ZESZYTY NAUKOWE POLITECHNIKI ŚLĄSKIEJ 1997
Seria: MATEMATYKA-FIZYKA z. 82 Nr kol. 1360

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AFFINE CONNECTIONS ON MANIFOLDS WITH CERTAIN (f,g) – STRUCTURE

Summary. The subject of the paper is the determination of the affine connections compatible with the (f,g)-structures defined by a tensor field f of type (l,l), having the property $f^{3s}+f^{s}=0$, $s\geqslant 1$, and by a Riemannian structure g which satisfies an additional condition.

KONEKSJE AFINICZNE NA ROZMAITOŚCIACH Z PEWNĄ (f,g) – STRUKTURĄ

Streszczenie. W pracy wyznaczono koneksje afiniczne zgodne z (f,g)-strukturą zdefiniowaną polem tensorowym f typu (l,l), mającym własność $f^{3s}+f^{s}=0$, $s\geqslant 1$, i strukturą g Riemanna spełniającą pewne dodatkowe warunki.

1. Introduction

In this paper we study the (f,g)-structures determined by a tensor field f of type (l,l) so that $f^{3s}+f^{s}=0$, where s is a natural number, $s\geqslant 1$ and g is a Riemannian structure satisfying the condition $g(f^{s}X,f^{s}Y)=-g(f^{2s}X,Y)$ for every X, Y.

The case of the (f,g)-structures with $f^{2}+f=0$ was studied by R. Miron and Gh. Atanasiu in [1].

2. (f,g)-structures

Let M be a Riemannian manifold with the Riemannian metric g, $\mathcal{C}(M)$ the affin modul of the affine connections on M, $\mathcal{F}(M)$ the algebra of all differentiable funcitons on M and

$\mathcal{T}_p^r(M)$ the $\mathcal{F}(M)$-module of the tensor fields of type (r,p).

All the objects are of class C^∞.

Definition 1. *We call f-structure on M, a non-null field of tensors $f \in \mathcal{T}_l^l(M)$, of rank q, so that*

$$f^{3s} + f^s = 0.$$

If we put

$$l = -f^{2s}, \quad m = f^{2s} + I, \tag{1}$$

I denoting the identity operator (1) then we have

$$f^s l = l f^s = f^s, \quad f^s m = m f^s = 0, \quad f^{2s} l = l f^{2s} = -l. \tag{2}$$

Thus the operators l and m are complementary projection operators on m, because

$$l^2 = l, \quad m^2 = m, \quad lm = ml = 0, \quad l + m = I. \tag{3}$$

The Riemannian structure g can be considered a $\mathcal{T}_l^0(M)$-valued diferential l-form, i.e., $g : \mathcal{T}_0^l(M) \longrightarrow \mathcal{T}_l^0(M)$, $g(X) = g_X$, where $g_X(Y) = g(X,Y)$ for every $X, Y \in \mathcal{T}_0^l(M)$. If $f \in \mathcal{T}_l^l(M)$, then ${}^t f$ is the transpose of f, ${}^t f : \mathcal{T}_l^0(M) \longrightarrow \mathcal{T}_0^l(M)$, ${}^t f(z) = z \circ f$.

Definition 2. *We call by (f,g)-structure on M, a couple made up of a f-structure and a Riemannian structure g so that*

$${}^t f^s \circ g \circ f^s = g \circ m. \tag{4}$$

Theorem 1. *Let M be a paracompact differential manifold with a f-structure. Then, there is a (f,g)-structure.*

Proof. In truth, if $\tilde{g}$ is a Riemannian metric fixed on M, then

$$g(X,Y) = \frac{1}{2}\left(\tilde{g}(f^s X, f^s Y) + \tilde{g}(f^{2s} X, f^{2s})\right) \tag{5}$$

verifies the condition (4)

Thus we can state.

