ZESZYTY NAUKOWE POLITECHNIKI ŚLĄSKIEJ 1997
Seria: MATEMATYKA-FIZYKA z. 82 Nr kol. 1360

Aleksander Michał NAWRAT

APPLICATION OF SECOND LAPUNOV METHOD TO RESEARCH OF STABILITY ORDINARY DIFFERENTIAL EQUATIONS WITH DELAY

Summary. In this paper we fix our attention on stability of differential equations with delay using second Lapunov method. This is very important problem, because not always we know the exactly solution, but we know that the solution exists and is only one. For furthermore information about this solution without knowing exactly formula we using the stability theory. In this paper we fix our attention on second Lapunov method, which gives us very important mathematical tool to research the stability and asymptotically stability of differential equations with delay. In introduction we give some definitions, which we use in the main part of this paper. In the main part we give and prove some theorems, which will have applications to the stability of differential equations with delay. On the end we give some examples of practical use of before mentioned theorems.

ZASTOSOWANIE DRUGIEJ METODY LAPUNOWA DO BADANIA STABILNOŚCI RÓWNAŃ RÓŻNICZKOWYCH Z ODCHYLONYM ARGUMENTEM

Streszczenie. W pracy skupiliśmy naszą uwagę na badaniu stabilności równań różniczkowych z odchylonym argumentem za pomocą uogólnień drugiej metody Lapunowa. Jest to jeden z ważnych problemów, gdyż nie zawsze znamy jawne rozwiązanie danego równania, a jedynie wiemy o nim tyle, że rozwiązanie istnieje i jest jedyne. Do zdobycia dalszych informacji na temat tego rozwiązania bez znajomości jawnego wzoru służy nam właśnie teoria stabilności. Przybliżymy tutaj drugą metodę Lapunowa, która daje poważne narzędzie matematyczne, użyteczne przy badaniu stabilności oraz asymptotycznej stabilności równań różniczkowych z odchylonym argumentem. We wstępie do tej pracy skoncentrujemy się na przybliżeniu potrzebnych nam pojęć i definicji, których będziemy używali w dalszej części pracy.

Mathematics Subject Classification: 34K20. Key words and phrases: differential equation with delay, second Lapunov method, stability

W głównej części pracy podamy i udowodnimy twierdzenia, które znajdą poważne zastosowania w badaniach nad stabilnością takich równań. Na samym końcu naszej pracy podamy przykłady praktycznego zastosowania tej teorii dla konkretnych równań różniczkowych z opóźnionym argumentem.

1. Introduction

For a simple ordinary differential equation with delay of the form:

$$\dot{x}(t) = f\left(t, x(t), x[t-\tau_1(t)], \ldots, x[t-\tau_m(t)]\right), \tag{1}$$

primary initial-value problem is based on determining continuous solution $x(t)$ of equation (1) for $t \geqslant 0$, with the condition, that $x(t) = \varphi(t)$ on initial set E_{t_0}, set from point t_0 and from value $t - \tau_i(t)$, $(i = 1, \ldots, m)$, which are smaller than t_0 for $t \geqslant t_0$, where $\varphi(t)$ is given continuous function called the initial function.

We denote by $x_\varphi(t)$ the solution of equation (1) with defined initial function $\varphi(t)$. Now we will introduce some basic definitions concerning ordinary differential equations with delay of the form (1).

Definition 1. *The solution x_φ of equation (1) is said to be stable, if for every $\epsilon > 0$ there exists $\delta(\epsilon) > 0$, such that from inequality $||\varphi(t) - \psi(t)|| < \delta(\epsilon)$ on the initial set it follows the inequality $||x_\varphi(t) - x_\psi(t)|| < \epsilon$ for $t \geqslant t_0$, where $\varphi(t)$ is an arbitrary continuous initial function.*

Definition 2. *The solution $x_\varphi(t)$ of the equation (1) is said to be asymptotically stable, if:*
(1) it is stable;
(2) for each $t_0 \geqslant 0$ there is a $\delta_1 > 0$ such that every solution $x_\psi(t)$ which satisfies $||\varphi(t) - \psi(t)|| < \delta_1||$ will also satisfy

$$\lim_{t\to\infty} [x_\varphi(t) - x_\psi(t)] = 0,$$

where $\varphi(t)$ is arbitrary continuous initial function.

