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A NEW VERSION OF GRONWALL-BELLMAN LEMMA AND ITS APPLICATIONS TO DIFFERENTIAL EQUATIONS WITH DELAY

Summary. In this paper we fix our attention on differential equations with delay. We prove some theorems and a lemma, which modifiy the Gronwall-Bellman Lemma well known for ordinary differential equations theory. We also prove some integral inequalities, which may find a practical application. We apply Lebesgue measure, which simplifies the proofs. We also use classical Gronwall-Bellman Lemma, which is a starting point for this paper. The main result of this paper may find an application to research of stability of differential equations with delay.

PEWNA NOWA WERSJA LEMATU GRONWALLA-BELLMANA I JEJ ZASTOSOWANIE DO RÓWNAŃ RÓŻNICZKOWYCH Z OPÓŹNIONYM ARGUMENTEM

Streszczenie. W pracy skoncentrowaliśmy się na badaniu równań różniczkowych z opóźnionym argumentem. W tym celu udowodniliśmy trzy twierdzenia i lemat, który jest pewną modyfikacją dobrze znanego z równań różniczkowych lematu Gronwalla-Bellmana. Udowodniono też kilka nierówności całkowych, które mogą znaleźć praktyczne zastosowania. Ze względów praktycznych zastosowaliśmy miary Lebesgue'a, które znacznie uprościły poniższe dowody. Użyty został również klasyczny lemat Gronwalla-Bellmana, który był punktem wyjścia dla powstania tej pracy. Główne rezultaty tej pracy to twierdzenia, które mogą mieć zastosowanie do badania stabilności równań różniczkowych z opóźnionym argumentem.

Mathematics Subject Classification: 34K20. Key words and phrases: differential equations with delay, Gronwall-Bellman stability

1. Introduction

From historical ground some differential equations with delay can be found in papers of L. Euler, but systematic research of these equations began in XX century. In that time parts of applied mathematics, like control theory, used very often these equations. In years 1950-1970 these equations becam more important in the different research areas like applied physics, some technical aspects or even in economy and biology.

We start with some definitions, which characterize the differential equations with delay.

Definition 1. *Differential equation with deviation is the differential equation in which the unknown function appears with different value of arguments.*

For example:

$$\dot{x}(t) = f(t, x(t), x[t - \tau(t)]), \tag{1}$$

$$\dot{x}(t) = f(t, x(t), x(t - \tau_1), x(t - \tau_2)), \tag{2}$$

$$\ddot{x}(t) = f(t, x(t), \dot{x}(t), x[t - \tau(t)], \dot{x}[t - \tau(t)]). \tag{3}$$

Definition 2. *Differential equation with deviation we call differential equation with delay, if the highest degree of a derivative of the unknown function appears by the same argument and this argument is not higher then the other arguments from unknown function and is derivates which comes to the equation.*

For example that can be equations like 1, 2, 3 but with added condition on the delay function like this:

$$\tau(t) \geqslant 0, \quad \tau_1 > 0 \quad \wedge \quad \tau_2 > 0, \quad t \geqslant 0.$$

For a simple ordinary differential equation with delay of the form:

$$\dot{x}(t) = f(t, x(t), x[t - \tau_1(t)], \ldots, x[t - \tau_m(t)]), \tag{4}$$

primary initial-value problem is based on determining a continuous solution $x(t)$ of equation (4) for $t \geqslant t_0$, with the condition, that $x(t) = \varphi(t)$ on initial set E_{t_0}, set from point t_0 and from value $t - \tau_i$, $(i = 1, \ldots, m)$, which are smaller than t_0 for $t \geqslant t_0$, where $\varphi(t)$ is given continuous function called the initial function.

We denote by $x_\varphi(t)$ the solution of the equation (4) with a given initial function $\varphi(t)$. Now we introduce some basic definitions concerning ordinary differential equations with delay of the form (4).

