $\psi = \sum_{v \in Z} (\varphi, e_v) e_v$ in P_k . Since the norms $\|\cdot\|$ and $\|\cdot\|_k$ are compatible therefore $\psi = \varphi$ in P_k . Thus the proof of theorem is finished. ■

Because $\sum_{v \in Z} |(\varphi, e_{vk})_k|^2 \leq \|\varphi\|_k^2$, therefore by (8) we have

$$\sum_{v \in Z} (v^2 + 1)^k |c_v|^2 < \infty, \quad c_v := (\varphi, e_v). \quad (9)$$

Note that

$$\varphi = \sum_{v \in Z} c_v e_v = \sum_{v \in Z} (v^2 + 1)^{\frac{k}{2}} c_v e_{vk}.$$

By virtue of the completeness of P_k and (9), we infer that the series $\sum_{v \in Z} (v^2 + 1)^{\frac{k}{2}} c_v e_{vk}$ converges to a function ψ belonging to P_k , therefore φ is in P_k .

We have proved the fact which can be stated as a separate

Theorem 2. *A function φ belonging to $L^2_{2\pi}$ is in P_k if and only if (9) holds. ■*

Now we describe the Fourier expansions of linear continuous forms belonging to the dual space P_{-k} of P_k .

Theorem 3. *A linear form Λ belonging to P' is in P_{-k} if and only if*

$$\sum_{v \in Z} \frac{|C_v|^2}{(v^2 + 1)^k} < \infty, \quad C_v = \Lambda(e_v). \quad (10)$$

Proof. Let $\Lambda \in P_{-k}$ then there exists a function ψ in P_k such that

$$\Lambda(\varphi) = (\psi, \varphi)_k \quad \text{for each } \varphi \in P_k. \quad (11)$$

The function ψ has the following Fourier representation

$$\psi = \sum c_{vk} e_{vk}, \quad c_{vk} = (\psi, e_{vk})_k. \quad (12)$$

Moreover, we have

$$\|\Lambda\|_{P_{-k}}^2 = \|\psi\|_k^2 = \sum_{v \in Z} |c_{vk}|^2. \quad (13)$$

In view of (8) we obtain

$$\Lambda(\varphi) = \left(\sum_{v \in Z} c_{vk} e_{vk}, \varphi \right)_k = \sum_{v \in Z} c_{vk} (v^2 + 1)^{\frac{k}{2}} (e_v, \varphi).$$

Taking $\varphi = e_\mu$ in the previous equality we get

$$\Lambda(e_\mu) = (\mu^2 + 1)^{\frac{k}{2}} c_{\mu k}. \quad (14)$$

Finally we have

$$\Lambda(\varphi) = \sum_{v \in Z} \Lambda(e_v) (e_v, \varphi), \quad \varphi \in P_k. \quad (15)$$

The numbers $\Lambda(e_v)$, $v \in Z$ are called the Fourier coefficients of Λ . From (13) and (14) it follows the following equality

$$\|\psi\|_k^2 = \sum_{v \in Z} \frac{|C_v|^2}{(v^2 + 1)^k}, \quad C_v = \Lambda(e_v). \quad (16)$$

Therefore (10) holds.

It remains to prove that (10) implies the existence of a function ψ belonging to P_k such that (11) holds. It is easy to check that the function has the required properties. ■

2. An application of the theory of periodic distributions can be made to the Dirichlet problem on the unit disc. The Dirichlet problem for a distributional data one can formulate as follows:

A periodic distribution $\Lambda \in P'$ is given and it is required to find a harmonic function $u(r, t)$ under polar variables defined on the unit disc such that $(r, \cdot) \rightarrow \Lambda$ as $r \rightarrow 1$ with respect to a suitable topology.

Theorem 4. For each periodic distribution Λ in P_{-k} there exists a harmonic function u in the open unit disc U such that its polar representation has the following property:

$$u(r, \cdot) \rightarrow \Lambda, \quad \text{when } r \rightarrow 1 \quad \text{under the topology } \sigma(P_{-k}, P_k) ([1], p.338).$$

Proof. Let $\Lambda \in P_{-k}$ and $C_v = \Lambda(e_v)$, $v \in Z$. We know that

$$\|\Lambda\|_{P_{-k}}^2 = \sum_{v \in Z} \frac{|C_v|^2}{(v^2 + 1)^k}.$$

From this it follows that $\limsup_{|v| \rightarrow \infty} |C_v|^{\frac{1}{|v|}} \leq 1$. Therefore the series $\sum_{v \in Z} C_v z^{|v|}$ converges almost uniformly in the open unit disc U . The complex function

$$v(x, y) = g(z) = \sum_{v=0}^{\infty} C_v z^v + \sum_{v=1}^{\infty} C_{-v} (\bar{z})^v, \quad z = x + iy$$