Notice 1. *By changing variables: $y(t) = x(t) - x_\varphi(t)$ we can convert research of stability of an optional solution $x_\varphi(t)$ of the equation (1) to research of stability of the trivial solution$y(t) \equiv 0$.*

From this reason we consider only stability of trivial solution.

Now we introduce following norms in the spaces C_0 and L_2 on $[-\tau, 0]$.

$$||x||_\tau := \sup_{\substack{-\tau \leqslant s \leqslant 0 \\ 1 \leqslant i \leqslant 0}} |x_i(s)|,$$

$$||x||_{\tau^2} := \left[\int_{-\tau}^{0} \sum_{i=1}^{n} x_i^2(s) ds \right]^{\frac{1}{2}}.$$

Let U_ε be a $\varepsilon-$neighborhood in matric C_0 $x_i \equiv 0$ $(i = 1, \ldots, n)$ equilibrum point, and S_ε be a $\varepsilon-$sphere with center in this point.

Definition 3. *The functional*

$$V\left[t, x_1(s), \ldots, x_n(s)\right] = V[t, x(s)], \tag{2}$$

$-\tau \leqslant s \leqslant 0$, $t \geqslant t_0$, *we call positively (negatively) defined, if there exists for* $r \neq 0$ *such a function* $\varphi(r) > 0$ *and*

$$V\left[t, x(s)\right] \geqslant \varphi[||x(s)||_\tau], \ (V\left[t, x(s)\right] \geqslant -\varphi[||x(s)||_\tau]).$$

Definition 4. *The functional (2) have infinitesimal upper limit, if there exists for* $r \neq 0$ *a continuous function* $\varphi_1(r)$ *such that*

$$V\left[t, x(s)\right] \geqslant \varphi_1[||x(s)||_\tau].$$

2. The main Part

Theorem 1. ([5] about stability) *If there exists a continuous positivly defined functional:*

$$V[t, x(s)], \ -\tau \leqslant s \leqslant 0, \ t \geqslant t_0, \ ||x||_\tau < H, \ H > 0, \ V[t, 0] = 0,$$

with non-positive derivative along integral curve:

$$\frac{dV[t, x_\phi(t+s)]}{dt} \leqslant 0,$$

and where $x_\phi(t+s)$ *is solution from the system:*

$$\dot{x}_i(t) = f(t, x_1[t - \tau_{11}], \ldots, x_1[t - \tau_{1n}], \ldots, x_n[t - \tau_{n1}], \ldots, x_n[t - \tau_{nn}]), \tag{3}$$

Then the trivial solution of the system (3) is stable.

Proof. For a given ε $(0 < \varepsilon < H)$, we can take $\delta(\varepsilon) \leqslant$ *varepsilon* such a:

$$\inf_{||x(u+s)||_\tau=\varepsilon} V[t, x(u+s)] > \sup_{||x(z+s)||_\tau<\delta(\varepsilon)} V[t_0, x(z+s)], \tag{4}$$

for $z \geqslant t_0$, $u \geqslant t_0$, $-\tau \leqslant s \leqslant 0$. As the functional $V[t, x(s)]$ is positive defined, and $V[t_0, x(s)]$ is continuous in neighbourhood $x(s) \equiv 0$, there exist possibility to choose such $\delta(\varepsilon)$, that:

$$\inf_{||x(u+s)||_\tau=\varepsilon} V[t, x(u+s)] > 0.$$

We take $\delta(\varepsilon)$, that an arbitrary initial function $\Phi(t)$, satisfied the condition:

$$||\Phi(t-0+s)||_\tau < \delta(\varepsilon),$$

and define solution $x_\Phi(t)$, $t \geqslant t_0$, such that

$$||x_\Phi(z+t)||_\tau < \varepsilon.$$

Because along trajectory, the function, in which along integral curve transform functional V, is non-increasing. Therefore value $||x_\Phi||_\tau$ can be on the ground of inequality (4) equal ε. The proof of the Theorem 1 is complete.■