Definition 3. *The solution $x_\varphi(t)$ of the equation (4) is said to be stable, if for every $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$, such that from inequality $||\varphi(t) - \psi(t)|| \leqslant \delta(\varepsilon)$ on the initial set it follows the inequality $||x_\varphi(t) - x_\psi(t)|| \leqslant \varepsilon$ for $t \geqslant t_0$, where $\varphi(t)$ is an arbitrary continuous initial function.*

Definition 4. *The solution $x_\varphi(t)$ of the equation (4) is said to be asymptotically stable, if:*
(1) it is stable,
(2) for each $t_0 \geqslant 0$ there is a $\delta_1 > 0$ such that every solution $x_\psi(t)$, which satisfies $||\varphi(t) - \psi(t)|| < \delta_1$, will also satisfy

$$\lim_{t\to\infty}[x_\varphi(t) - x_\psi(t)] = 0,$$

where $\varphi(t)$ is an arbitrary continuous initial function.

Notice 1. *By changing variables: $y(t) = x(t) - x_\varphi(t)$ we can convert research of stability of an optional solution x_φ of the equation (4) to research of stability of the trivial solution $y(t) \equiv 0$.*

From this reason we consider only stability of the trivial solution.

First we use the classical Gronwall-Bellman Lemma which is the starting point for this paper.

Lemma 1. (Gronwall-Bellman [2]) *Let some constant $C \in \Re$ be given, and let u, v, and $v(t) \geqslant 0$ for $t \in I$, be continuous functions define on interval I. If for a certain $t_0 \in I$ and for any $t \in I$, $t \geqslant t_0$, holds:*

$$u(t) \leqslant C + \int_{t_0}^{t} u(s)v(s)ds,$$

then for any $t \in I$ holds:

$$u(t) \leqslant C \exp\left\{\int_{t_0}^{t} v(s)ds\right\}.$$

The proof of this Lemma can be found in works [1] and [2]. Now we prove some modernizations of this Lemma, Lemma 2 and Theorems 1 and 2, which are some modifications from well known theorems from ordinary differential equations. This Theorems can have applications in research of stability in the sense of first Lapunov method also well known from ordinary differential equations.

2. Main result

Now we prove some modernization of the well known Gronwall-Bellman Lemma.

Lemma 2. *Let some constans C, $\Re$ be given and some continuous functions u, v, and $v(t) \geqslant 0$ for $t \in [A, B]$ define on $[A, B]$. Besides let $u(t)$ be non-negative function equal zero outside of $[A, B]$. Let for a measurable set $E \subset [A, B]$ the following condition will be satisfied:*

$$m\gamma(E) \leqslant Nm(E),$$

where $\gamma(E)$ is set of those $t \in [A, B]$, for which $t - \tau(t) \in E$ and $m(E)$ is a Lebesgue measure on set E.

If for a certain $t_0 \in [A, B]$ and for any $t \in [A, B]$, $t > t_0$ the following condition holds:

$$u(t) \leqslant C + \int_{t_0}^{t} u(s - \tau(s))v(s)ds,$$

then for any $t \in [A, B]$ it follows:

$$u(t) \leqslant C \exp\left\{N \int_{t_0}^{t} v(s)ds\right\}.$$

Proof. We note that the above assumption imply the following inequality:

$$\int_{A}^{B} u(s - \tau(s))ds \leqslant N \int_{A}^{B} u(s)ds.$$

Sufficiency. We build upper Lebesgue grand total for both integral:

$$S = \sum_{k} y_{k+1} m e_k,$$

$$S' = \sum_{k} y_{k+1} m e'_k,$$

where

$$e_k - \text{ is the set of those } t \in [A, B], \text{ for which } y_k \leqslant u(t - \tau(t)) < y_{k+1},$$

$$e'_k - \text{ is the set of those } t \in [A, B], \text{ for which } y_k \leqslant u(t) < y_{k+1}.$$

It is easy to prove that:

$$e_k \subset \gamma(e'_k).$$

In that case:

$$me_k \leqslant m\gamma(e'_k) \leqslant Nme'_k,$$

and

$$S \leqslant NS'.$$

Limiting transition finish proof of sufficency.

Necessity. Let for some $E \subset [A, B]$:

$$m\gamma(E) > NmE.$$

We define $u(t)$ by:

$$f(x) = \begin{cases} 1 & t \in E \\ 0 & t \notin E \end{cases}$$

Therefore:

$$\int_A^B u(s - \tau(s))ds = m\gamma(E) > NmE = N\int_A^B u(s)ds.$$

So our inequality is proved.■

This earlier proof Lemma 2 we use in the proof of the next Theorem.

