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ON SOME PROPERTIES OF RESIDUALLY FINITE GROUPS*

Summary. We collect known properties of residually finite groups and show that the question of A. Shalev has a positive answer for residually finite groups.

O PEWNYCH WŁASNOŚCIACH GRUP REZYDUALNIE SKOŃCZONYCH

Streszczenie. Przedstawiamy znane własności grup rezydualnie skończonych (inaczej: skończenie aproksymowalnych) i pokazujemy, że w tej klasie grup problem A. Shaleva ma rozwiązanie.

1. Properties of residually finite groups

By $\mathfrak{N}_G$ we denote the family of all subgroups of finite index in a group G:

$$\mathfrak{N}_G = \{H;\ H < G,\ |G : H| < \infty\}.$$

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The set $\mathfrak{N}_G$ depends on the group G and always contains it. If there is no other subgroup we call $\mathfrak{N}_G$ singular. For example for finitely generated infinite simple group, $\mathfrak{N}_G$ is singular (a simple group has no proper normal subgroups, hence from Lemma 1(3) below, has no proper subgroups of finite index). We note the following properties:

Lemma 1.

1. *Let G be a finitely generated group. Then the number of subgroups of G having any fixed finite index n is finite (M. Hall, see [7]).*

2. *The set $\mathfrak{N}_G$ is closed under finite intersections. An intersection of an infinite number of subgroups in $\mathfrak{N}_G$ does not belong to $\mathfrak{N}_G$.*

3. *Every $H \in \mathfrak{N}_G$ contains a normal subgroup of G of finite index in G, and hence G posses a finite homomorphic image.*

Proof. 2. By Theorem of Poincaré [12, 1.3.12] a finite intersection of groups in $\mathfrak{N}_G$ is a group in $\mathfrak{N}_G$. To prove the second part we assume that an intersection of an infinite family has a finite index – $\bigcap_{s\in S} H_s = K$, $|S| = \infty$. Since $|G : K| = |G : H_s||H_s : K|$, then $|G : H_s|$ divides $|G : K|$. Thus there is only a finite number of divisors of $|G : K|$, and for each divisor, say d_i, by (1) there is only a finite number of groups of index d_i. This implies that $|S| < \infty$, which is a contradiction.
3. By Theorem of M. Hall [7, 4.7] every subgroup $H \in \mathfrak{N}_G$ contains a normal subgroup $N \lhd G$, $N \in \mathfrak{N}_G$ (the intersection of conjugates). Since for such a subgroup N there exists a homomorphism φ such that $Ker\varphi = N$, then G^φ is finite. ∎

Theorem 1. *Properties of subgroups in $\mathfrak{N}_F$ for a free finitely generated group F (see [7, I,3]):*

1. *If F is free of finite rank > 1 and $H \in \mathfrak{N}_F$, then H has finite rank and:*
$$|F : H| = \frac{rank(H) - 1}{rank(F) - 1}.$$
2. *Let A be a finite subset of F and K be a finitely generated subgroup of F disjoint from A. Then K is a free factor of a subgroup $H \in \mathfrak{N}_F$ disjoint from A.*
3. *If a finitely generated subgroup K of F contains a non-trivial normal subgroup of F, then K has a finite index in F.*
4. *Every non-trivial finitely generated normal subgroup of F is in $\mathfrak{N}_F$.*
5. *If H is a finitely generated subgroup of a free group F which has a non-trivial intersection with every non-trivial normal subgroup of F, then H is in $\mathfrak{N}_F$.*
6. *If H has a finite index in a free group F, then H has a non-trivial intersection with every non-trivial subgroup of F.*

Theorem 2. *For a group G the following conditions are equivalent:*

1. *For every nontrivial $g \in G$ there exists $H \in \mathfrak{N}_G$, $H \triangleleft G$ such that $g \notin H$.*
2. *The intersection of all subgroups in $\mathfrak{N}_G$ is trivial.*
3. *For every two different elements $g, h \in G$ there exists a homomorphism φ with a finite image, such that $g^\varphi \neq h^\varphi$.*

4. *For every nontrivial $g \in G$ there exists a homomorphism φ of G, with a finite image, such that $g^\varphi \neq 1$.*

5. *A group G is isomorphic to a subcartesian product of its finite quotient groups.*

Proof. We show the equivalence in several steps:

1. (1) $\Leftrightarrow$(2)

If all subgroups intersect in the unity, then for every element $g \in G$ there exists at least one subgroup N not containing g. By (3) in Lemma 1 there exists $H < N, H \triangleleft G$ and $g \notin H$ as required. Conversely, if for every element g there exists a subgroup in $\mathfrak{N}_G$ which does not contain g, then all subgroups in $\mathfrak{N}_G$ can intersect only in unity.