Theorem 2. *If there exists a continuous positively defined functional* $V[t, x(s)]$ *for* $t \geqslant t_0$ *and* $||x||_\tau < H$, $H > 0$ *which has infinitesimal upper limit and such that derivative of* $V[t, x_\Phi(t+s)]$ *towards* t *is negatively defined, then trivial solution of system (3) is uniformly asymtotically stable.*

Proof. Because:

$$\lim_{x\to 0^+} \varphi(x) = 0, \text{ and } \varphi(r) > 0 \text{ for } r > 0,$$

then for any $\varepsilon > 0$, such that for $z \geqslant t_0$, $-\tau \leqslant s < 0$,

$$\varphi_1\left[||x(z+s)||_\tau\right] < \frac{1}{2}\varphi(\epsilon),$$

for $||x(z+s)||_\tau < \delta(\varepsilon)$. Hence for $t_1 \geqslant t_0$ and $||x(z+s)||_\tau < \delta(\varepsilon)$ we obtain:

$$V[t_1, x(z+s)] \leqslant \varphi_1\left[||x(z+s)||_\tau\right] < \frac{1}{2}\varphi(\varepsilon) < \varphi(\varepsilon).$$

For $t_2 \geqslant t_0$ and $||x(u+s)||_\tau = \varepsilon$, $u \geqslant t_0$ we have that:

$$\varphi(\varepsilon) \leqslant V[t_2, x(u+s)],$$

and hence

$$\sup_{||x(z+s)||_\tau<\delta(\varepsilon)} V[t_1, x(z+s)] \leqslant \frac{1}{2}\varphi(\varepsilon) < \varphi(\varepsilon) < \inf_{||x(u+s)||_\tau=\varepsilon} V[t_2, x(u+s)].$$

We obtain

$$\sup_{||x(z+s)||_\tau<\delta(\varepsilon)} V[t_1, x(z+s)] < \inf_{||x(u+s)||_\tau=\varepsilon} V[t_2, x(u+s)], \tag{5}$$

For $||\Phi(t_0+s)||_\tau < \delta\epsilon$, function $V(t) = V[t, x_{\Phi(t+s)}]$ is monotone decreasing relative to variable t. Therefore we have that trajectory $x = x_{\Phi(t)}$ for $t \geqslant t_0$ remains in domain $||x_\Phi||_\tau < \varepsilon$.

We show that in oder case, we come to contradiction. We consider that for any $t_1 \geqslant t_0$:

$$||x_\Phi(t_1+s)||_\tau \geqslant \varepsilon,$$

so that

$$||x_\Phi(t_1+s)||_\tau > \varepsilon,$$

hence x_Φ is continuous and $||\cdot||_\tau$ is continuous function, that exist $\bar{t} \geqslant t_0$ so that

$$||x_\Phi(\bar{t}+s)||_\tau = \varepsilon,$$

Because (V is decreasing with respect to t), so that

$$\inf_{||x_\Phi(\bar{t}+s)||_\tau=\varepsilon} V[\bar{t}, x_\Phi(\bar{t}+s)] \leqslant V[\bar{t}, x_\Phi(\bar{t}+s)] \leqslant V[t_0, x_\Phi(t_0+s)] \leqslant$$

$$\leqslant \sup_{||x_\Phi(t_0+s)||_\tau<\delta(\varepsilon)} V[t_0, x_\Phi(t_0+s)] < V[t_0, x_\Phi(t_0+s)].$$

Hence

$$\inf_{||x_\Phi(\bar{t}+s)||_\tau=\varepsilon} V[\bar{t}, x_\Phi(\bar{t}+s)] \leqslant \sup_{||x_\Phi(t_0+s)||_\tau<\delta(\varepsilon)} < V[t_0, x_\Phi(t_0+s)],$$

what contradicts (5). Therefore for every $t \geqslant t_0$

$$||x_\Phi(t+s)||_\tau < \varepsilon,$$

then $x \equiv 0$ is stable.