Theorem 1. *Let function $u(t)$, continuous and positive for any t, $\bar{t} \in [A, B]$ satisfy an inequality:*

$$u(t) \leqslant u(\bar{t}) + \int_t^{\bar{t}} u(s - \tau(s))v(s)ds, \tag{5}$$

where $v(t)$ is continuous function in interval $[A, B]$ and $v(t) \leqslant 0$ for this $t \in [A, B]$. Let for a measurable set $E \subset [A, B]$ the following condition is satisfied:

$$m\gamma(E) \leqslant Nm(E)$$

where $\gamma(E)$ is set of this $t \in [A, B]$, for which $t - \tau(t) \in E$ and mE Lebesgue measure from set E.

Then for this $A \leqslant t_0 \leqslant t \leqslant B$ is fulfilled this estimation:

$$u(t_0)\exp\left\{-N\int_{t_0}^{t} v(s)ds\right\} \leqslant u(t) \leqslant u(t_0)\exp\left\{N\int_{t_0}^{t} v(s)ds\right\}. \tag{6}$$

Proof. From the inequality (5) for $t \geqslant \bar{t}$, we have:

$$u(t) \leqslant u(\bar{t}) + \int_t^{\bar{t}} u(s - \tau(s))v(s)ds.$$

Hence on the ground from Lemma 2 we obtain:

$$u(t) \leqslant u(\bar{t}) + \exp\left\{\int_t^{\bar{t}} v(s)ds\right\}. \tag{7}$$

Similiarly, from the inequality (5) for $t \leqslant \tilde{t}$, we have:

$$u(t) \leqslant u(\tilde{t}) + \int_{t}^{\tilde{t}} u(s-\tau(s))v(s)ds \leqslant u(\tilde{t}) + N \int_{t} \tilde{t}u(s)v(s)ds =$$

$$= u(\tilde{t}) + \int_{t} \tilde{t}u(s)Nv(s)ds \leqslant u(\tilde{t}) \exp\left\{\int_{t}^{\tilde{t}} Nv(s)ds\right\} = u(\tilde{t}) \exp\left\{N \int_{t}^{\tilde{t}} v(s)ds\right\}.$$

Hence

$$u(t) \leqslant u(\tilde{t}) \exp\left\{N \int_{t}^{\tilde{t}} v(s)ds\right\},$$

we obtain

$$u(t) \exp\left\{-N \int_{t}^{\tilde{t}} v(s)ds\right\} \leqslant u(\tilde{t}).$$

Now we change $t \to \tilde{t}$ and $\tilde{t} \to t$ and hence

$$u(\tilde{t}) \exp\left\{-N \int_{\tilde{t}}^{t} v(s)ds\right\} \leqslant u(t). \tag{8}$$

Substituting $\tilde{t} = t_0$ in the inequalities (7) and (8) we obtain estimation (6). The proof of the Theorem 1 is complete.■

To prove the next Theorem we need the following Definition:

Definition 5. *Let function $f(t) : [t_0, \infty) \to C$, then number defined by:*

$$\kappa[f] = \overline{\lim_{t\to\infty}} \frac{1}{t} \cdot \ln|f(t)|,$$

we will call as index from expotetial increase or characteristic exponent.

Theorem 2. *Consider linear homogeneous system:*

$$\dot{x}(t) = A(t) \cdot x(t - \tau(t)), \tag{9}$$

where $A(t)$ is continuous square matrix grade n in interval $[A, \infty)$. Let for a measurable set $E \subset [A, B]$ the following condition be satisfied:

$$m\gamma(E) \leqslant Nm(E),$$

where $\gamma(E)$ is set of this $t \in [A, B]$, for which $t - \tau(t) \in E$ and mE Lebesgue measure from set E. If matrix $A(t)$ is bounded, that means if:

$$||A(t)|| \leqslant C \leq \infty,$$

then every non-vanishing real or complex solution $x = x(t)$, $(A \leq t_0 \leqslant t \leq \infty)$ systems (9) have finite characteristic exponent.