2. (1) $\Leftrightarrow$(4)

Let (1) holds. Since every normal subgroup is a kernel of a certain homomorphism, then $\forall g \in G,\ \exists \varphi,\ g \notin Ker\varphi$, which means that $g^\varphi \neq 1$ and (4) holds. Conversely, let $\forall g,\ \exists \varphi,\ g^\varphi \neq 1,\ |G^\varphi| < \infty$. Then $G^\varphi \simeq G/Ker\varphi,\ g \notin Ker\varphi \in \mathfrak{N}_G$. We put $H := Ker\varphi$ and (1) holds.

3. (3) $\Leftrightarrow$(4)

(4) follows immediately if substitute h by 1 in (3). Conversely, if we take gh^{-1} instead of g in (4), we obtain that $(gh^{-1})^\varphi \neq 1$, and hence $g^\varphi \neq h^\varphi$ as required.

4. (2) $\Leftrightarrow$(5)

Let $H_i \in \mathfrak{N}_G,\ i \in I$. We define a homomorphism φ of G into a cartesian product of its finite quotient groups in the following way: $\forall g \in G,\ g \to (gH_1, gH_2, \ldots)$. Then G^φ is a subcartesian product of finite quotient groups of G. We shall prove that if (2) holds, then φ is a monomorphism. Let $g^\varphi = h^\varphi$, then $(gh^{-1})^\varphi = 1$, so $(gh^{-1}H_1, gh^{-1}H_2, \ldots) = (H_1, H_2, \ldots)$, which implies that $\forall i,\ gh^{-1} \in H_i$. Since $\bigcap_{H \in \mathfrak{N}_G} H = \{1\}$, then $gh^{-1} = 1$, which means, that φ is an injection, as required. Conversely, let G^φ be a subcartesian product of $G/H_i,\ H_i \triangleleft G,\ H_i \in \mathfrak{N}_G$. Then $Ker\varphi = \bigcap_{H_i \in \mathfrak{N}_G} H_i$. Since

φ is an injection then $Ker\varphi = 1$ and hence the intersection of normal subgroups in $\mathfrak{N}_G$ is trivial, which means that the intersection of all subgroups in $\mathfrak{N}_G$ is also trivial.■

Definition 1. *A group G is called* **residually finite** *if it satisfies one of the properties given in Theorem 2.*

Example 1. *A finite group is residually finite. Let G be finite. Taking $\varphi = id$ in (3) we obtain that G is residually finite.*

Example 2. *Let $G = \langle x \rangle$ be an infinite cyclic group. So G is isomorphic to* **Z**. *Then $\forall g \in Z$, $\exists n \in N$ such that $(g, n) = 1$, which means that $g \notin nZ$. Since $Z/nZ = Z_n$ and $|Z_n| = n$, then G satisfies (1) in Theorem 1 and hence is residually finite.*

Now we give a few properties known from the literature about classes of groups which are residually finite and then about a subclass of residually finite groups.

Lemma 2.

1. *An extension of a residually finite group by a finite group is a residually finite group [6].*
2. *A subgroup of a residually finite group is a residually finite group [6].*

Theorem 3. *Classes of residually finite groups:*

1. *Every free group is residually finite (see [12, 6.1.9]).*
2. *(Finitely generated free)-by-cyclic groups are residually finite [2].*
3. *Every finitely generated nilpotent group is residually finite [6]. Infinitely generated ones do not have to be residually finite – e.g. a group of type C_{p^∞} for any prime p is not residually finite.*

4. *Finitely generated soluble groups are not necessarily residually finite [1].*

5. *Every finitely generated abelian-by-nilpotent group is residually finite. In particular, f.g. metabelian groups are residually finite [6].*

6. *Every finitely generated complex linear group is residually finite [4].*

Theorem 4. (Properties of residually finite groups).
Let G be any residually finite group and G_{fg} be a finitely generated residually finite group. Then:

1. *$Aut(G_{fg})$ is residually finite (Baumslag, see [7, 4.8]).*

2. *G_{fg} is hopfian (Mal'cev, see [7, 4.10]).*

3. *A semidirect product of G_{fg} by G is residually finite (Mal'cev, [9]).*

4. *If G is n-Engel, then it is nilpotent [15, 1.2].*

5. *If G satisfies a positive law, then it is nilpotent-by-(finite exponent) [3].*

6. *There exists a group G which gives a negative answer to Burnside problem [5]: Are finitely generated periodic groups finite?*

We show below that the question of A. Shalev has positive answer in the class of residually finite groups.