We take arbitrary small η and we assert $\delta_1(\eta) > 0$ such that:

$$\sup_{||x(z+s)||_\tau<\delta_1(\eta)} V[t, x(z+s)] < \inf_{||x(u+s)||_\tau=\eta} V[t, x(u+s)].$$

Taking such a $\delta_1(\eta) \leqslant ||x_\Phi||_\tau < H$ we come to contradiction, because in this case:

$$\frac{dV[t, x_\Phi(t)]}{dt} \leqslant -\alpha < 0,$$

hence

$$V[t, x_\Phi(t)] - V[t_0, x_\Phi(t_0)] \leqslant -\alpha[t - t_0], \tag{6}$$

(from (6) we obtain $V[t, x_\Phi(t)] < 0$, for $t > \frac{1}{\alpha}(V[t_0, x_\Phi(t_0)] + \alpha t_0)$). Therefore, there exists a point:

$$t^* < \frac{1}{\alpha}(V[t_0, x_\Phi(t_0)] + \alpha t_0), \tag{7}$$

such that:

$$||x_\Phi(t^*)||_\tau < \delta_1(\eta),$$

and hence for $t > t^* + \tau$, we have:

$$||x_\Phi(t)||_\tau < \eta.$$

From this reason that η is arbitrary, the proof of asymptotic stability is finished.

Because $\delta(\eta)$ is not relative from t_0 and from estimation (7), asymptotic stability is uniformly. The proof of the Theorem 2 is complete.∎

Theorem 3. *Let for $t_0 \leqslant t \leqslant t_0 + H$ $(H > 0)$ all function $\tau_i(t)$ be continuous and non-negative, function f in neighbourhood point*

$$(t_0, \varphi(t_0), \varphi(t_0 - \tau - 1(t_0)), \dots, \varphi(t_0 - \tau_m(t_0))),$$

be continuous and satisfied Lipschitz condition relative to all arguments without first, and initial function $\varphi(t)$ be continuous on set E_{t_0}. Let furthermore exist functional satisfied conditions:

(1) $V[t, x(s)] \leqslant W_1(||x(0)||) + W_2(||x(s)||_{\tau^2})$,
(2) $V[t, x(s)] \geqslant W(||x(0)||)$,
(3) $\limsup\limits_{\delta \to 0^+} \frac{\Delta V}{\Delta t} \leqslant -\varphi(||x(0)||)$,

where functions $W_1(r)$ and $W_2(r)$ for $r \geqslant 0$ are continuous and ascending, and $W_1(0) = W_2(0) = 0$, and functions $W(r)$ and $\varphi(r)$ are continuous and positive for $r > 0$. Then trivial solution of the system

$$\dot{x}_i(t) = f_i(t, x_1(t), \dots, x_n(t), x_1[t - \tau_1(t)], \dots, x_1[t - \tau_n(t)], \dots,$$
$$\dots, x_n[t - \tau_1(t)], \dots, x_n[t - \tau_n(t)]) \qquad (i = 1, \dots, n),$$

where $\tau \geqslant \tau_i(t) \geqslant 0$ is asymtotically stable.

Proof. For given $\varepsilon > 0$ we take a value $\delta > 0$ such a:

$$W_1(\delta) + W_2(\delta\eta\tau) < W(\varepsilon),$$

then on the ground of conditions (1) and (2) we obtain:

$$V[t_0, \Phi(s)] < W(\varepsilon), \tag{8}$$

for $||\Phi(t+s)||_{tau} < \delta$. On the ground of the condition (3) along the trajectory function

$$V[t, x_\Phi(t+s)] = V(t),$$

is not increasing, then on the ground of inequality (8) follow that:

$$\forall_{t \geqslant t_0} V[t, x_\Phi(t+s)] < W(\varepsilon). \tag{9}$$

Therefore on the ground of condition (2) we obtain:

$$||x_\Phi(t)||_\tau < \varepsilon, \text{ for } t \geqslant t_0.$$

Really, because for any $\bar{t} \geqslant t_0$,

$$||x_\Phi(\bar{t}+s)||_\tau \geqslant \varepsilon,$$

that would exist such a $u \geqslant t_0$, that $||x_\Phi(u)|| = \varepsilon$ and $||x_\Phi(u+s)||_\tau = \varepsilon$, then we would have inequalities:

$$W(\varepsilon) = W(||x(u)||) \leqslant V[t, x_\Phi(u+s)] < W(\varepsilon),$$

what is impossible. Hence $||x_\Phi(t+s)||_\tau < \varepsilon$ for every $t \geqslant t_0$, that proofs stability of trivial solution. Proof of asymtotically stability is completely similiar to proof of the Theorem 2. The proof of Theorem 3 is complete.∎