Proof. Let $x = col[x_1, \dots, x_n]$ be non-vanishing solution of the linear homogeneous systems (9), and t_0 and t be points from interval (A, ∞). From equation (9) we have:

$$x(t) = x(t_0) + \int_{t_0}^{t} A(s)x(s - \tau(s))ds,$$

hence

$$||x(t) \leqslant ||x(t_0)|| + \int_{t_0}^{t} ||A(s)|| \cdot ||x(s - \tau(s))|| \cdot |ds|.$$

Hence on the ground from Theorem 1 we obtain:

$$||x(t_0)|| \exp\left\{-N \int_{t_0}^{t} ||A(s)|| \cdot ds\right\} \leqslant ||x(t)|| \leqslant ||x(t_0)|| \cdot \exp\left\{N \int_{t_0}^{t} ||A(s)|| \cdot ds\right\}.$$

Taking this fact into consideration:

$$\kappa\left[\frac{||x(t)||}{||x(t_0)||}\right] = \kappa[x(t)],$$

we obtain

$$\kappa\left[\exp\left\{-N \int_{t_0}^{t} ||A(s)|| \cdot ds\right\}\right] \leqslant \kappa[x(t)] \leqslant \kappa\left[\exp\left\{N \int_{t_0}^{t} ||A(s)|| \cdot ds\right\}\right],$$

or

$$-\underline{A} \leqslant \kappa[x(t)] \leqslant \overline{A},$$

where

$$\underline{A} = \varliminf_{t \to \infty} \frac{N}{t} \int_{t_0}^{t} ||A(s)|| \cdot ds,$$

$$\overline{A} = \varlimsup_{t \to \infty} \frac{N}{t} \int_{t_0}^{t} ||A(s)|| \cdot ds,$$

Since matrix $A(t)$ is bounded from assumption, that all characteristic exponent non-vanishing solution $x(t)$ of the linear homogeneous system (9) lie in interval $[-\underline{A}, \overline{A}]$ include in interval $[-C, C]$. This completes the proof of Theorem 2.∎

Theorem 3. *The fundamental solution system of the homogeneous linear differential equation:*

$$\dot{x}(t) = A(t) \cdot x(t - \tau(t)), \tag{10}$$

is normal if and only if is incomprassible.

Proof.

Necessity. We assume that the system is incompressible and we show that it is normal.

Hip. Assume that it is othervise, that means that exists fundamental solution system:

$$Y(t) = \{y^1(t), \dots, y^n(t)\},$$

such that:

$$\Sigma_Y \leq \Sigma_X, \tag{11}$$

where

$$\Sigma_Y = \sum_{s=1}^{m} \nu_s \cdot \alpha_s,\ \Sigma_X = \sum_{x=1}^{m} \eta_s \cdot \alpha_s,\ \sum_{s=1}^{m} \nu_s = \sum_{s=1}^{m} \eta_s = n,$$

but α_s that are characteristic exponents from non-vanishing solutions from differential equation (10) and ν_s, η_s, $(s = 1, \dots, m)$, that are numbers of linear indepedent solutions with characteristic exponent α_s included correspondingly in fundamental systems $Y(t)$ and $X(t)$. We will dispose the solutions from $Y(t)$ and $X(t)$ in the sequence of increase their characteristic exponents.

Let $N'_s = \sum_{k=1}^{s} \eta_k$ $(s = 1, \dots, m)$ be a maximal number of linear indepedent solutions which have characteristic exponent α_s. We follow the new symbol:

$$N'_s = \sum_{k=1}^{s} \nu_s,\ (s = 1, \dots, m).$$

Theorem A [4]. *The number N_s is equal to dimension of the linear subspace M_s, that means:*

$$N_s = \dim M_s,\ (s = 1, \dots, m).$$

Theorem B [4]. *Let be the fundamental system $X(t)$ incompressible, η_s $(s = 1, \dots, m)$ be the number of solutions with characteristic exponent α_s, and N_s be maximal number of linear indepedent solutions with characteristic exponent α_s, that holds this equality:*

$$\eta_1 + \eta_2 + \dots + \eta_s = N_s,\ (s = 1, \dots, m).$$

The proofs of these Theorems can be found in every course book for Ordinary Differential Equations.