of real arguments x and y is a harmonic function in U . Moreover, the series $\sum_{v \in Z} C_v r^{|v|} e^{ivt}$ converges uniformly with respect to $t \in \mathbb{R}$ for $0 \leq r < 1$. Put $u(r, t) := \sum_{v \in Z} C_v r^{|v|} e^{ivt}$. The function $u(\cdot, \cdot)$ may be regarded as the polar representation of the harmonic function $v(\cdot, \cdot)$. It is easy to see that $u(r, \cdot) \in P$ for $0 \leq r < 1$. From this it follows that the following equality

$$u(r, \cdot) = M^k \sum_{v \in Z} \frac{C_v r^{|v|}}{(v^2 + 1)^k} e^v \quad (17)$$

holds. We have to prove that

$$\Lambda_r(\varphi) := (u(r, \cdot), \varphi) \rightarrow \Lambda(\varphi) \quad \text{as } r \rightarrow 1 \quad \text{and } \varphi \in P_k.$$

In view of (17) we infer that $(u(r, \cdot), \varphi) = \left(M^k \sum_{v \in Z} \frac{C_v r^{|v|}}{(v^2 + 1)^k} e^v, \varphi \right)$. Put

$$\psi_r := \sum_{v \in Z} \frac{C_v r^{|v|}}{(v^2 + 1)^k} e^v = \sum_{v \in Z} \frac{C_v r^{|v|}}{(v^2 + 1)^{\frac{k}{2}}} e^{vk} \quad \text{for } 0 \leq r < 1.$$

In view of (8) we have $\Lambda_r(\varphi) = (u(r, \cdot), \varphi) = (\psi_r, \varphi)_k$. By (16) we obtain

$$\|\Lambda_r\|_{P_{-k}}^2 = \|\psi_r\|_k^2 = \sum_{v \in Z} \frac{|C_v|^2 r^{2|v|}}{(v^2 + 1)^k} e^{2vk} \leq \|\Lambda\|_{P_{-k}}^2. \quad (18)$$

Because of (18) and the linear density of $\{e_v\}$ in P_k ([1], p.150) what is left is to show that $\Lambda_r(e_\mu) \rightarrow \Lambda(e_\mu)$ when $r \rightarrow 1$ for $\mu \in Z$. In fact,

$$\Lambda_r(e_\mu) = \left(\sum_{v \in Z} C_v r^{|v|} e_v, e_\mu \right) = r^{|\mu|} C_\mu \rightarrow C_\mu = \Lambda(e_\mu).$$

The proof is now complete. ■

3. Example. Let $\delta(\varphi) := \varphi(0)$. Let $\varphi \in P_1$ and $c_v = (\varphi, e_v)$, $v \in Z$, then by (9) we have $\sum_{v \in Z} |c_v| < \infty$. This implies that the Fourier series of φ converges uniformly to φ , therefore φ is continuous. From this it follows that the symbol $\delta(\varphi)$ is meaningful for φ in P .

Note that $C_v = \delta(e_v) = 1$ for $v \in Z$. From this and theorems 2 and 3 it follows that $\delta(\varphi) = (\psi, \varphi)_1$, where $\psi = \sum_{v \in Z} \frac{1}{v^2+1} e_v$. The function

$$u(r, t) = \sum_{v \in Z} r^{|v|} e^{ivt} = P_r(t)$$

(Poisson kernel) is the polar representation of the solution of the Dirichlet problem on the unit disc corresponding to the boundary data δ .

By Theorem 4 we obtain the following

Theorem 5. $P_r(\cdot) \rightarrow \delta$ as $r \rightarrow 1$ under the topology $\sigma(P_{-1}, P_1)$. ■

Remark. Another similar approach to the theory of periodic distributions was presented in [5], p.260.

References

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Streszczenie

W pracy została zwięzle przedstawiona teoria dystrybucji okresowych. P oznacza przestrzeń wektorową zespółonych funkcji gładkich 2π -okresowych, wyposażoną w topologię projektynową wyznaczoną przez przeliczalną rodzinę zgodnych norm hilbertowskich $\|\cdot\|_k^2 = (M^k \cdot, \cdot)$, $M = -\frac{d^2}{dt^2} + 1$. P' oznacza przestrzeń sprzężoną do przestrzeni P . Elementy przestrzeni P' nazywamy dystrybucjami okresowymi. Symbolem P_k , $k = 0, 1, \dots$ oznaczamy uzupełnienie przestrzeni P względem normy $\|\cdot\|_k$. P_{-k} oznacza przestrzeń sprzężoną do przestrzeni P_k . W pracy zostało udowodnione, że element Δ należący do P' jest elementem przestrzeni P_k , $k \in Z$ wtedy i tylko wtedy, gdy $\sum_{v \in Z} (v^2 + 1)^k |C_v|^2 < \infty$, gdzie C_v są współczynnikami Fouriera elementu Δ .

W drugiej części pracy został rozwiązany problem Dirichleta dla równania Laplace'a na kole jednostkowym z dystrybucyjnymi warunkami brzegowymi.