Example 1. Solution $x \equiv 0$ equation

$$\dot{x} + ax(t) + b(t)x(t-\tau) = 0,$$

where a and τ are constans, $\tau > 0$, $b(t)$ is continuous function, is asymptotically stable if $|b(t)| < a$.

We consider functional:

$$V[t, x(s)] = x^2(t) + 2\alpha \int_{-\tau}^{0} x^2(t+s)ds, \ \alpha > 0.$$

For $\alpha > 0$ this functional satisfies first two conditions from Theorem 3. Indeed:
1.$V[t, x(s)] \leqslant Cr^2$, $(C > 1 + 2\alpha\tau)$, $r = ||x||$ for relative large C.
2.$V[t, x(s)] \geqslant r^2$.
We have only to proof third condition:

$$\frac{dV}{dt} \leqslant -W(r), \ x(r) > 0 \text{ for } r \neq 0.$$

$$\frac{dV}{dt} = 2x(t)\dot{x}(t) + 2\alpha \int_{-\tau}^{0} 2x(t+s)\dot{x}(t+s)ds =$$

$$= 2x(t)[-ax(t) - b(t)x(t-\tau)] + 4\alpha \int_{-\tau}^{0} x(t+s)\dot{x}(t+s)ds.$$

Now we count:

$$\int_{-\tau}^{0} x(t+s)\dot{x}(t+s)ds = x^2(t+s)|_{-\tau}^{0} - \int_{-\tau}^{0} x(t+s)\dot{x}(t+s)ds,$$

then

$$2\int_{-\tau}^{0} x(t+s)\dot{x}(t+s)ds = x^2(t) - x^2(t-\tau),$$

hence

$$\int_{-\tau}^{0} x(t+s)\dot{x}(t+s)ds = \frac{1}{2}[x^2(t) - x^2(t-\tau)],$$

Putting this formula on $\frac{dV}{dt}$, we obtain:

$$\frac{dV}{dt} = 2x(t)[-ax(t) - b(t)x(t-\tau)] + 2\alpha[x^2(t) - x^2(t-\tau)] =$$

$$= -2[(a-\alpha)x^2(t) + b(t)x(t)x(t-\tau) + \alpha x^2(t-\tau)].$$

Quadratic form, included in square brackets is positively defined for

$$(a-\alpha)\alpha - \frac{1}{4}b^2(t) > 0.$$

Maximum from left side is reached for $\alpha = \frac{1}{2}a$. We obtain inequalities $a > 0$ (because $\alpha > 0$) and $a > |b(t)|$. Function $W(r)$ we can take r^2.■

Example 2. We consider stability of the trivial solution of the equation:

$$\ddot{x} + \varphi[t, \dot{x}(t)] + f[x(t-\tau(t))] = 0, \tag{10}$$

where function f has continuous derivative and satisfied conditions:

$$\frac{f(x)}{x} > a > 0, \ |f'(x)| < N, \text{ for } x \neq 0, \tag{11}$$

$\varphi(t,y)$ and $\tau(t)$ are continuous, periodical function for variable t, by the way

$$\frac{\varphi(t,y)}{y} > b > 0, \text{ for for } y \neq 0,$$

and

$$0 \leqslant \tau(t) \leqslant \tau. \tag{12}$$

For $t > \tau$, (if $t_0 = 0$), equation (10) we can replace systems in the following form:

$$\begin{cases} \dot{x}(t) = y(t) \\ \dot{y}(t) = -\varphi[t, y(t)] - f[x(t)] + \int\limits_{-\tau(t)}^{0} f'[x(t+s)]y(t+s)ds \end{cases} \tag{13}$$