On the ground of Theorem A and B we obtain:

$$N'_1 \leqslant N_1, N'_2 \leqslant N_2, \dots, N'_{m-1} \leqslant N_{m-1}, N'_m = N_m = n,$$

where

$$\nu_s = N'_s - N'_{s-1}, \eta_s = N_s - N_{s-1},\ (s = 1, \dots, m; N'_0 = N_0 = 0).$$

Hence, we obtain

$$\Sigma_Y = \sum_{s=1}^{m} \nu_s \cdot \alpha_s = \sum_{s=1}^{m} (N'_s - N'_{s-1}) \cdot \alpha_s = N'_m \alpha_m - \sum_{s=1}^{m-1} N'_s(\alpha_{s+1} - \alpha_s) \geqslant$$

$$\geqslant N_m \alpha_m - \sum_{s=1}^{m-1} N_s \cdot (\alpha_{s+1} - \alpha_s) = \sum_{s=1}^{m} \eta_s \cdot \alpha_s = \Sigma_X .$$

This is contradiction with inequality (11). On the ground of this, when the system $X(t)$ is incompressible that is normal.

Sufficency. We assume that the system $X(t)$ is normal and we show that it is incompressible.

Hip. Assume, that it is otherwise, that means that exists linear combination

$$y = \sum_{i=1}^{p} c_i x^{(i)}(t), \ (c_p \neq 0), \tag{12}$$

such that

$$[y] \leq \max_i \kappa[x^{(i)}(t)] = \kappa[x^{(p)}(t)], \tag{13}$$

We consider system of solution:

$$Y = \{x^{(1)}(t_0, \ldots, x^{(p-1)}(t), y(t), x^{(p+1)}(t), \ldots, x^{(n)}(t)\}$$

System Y is fundamental. Let

$$\sum_{i \neq p} a_i \cdot x^{(i)}(t) + a_p \cdot y(t) \equiv 0, \ \sum_{i=1}^{n} |a_i| \not\equiv 0, \tag{14}$$

On the ground of linear indepedence of solutions $x^{(i)}(t)$ we have $a_p \neq 0$. Substituting in equation (14) the equation (12) we obtain

$$\sum_{i \leq p} (a_i + a_p c_i) x^{(i)}(t) + a_p c_p x^{(p)}(t) + \sum_{i \leq p} a_i x^{(i)}(t) \equiv 0.$$

From this we obtain, that $a_p c_p = 0$ what is impossible, because $c_p \neq 0$ and $a_p \neq 0$. That follows that our system Y is fundamental.

On the ground of (13) we obtain, that $\Sigma_Y \leq \Sigma_X$ and that is contradiction, that system X is normal. Then every normal system is incompressible. This completes the proof of Theorem 3.■

The main results this paper are Theorems 1, 2 and 3, which may found application to research of the stability of the differential equations with delay.

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Streszczenie

W pracy skoncentrowaliśmy się na badaniu równań różniczkowych z opóźnionym argumentem. W tym celu udowodniliśmy trzy twierdzenia i lemat, który jest pewną modyfikacją dobrze znanego z równań różniczkowych lematu Gronwalla-Bellmana. Udowodniono też kilka nierówności całkowych, które mogą znaleźć praktyczne zastosowania. Ze względów praktycznych zastosowaliśmy miary Lebesgue'a, które znacznie uprościły poniższe dowody. Użyty został również klasyczny lemat Gronwalla-Bellmana, który był punktem wyjścia dla powstania tej pracy.

Główne rezultaty tej pracy to twierdzenia, które mogą mieć zastosowanie do badania stabilności równań różniczkowych z opóźnionym argumentem.

Lemat 2. *Niech będą dane pewne stałe* m, $N \in \Re$ *i pewne ciągłe i określone funkcje na* $[A, B]$, u, v, *przy czym* $v(t) > 0$ *dla* $t \in [A, B]$. *Niech ponadto* $u(t)$ *będzie funkcją nieujemną równą zeru poza* $[A, B]$. *Niech dla dowolnego zbioru mierzalnego* $E \subset [A, B]$ *będzie spełniony warunek:*

$$m\gamma(E) \leqslant Nm(E),$$

gdzie $\gamma(E)$ *zbiór tych* $t \in [A, B]$, *dla których* $t - \tau(t) \in E$ *i* mE *miara Lebesgue'a zbioru* E. *Jeżeli dla pewnego ustalonego* $t_0 \in [A, B]$, $t \geqslant t_0$ *spełniona jest nierówność:*

