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ON SOME CHARACTERIZATION OF EHRESMANN'S PSEUDOGROUP

Summary. In [1] it was proved that the set of all diffeomorphisms of a quasi-algebraic space is the Ehresmann's pseudogroup. We notice that for proving this theorem we did not use properties of a quasi-algebraic space, so it is possible to generalize this theorem and formulate it for any set of functions. So we can say that the set of all diffeomorphisms of any set of functions is the Ehresmann's pseudogroup. We also show in this paper that in this way we can get any pseudogroup. So we get some charakterization of a pseudogroup.

O PEWNEJ CHARAKTERYZACJI PSEUDOGRUPY EHRESMANNA

Streszczenie. W artykule tym podajemy charakteryzację pseudogrupy Ehresmanna za pomocą zbioru dyfeomorfizmów rodziny funkcji. Stwierdzamy, że zbiór wszystkich dyfeomorfizmów dowolnej rodziny funkcji jest pseudogrupą Ehresmanna, korzystając z faktu, że analogiczne twierdzenie dla przestrzeni K-quasi-algebraicznych można uogólnić. Pokazujemy również w tej pracy, że każdą pseudogrupę Ehresmanna można otrzymać w powyższy sposób.

In [1] it was proved that the set of all diffeomorphisms of a quasi-algebraic space is the Ehresmann's pseudogroup. We notice that for proving this theorem we did not use properties of a quasi-algebraic space, so it is possible to generalize this theorem and formulate it for any set of functions. So we can say that the set of all diffeomorphisms of any set of functions is the Ehresmann's pseudogroup. We also show in this paper that in this way we can get any pseudogroup. So we get some charakterization of a pseudogroup.

According to [1] we adopt the following definitions and notations generalizing some of them. For any function g we denote the domain of g by D_g. If f is a function, then $g \circ f$ stands for the function defined on the set $f^{-1}(D_g)$ being the counter image of D_g given by f, i.e. $f^{-1}(D_g)$ is the set of all x of D_f such that $f(x) \in D_g$, by the formula

$(g \circ f)(x) = g(f(x))$ for $x \in f^{-1}(D_g)$. If S is a set, then $id|S$ stands for the identity function defined on S. For any set A of functions we will denote $\bigcup_{\alpha \in A} D_\alpha$ by $\underline{A}$ and the smallest topology on $\underline{A}$, such that all sets $D_\alpha, \alpha \in A$ are open, by $top\ A$.

Let A and B be sets of functions and f be a function which maps $\underline{A}$ into $\underline{B}$. We will say that f maps smoothly A into B, what we denote in form

$$f : A \longrightarrow B \tag{1}$$

iff for any $\beta \in B$ we have $\beta \circ f \in A$. We say that f is a diffeomorphism of A onto B, what we denote in form

$$f : A \xrightarrow{\approx} B \tag{2}$$

iff $f : A \longrightarrow B$ is one-to-one and $f^{-1} : B \longrightarrow A$.

For any set A of functions and any subset M of $\underline{A}$ let us define

$$A|M = \{\alpha|D_\alpha \cap M; \alpha \in A\} \tag{3}$$

$$A_M = \left\{\beta; \beta \text{ is a function } \wedge \bigwedge_{\beta \in D_\beta} \bigvee_{\alpha \in A} \quad \bigvee_{U \in (top\ A)|M} p \in U \subset D_\beta \cap D_\alpha \wedge \beta|U = \alpha|U \right\} \tag{4}$$

Here $(top\ A)|M$ stands for the topology on M induced by $top\ A$ and $\beta|U$ stands for the restriction of β to U.

For any sets A and B of functions let us define $Diff(A, B)$ as the set of all diffeomorphisns of A onto B and

$$Diff\ A = \bigcup_{U,V \in top\ A} Diff\ (A_U, A_V) \tag{5}$$

As it was stated in the introduction we can generalize the theorem 4.1. from [1] and we can formulate it for any set of functions because we did not use properties of a quasi-algebraic space for proving it.

Theorem 1. *If $\underline{A} \neq \emptyset$ then the set $Diff\ A \backslash \{\emptyset\}$ forms Ehresmann's pseudogroup on the topological space $(\underline{A},\ top\ A)$.*

To get a characterization of a pseudogroup we have to show that any Ehresmann's pseudogroup can be obtained get in this way.

Theorem 2. *For any Ehresmann's pseudogroup Γ on a topological space S the equality $Diff\ \Gamma \backslash \{\emptyset\} = \Gamma$ holds.*

Proof. It is obvious that this topology can not be different from $\{D_f\}_{f \in \Gamma} \cup \{\emptyset\}$. It means that it equals $top\ \Gamma$, too.

Let $f \in Diff\ \Gamma$ and $f \neq \emptyset$, so there exist non-empty U, $V \in top\ \Gamma$, that $f \in Diff\ (\Gamma_U, \Gamma_V)$. From the fact that f belongs to $Diff\ (\Gamma_U, \Gamma_V)$ we get that for every $g \in \Gamma_V$, $g \circ f \in \Gamma_U$. If we put $g = id|V$, we will get that $f \in \Gamma_U$. But as Γ is Ehresmann's pseudogroup $\Gamma|U \subset \Gamma$. In [1] it was shown that for any quasi-algebraic space A and any subset M of $\underline{A}$ the equality $A_M = \{\bigcup D;\ D \subset A|M \wedge \bigcup D$ is a function$\}$ holds. The same equality holds for any Ehresmann's pseudogroup Γ because the set of D_f, $f \in \Gamma$, forms not only the basis for the topology $top\ \Gamma$ but $\{D_f\}_{f \in \Gamma} \cup \{\emptyset\} = top\ \Gamma$. That means that $f = \bigcup D$ where $D \subset \Gamma$. It follows that f, as one-to-one function belongs to Γ.

To prove the other inclusion let us assume that f belongs to Γ. It follows from the definition of Ehresmann's pseudogroup that $f : \Gamma|D_f \to \Gamma|f(D_f)$. From this we get that $f : \Gamma_{D_f} \to \Gamma_{f(D_f)}$ (see [1]). As D_f and $f(D_f)$ are open sets which means that $f \in Diff\ \Gamma$. Since f is a non-empty function, we get the other inclusion.

References

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Streszczenie

W [1] zostało udowodnione, że zbiór wszystkich dyfeomorfizmów przestrzeni K-quasi--algebraicznych tworzy pseudogrupę Ehresmanna. Można zauważyć, że dowodząc tego twierdzenia, nie korzystaliśmy z własności przestrzeni K-quasi-algebraicznej, więc możliwe jest uogólnienie tego twierdzenia i sformułowanie go dla dowolnego zbioru funkcji. Możemy więc powiedzieć, że zbiór wszystkich dyfeomorfizmów dowolnego zbioru funkcji

jest pseudogrupą Ehresmanna. Pokazujemy także w pracy, że w ten sposób można otrzymać każdą pseudogrupę, tworząc zbiór jej wszystkich dyfeomorfizmów. Otrzymujemy więc pewną charakteryzację pseudogrupy.