We can obtain (10)-th equation, when we put the first equation to the second equation in the system (13), so we obtain:

$$\ddot{x} = -\varphi[t, \dot{x}(t)] - f[x(t)] + \int\limits_{-\tau(t)}^{0} f'[x(t+s)]\dot{x}(t+s)ds,$$

hence:

$$\ddot{x}(t) + \varphi[t, \dot{x}(t)] + f[x(t)] - [f[x(t+s)]]|_{-\tau(t)}^{0} = 0,$$

$$\ddot{x}(t) + \varphi[t, \dot{x}(t)] + f[x(t)] - f[x(t)] + f[x(t - \tau(t))] = 0,$$

therefore

$$\ddot{x}(t) + \varphi[t, \dot{x}(t)] + f[x(t - \tau(t))] = 0,$$

We consider functional:

$$V[x(s), y(s)] := 2\int\limits_{0}^{x} f(s)ds + y^2 + \frac{a^2}{\tau^2}\int\limits_{-\tau}^{0}\left[\int\limits_{s_1}^{0} y^2(s)ds\right]ds_1.$$

We count $\lim_{\Delta t \to +0} \frac{\Delta V}{\Delta t}$ along the trajectory from equation (10). We obtain:

$$\lim_{\Delta t \to +0} \frac{\Delta V}{\Delta t} = 2\left(\int\limits_{0}^{x} f(s)ds\right)' + 2yy' + \frac{a^2}{\tau^2}\left(\int\limits_{-\tau}^{0}[\int\limits_{s_1}^{0} y^2(s)ds]ds_1\right)' =$$

$$= 2yf(x) + 2y[-\varphi(t, y) - f(x) + \int\limits_{-\tau(t)}^{0} f'[x(t+s)]y(t+s)ds] + \frac{a^2}{\tau^2}\int\limits_{s_1}^{0} y^2(s)|_{s_1}^{0} ds_1 =$$

$$= 2yf(x) - 2y\varphi(t, y) - 2yf(x) + 2y\int\limits_{-\tau(t)}^{0} f'[x(t+s)]y(t+s)ds + \frac{a^2}{\tau^2}\int\limits_{-\tau}^{0}[y^2(t) - y^2(t+s)]ds =$$

$$= -2y\varphi(t, y) + 2y\int\limits_{-\tau(t)}^{0} f'[x(t+s)]y(t+s)y(t)ds + \frac{a^2}{\tau^2}\int\limits_{-\tau}^{0}[y^2(t) - y^2(t+s)]ds.$$

From conditions (11) and (12) we obtain estimation:

$$\lim_{\Delta t \to +0} \frac{\Delta V}{\Delta t} =$$

$$= -2y\varphi(t,y) + 2y \int_{-\tau(t)}^{0} f'[x(t+s)]y(t+s)y(t)ds + \frac{a^2}{\tau^2}\int_{-\tau}^{0}[y^2(t) - y^2(t+s)]ds <$$

$$< 2\left|\int_{-\tau(t)}^{0} f'[x(t+s)]y(t+s)y(t)ds\right| + \frac{a}{\tau}\int_{-\tau}^{0}[y^2(t) - y^2(t+s)]ds <$$

$$< 2\int_{-\tau}^{0} N|y(t+s)y(t)|ds + \frac{a}{\tau}\int_{-\tau}^{0}[-y^2(t) - y^2(t+s)]ds =$$

$$= -\int_{-\tau}^{0}\left[\frac{a}{\tau}y^2(t) + 2N|y(t+s)y(t)| + \frac{a}{\tau}y^2(t+s)\right]ds. \tag{14}$$

Our functional shuld fulfilled conditions from Theorem 3, than we need that it satisfied:

$$\tau < \frac{a}{N}. \tag{15}$$

Under this conditions form of variable $y(t)$, $y(t+s)$ be positive. Next under condition (15) $\lim_{\Delta t \to +0} \frac{\Delta V}{\Delta t}$ along the trajectory of system (13) is non-positive and $\frac{\Delta V}{\Delta t} = 0$.