$$u(t) \leqslant C + \int_{t_0}^{t} u(s - \tau(s))v(s)ds,$$

wówczas dla $t \in [A, B]$ *spełniona jest taka nierówność:*

$$u(t) \leqslant C \exp\left\{ N \int_{t_0}^{t} v(s)ds \right\}.$$

Twierdzenie 1. *Niech funkcja* $u(t)$, *ciągła i dodatnia dla dowolnych wartości* $t, \bar{t} \in [A, B]$ *spełnia nierówność:*

$$u(t) \leqslant u(\bar{t}) + \int_{t_0}^{\bar{t}} u(s - \tau(s))v(s)ds,$$

gdzie $v(t)$ *jest funkcją ciągłą w przedziale* $[A, B]$ *i* $v(t) \geqslant 0$ *dla* $t \in [A, B]$. *Niech dla dowolnego zbioru mierzalnego* $E \subset [A, B]$ *będzie spełniony warunek:*

$$m\gamma(E) \leqslant Nm(E),$$

gdzie $\gamma(E)$ *zbiór tych* $t \in [A, B]$, *dla których* $t - \tau(E) \in E$ *i* mE *miara Lebesgue'a zbioru* E. *Wówczas dla* $A \leqslant t_0 \leqslant t \leqslant B$ *ma miejsce oszacowanie:*

$$u(t_0) \exp\left\{ -N \int_{t_0}^{t} v(s)ds \right\} \leqslant u(t) \leqslant u(t_0) \exp\left\{ N \int_{t_0}^{t} v(s)ds \right\}.$$

Twierdzenie 2. *Rozważmy układ jednorodny:*

$$\dot{x}(t) = A(t) \cdot x(t - \tau(t)),$$

gdzie $A(t)$ *oznacza macierz kwadratową stopnia* n, *ciągłą w przedziale* $[A, \infty)$. *Niech dla dowolnego zbioru mierzalnego* $E \in [A, B]$ *będzie spełniony warunek:*

$$m\gamma(E) \leqslant Nm(E),$$

gdzie $\gamma(E)$ *zbiór tych* $t \in [A, B]$, *dla których* $t - \tau(E) \in E$ *i* mE *miara Lebesgue'a zbioru* E. *Jeżeli macierz* $A(t)$ *jest ograniczona, tzn. jeżeli*

$$||A(t)|| \leqslant C < \infty,$$

to każde rzeczywiste lub zespolone rozwiązanie niezerowe $x = x(t)$, $(A < t_0 \leqslant t < \infty)$ *układu ma skończony wykładnik charakterystyczny.*

Twierdzenie 3. *Fundamentalny układ rozwiązań liniowego równania różniczkowego jednorodnego:*

$$\dot{x}(t) = A(t) \cdot x(t - \tau(t)),$$

jest normalny wtedy i tylko wtedy, gdy jest on nieściśliwy.

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Nicolae SOARE

AFFINE CONNECTIONS ON MANIFOLDS WITH CERTAIN (f, g) – STRUCTURE

Summary. The subject of the paper is the determination of the affine connections compatible with the (f, g)-structures defined by a tensor field f of type (l, l), having the property $f^{3s} + f^s = 0$, $s \geqslant 1$, and by a Riemannian structure g which satisfies an additional condition.

KONEKSJE AFINICZNE NA ROZMAITOŚCIACH Z PEWNĄ (f, g) – STRUKTURĄ

Streszczenie. W pracy wyznaczono koneksje afiniczne zgodne z (f, g)-strukturą zdefiniowaną polem tensorowym f typu (l, l), mającym własność $f^{3s} + f^s = 0$, $s \geqslant 1$, i strukturą g Riemanna spełniającą pewne dodatkowe warunki.

1. Introduction

In this paper we study the (f, g)-structures determined by a tensor field f of type (l, l) so that $f^{3s} + f^s = 0$, where s is a natural number, $s \geqslant 1$ and g is a Riemannian structure satisfying the condition $g(f^s X, f^s Y) = -g(f^{2s} X, Y)$ for every X, Y.

The case of the (f, g)-structures with $f^2 + f = 0$ was studied by R. Miron and Gh. Atanasiu in [1].

2. (f, g)-structures

Let M be a Riemannian manifold with the Riemannian metric g, $\mathcal{C}(M)$ the affin modul of the affine connections on M, $\mathcal{F}(M)$ the algebra of all differentiable funcitons on M and