Proposition 1. *For a (f,g)-structure on M and l, m defined by the relations (1) we have*

$$g \circ f^s = -{}^t f^s \circ g, \qquad g^{-l} \circ {}^t f^s = -f^s \circ g^{-l}, \tag{6}$$

$$g \circ m = {}^t m \circ g, \qquad g^{-l} \circ {}^t m = m \circ g^{-l}. \tag{7}$$

Definition 3. *The applications of A, $A' : \mathcal{T}_l^l(M) \longrightarrow \mathcal{T}_l^l(M)$ we call Obata operators associated to f they are defined by*

$$A(w) = \frac{1}{2}(w - wm - mw + 3mwm - f^s w f^s) \quad (8)$$
$$A'(w) = w - A(w). \quad (9)$$

We also consider the Obata operators B associated to g:

$$B(u) = \frac{1}{2}(u - g^{-l} \circ {}^t u \circ g), \quad B'(u) = \frac{1}{2}(u + g^{-l} \circ {}^t u \circ g). \quad (10)$$

Proposition 2. *For a (f, g)-structure on M and for A, A' and B, B' defined by (8) and (10) we have:*

1. *A and A' are complementary projections on $\mathcal{T}_l^l(M)$;*
2. *B and B' commute pairwise with A and A';*
3. *$A \circ B$ and $A' \circ B'$ are projections on $\mathcal{T}_l^l(M)$;*
4. *Ker $A' \cap$ Ker B' = Im$(A \circ B)$.*

In truth, by simple calculation, we obtain the result *1*. The affirmation *2* is true, because, taking into account the relations (6) we have

$$(A \circ B - B \circ A)(u) =$$
$$\frac{1}{4}\Big[(m \circ g^{-l} \circ {}^t u \circ g - g^{-l} \circ {}^t m \circ {}^t u \circ g) + (g^{-l} \circ {}^t u \circ g \circ m - g^{-l} \circ {}^t u \circ {}^t m \circ g)$$
$$-3(m \circ g^{-l} \circ {}^t u \circ g \circ m - g^{-l} \circ {}^t m \circ {}^t u \circ {}^t m \circ g)$$
$$+(f^s \circ g^{-l} \circ {}^t u \circ g \circ f^s - {}^t f^s \circ {}^t u \circ {}^t f^s \circ g)\Big] = 0, \quad \forall u \in \mathcal{T}_l^l(M).$$

Thus from $A \circ B = B \circ A$ we obtain

$$A \circ B' = B' \circ A, \quad A' \circ B' = b' \circ A'.$$

The above mentioned relations give us the possibility to formulate [4]:

Proposition 3. *The system of tensorial equations*

$$A'(u) = a, \quad B'(u) = b, \quad (11)$$

has a solution $u \in \mathcal{T}_l^l(M)$, if and only if

$$A(a) = 0, \quad B(b) = 0, \quad A'(b) = B'(a). \quad (12)$$

If the conditions (12) are fullfiled, then the general solution of the system (11) is $u = a + A(b) + A \circ B(w)$, $w \in \mathcal{T}_l^l(M)$.

3. (f, g)-affine connections

In the text below $\overset{\circ}{\nabla}$ will be an affine connection fixed on M and every tensor field $u \in \mathcal{T}_l^l(M)$, may be considered as a field of $\mathcal{T}_0^l(M)$-valued differential l-forms. So, if ∇ is an affine connection on M, then we shall denote by D and $\tilde{D}$ the associated connections, acting on the $\mathcal{T}_0^l(M)$-valued differential l-forms and respectively on the differential l-forms $g : \mathcal{T}_0^l(M) \longrightarrow \mathcal{T}_l^0(M)$:

$$D_X u = \nabla_X u - u\nabla_X, \ \tilde{D}_X g = {}^t\nabla_X g - g\nabla_X, \forall X \in \mathcal{T}_0^l(M) \tag{13}$$

where

$$({}^t\nabla_X g)(Y, Z) = Xg(Y, Z) - g(\nabla_X Y, Z), \forall X, Y, Z \in \mathcal{T}_0^l(M) \tag{14}$$

Definition 4. *An affine connection ∇ on M is called (f, g)-affine connection if*

$$D_X f = 0, \ \tilde{D}_X g = 0, \ \forall \in \mathcal{T}_0^l(M). \tag{15}$$

Of course, for every (f, g)-affine connection, we have:

$$D_X l = \nabla_X l - l\nabla_X = 0, \qquad D_X m = \nabla_X m - m\nabla_X = 0, \tag{16}$$

$$D_X f^k = \nabla_X f^k - f^k\nabla_X = 0, \qquad k \text{ natural number}, \forall X \in \mathcal{T}_0^l(M), \tag{17}$$

We see that D and $\tilde{D}$ commute with the operators A, A', B, B'. We take $\nabla_X = \overset{\circ}{\nabla}_X + V_X$, where $V_X y = V(X, Y)$ and $V \in \mathcal{T}_2^l(M)$ for every $X, Y \in \mathcal{T}_0^l(M)$ and we find such a tensor field V that the conditions (15) are satisfied.