From second equation of the system (13), identity $y(t) \equiv 0$ under $t \geqslant \tau$, can be fulfilled only under $x(t) \equiv 0$ for $t > t_0 > 0$. And then under condition (15) functional V satisfied condition of Theoram 3, hence follows asymptotically stable solution $x = 0$, $y = 0$ of the equation (10).∎

References

[1] R. Bellman, K. L. Cooke, *Asymptotic behaviour of differential-difference equations,* Memories of the American Mathematical Society **35**.

[2] R. Bellman, K. L. Cooke, *Differential-Difference Equations,* Acad. Press., New York – London 1963.

[3] Л. Е. Ельсгольц, *Качественные методы в математической анализе,* Moskwa 1955.

[4] L. E. Esgolc, *Równania różniczkowe z odchylonym argumentem,* PWN, Warszawa 1966.

[5] Х. Х. Красовскии, *Некоторые задачи теории устойчивости,* Физматгиз, Москва 1959.

Instytut Matematyki
Politechnika Śląska
ul. Kaszubska 23
44-100 Gliwice
e-mail: olek@zeus.polsl.gliwice.pl

Recenzent: Janusz Szopa

Streszczenie

W pracy skupiliśmy naszą uwagę na badaniu stabilności równań różniczkowych z odchylonym argumentem za pomocą uogólnień drugiej metody Lapunowa. Jest to jeden z ważnych problemów, gdyż nie zawsze znamy jawne rozwiązanie danego równania, a jedynie wiemy o nim tyle, że rozwiązanie istnieje i jest jedyne. Do zdobycia dalszych informacji na temat tego rozwiązania bez znajomosci jawnego wzoru służy nam właśnie teoria stabilności. Przybliżymy tutaj drugą metodę Lapunowa, która daje poważne narzędzie matematyczne, użyteczne przy badaniu stabilności oraz asymptotycznej stabilności równań różniczkowych z odchylonym argumentem.

We wstępie do tej pracy skoncentrowaliśmy się na przybliżeniu potrzebnych nam pojęć i definicji, które były użyteczne w dalszej części pracy. W głównej części pracy podaliśmy i udowodniliśmy twierdzenia, które znajdują poważne zastosowania w badaniach nad stabilnością takich równań. Szczególnie twierdzenie 3:

Twierdzenie 3. *Niech wszystkie funkcje $\tau_i(t)$ są funkcjami ciągłymi dla $t_0 \leqslant t \leqslant t_0 + H$ $(H > 0)$ nieujemnymi, funkcja f jest ciągła w otoczeniu punktu:*

$$(t_0, \varphi(t_0), \varphi(t_0 - \tau - 1(t_0)), \ldots, \varphi(t_0 - \tau_m(t_0))),$$

i spełnia warunek Lipschitza względem wszystkich argumentów z wyjątkiem pierwszego, a funkcja początkowa $\varphi(t)$ jest ciągła na zbiorze E_{t_0}. Niech ponadto istnieje funkcjonał spełniający warunki:

(1) $V[t, x(s)] \leqslant W_1(||x(0)||) + W_2(||x(s)||_{\tau^2})$,

(2) $V[t, x(s0)] \geqslant W(||x(0)||)$,

(3) $\limsup\limits_{\delta \to 0^+} \frac{\Delta V}{\Delta t} \leqslant -\varphi(||x(0)||)$,

gdzie funkcje $W_1(r)$ *i* $W_2(r)$ *dla* $r \geqslant 0$ *są ciągłe i rosnące, przy czym* $W_1(0) = W_2(0) = 0$, *a funkcje* $W(r)$ *i* $\varphi(r)$ *są ciągłe i dodatnie dla* $r > 0$. *Wtedy rozwiązanie trywialne układu:*

$$\dot{x}_i(t) = f_i(t, x_1(t), \ldots, x_n(t), x_1[t - \tau_1(t)], \ldots, x_1[t - \tau_n(t)], \ldots,$$

$$\ldots, x_n[t - \tau_1(t)], \ldots, x_n[t - \tau_n(t)]) \qquad (i = 1, \ldots, n),$$

gdzie $\tau \geqslant \tau_i(t) \geqslant 0$ *jest asymptotycznie stabilne.*

Na samym końcu naszej pracy podaliśmy przykłady praktycznego zastosowania tego twierdzenia dla konkretnych równań różniczkowych z opóźnionym argumentem.