∇ will be an (f, g)-affine connection if and only if the field V verifies the system

$$V_X \circ f - f \circ V_X = -\overset{\circ}{D}_X f, \quad {}^tV_X \circ g + g \circ V_X = \overset{\circ}{\tilde{D}}_X g. \tag{18}$$

This system is equivalent with the system

$$A'(V_X) = \frac{1}{2}\left(-f^s(\overset{\circ}{D}_X f^s) - 2^t m(\overset{\circ}{D}_X l) + (\overset{\circ}{D}_X l)\right) \tag{19}$$

$$B'(V_X) = \frac{1}{2} g^{-l} \circ \overset{\circ}{\tilde{D}}_X g. \tag{20}$$

Applying the proposition 3, it becomes evident that the system (19) has solution and the general solution is

$$V_X = -\frac{1}{2}\left(-f^s(\overset{\circ}{D}_X f^s) - 2^t m(\overset{\circ}{D}_X l) + (\overset{\circ}{D}_X l)m\right) +$$

$$+\frac{1}{4}g^l\left\{\overset{\circ}{\tilde{D}}_X g - (\overset{\circ}{\tilde{D}}_X g)m - {}^tm \circ (\overset{\circ}{\tilde{D}}_X g) + 3^t m \circ (\overset{\circ}{\tilde{D}}_X g)m + {}^tf^s \circ (\overset{\circ}{\tilde{D}}_X g) \circ f^s)\right\} \tag{21}$$

$$+(A \circ B)(W_X), \quad W \in \mathcal{T}_2^1(M).$$

Then we obtain

Theorem 2. *There are (f,g)-affine connections: one of them is*

$$\nabla_X = \overset{\circ}{\nabla}_X + V_X \tag{22}$$

where $\overset{\circ}{\nabla}$ is an arbitrary affine connection fixed on M and V_X is given by (21), W being an arbitrary tensor field.

If $\overset{\circ}{\nabla}$ is the Levi-Civita connection of g, then we have $\tilde{\overset{\circ}{D}}_X g = 0$ and theorem 2 becomes

Theorem 3. *For every (f,g)-structure, the following affine connection*

$$\overset{c}{\nabla}_X = \overset{\circ}{\nabla}_X - \frac{1}{2}\left(-f^s \circ (\overset{\circ}{D}_X\, f^s) - 2^t m \circ (\overset{\circ}{D}_X\, l) + (\overset{\circ}{D}_X\, l) \circ m\right) \tag{23}$$

where $\overset{\circ}{\nabla}$ is the Levi-Civita cinnection of g, has the following characteristic:

1. $\overset{c}{\nabla}$ is dependent uniquely on f and g:

2. $\overset{c}{\nabla}$ is an (f,g)-affine connection.

Theorem 4. *The set of all the (f,g)-affine connections is given by*

$$\bar{\nabla}_X = \nabla_X + (A \circ B)(W_X), \quad W \in \mathcal{T}_2^1(M) \tag{24}$$

where ∇ is an (f,g)-affine connecton, in particular $\nabla = \overset{c}{\nabla}$.

Observing that (24) can be considered as a transformation of (f,g)-affine connections, we have:

Theorem 5. *The set of the transformations of (f,g)-affine connections with the multiplication of the applications is an abelian group, denoted $G(f,g)$, isomorphic to the additive group of tensors $W \in \mathcal{T}_2^1(M)$, which have the characteristic*

$$W_X \in Im(A \circ B) = Ker\ A' \cap Ker\ B', \quad X \in \mathcal{T}_0^1(M).$$

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Streszczenie

Praca omawia własności koneksji afinicznych z (f, g)-strukturą. Dowiedziono następujących własności.

Twierdzenie. Zbiór wszystkich koneksji (f, g)-afinicznych można opisać jako:

$$\bar{\nabla}_X = \nabla_X + (A \circ B)(W_X), \quad W \in \mathcal{T}_2^1(M),$$

gdzie ∇ jest koneksją (f, g)-afiniczną, w szczególności $\nabla = \overset{c}{\nabla}$.

Twierdzenie. Zbiór przekształceń koneksji (f, g)-afinicznych z mnożeniem przekształceń jest grupą abelową izomorficzną z pewną addytywną grupą tensorów.