2. Collapsing groups problem

By a **positive relation** on k variables $x_1, \ldots, x_k$ (k-ary relation) we mean a relation of the form

$$u(x_1, \ldots, x_k) = v(x_1, \ldots, x_k), \tag{1}$$

where the different words u and v are written without negative powers of variables.

We say that relation (1) is a **positive law (a semigroup identity)** in a set (a group etc.) S if it holds when substitute any elements in S for variables $x_1, \ldots, x_k$.

A group G is called n-collapsing if $\forall S = \{g_1, g_2, \ldots, g_n\}, g_i \in G$ (n fixed), $|S^n| < |S|^n$. A group is called collapsing if there exists n such that the group is n-collapsing.

In [14] A. Shalev posed a question if a collapsing group satisfies a positive law. Combining this with Corollary C [14] we obtain that the question has a positive answer for residually finite groups. Here we give the proof of that fact (omitted in the article). We use a result in [14] (cf. Theorem B).

Theorem 5. *There exist functions k, c such that every finite group T which is n-collapsing possesses a nilpotent normal subgroup N such that:*
(1) $exp(T/N)$ divides $k(n)$ and
(2) every 2-generator subgroup of N is nilpotent of class at most $c(n)$.

Positive laws defining nilpotent groups were found by Mal'cev [8]. We use here the positive laws given in [10]. In order to state this laws, we define a sequence of words P_i in the variables $x, y, z_1, z_2, \ldots$ inductively:

$$P_1(x,y) = xy, \quad P_2(x,y,z_1) = P_1(x,y)z_1P_1(y,x),$$

$$P_{i+1}(x,y,z_1,\ldots,z_i) = P_i(x,y,z_1,\ldots,z_{i-1})z_iP_i(y,x,z_1,\ldots,z_{i-1}).$$

A group is nilpotent of class at most c if and only if it satisfies the positive law:

$$P_c(x,y,z_1,\ldots,z_{c-1}) = P_c(y,x,z_1,\ldots,z_{c-1}).$$

The above law implies a nontrivial binary positive law if we put 1 for all z_i:

$$P_c(x,y,\underbrace{1,1,\ldots,1}_{c-1}) = P_c(y,x,\underbrace{1,1,\ldots,1}_{c-1}). \qquad (2)$$

Now we prove:

Theorem 6. *A residually finite collapsing group G satisfies a positive law of the type* $P_c(x^k, y^k, \underbrace{1, 1, \ldots, 1}_{c-1}) = P_c(y^k, x^k, \underbrace{1, 1, \ldots, 1}_{c-1})$.

Proof. Let $H \triangleleft G$, $H \in \mathfrak{N}_G$. If G is n-collapsing, then every one of its subgroups and homomorphic images is also n-collapsing [14]. So G/H is finite and n-collapsing. We denote $T := G/H$. Then by Theorem 5., T contains a normal subgroup, every 2-generator subgroup of which is nilpotent of class at most c, where c depends on n only and hence the normal subgroup in T satisfies the law (2). Again by Theorem 5., the quotient of T by this normal subgroup has exponent dividing k, where k depends on n only. This implies that for every $H \triangleleft G$, $H \in \mathfrak{N}_G$, G/H satisfies the binary positive law:

$$P_c(x^k, y^k, \underbrace{1, 1, \ldots, 1}_{c-1}) = P_c(y^k, x^k, \underbrace{1, 1, \ldots, 1}_{c-1}).$$

Since G is residually finite, then by property (5) in Theorem 1. it is a subcartesian product of the finite groups G/H, each of which satisfies the same law. So G also satisfies the same positive law as required.■

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Streszczenie

Grupa G nazywa się kolapsująca (collapsing), jeżeli:

$$\exists\, n\ \forall\, S = \{g_1, g_2, \ldots, g_n\},\ g_i \in G, \quad |S^n| < |S|^n.$$

W [14] A. Shalev postawił pytanie, czy grupa kolapsująca musi spełniać tożsamość półgrupową. Udowadniamy, że rezydualnie skończona grupa kolapsująca spełnia nietrywialną tożsamość postaci:

$$P_c(x^k, y^k, \underbrace{1, 1, \ldots, 1}_{c-1}) = P_c(y^k, x^k, \underbrace{1, 1, \ldots, 1}_{c-1}),$$

gdzie P_i definiuje się indukcyjnie [10]:

$$P_1(x, y) = xy, \quad P_2(x, y, z_1) = P_1(x, y) z_1 P_1(y, x),$$

$$P_{i+1}(x, y, z_1, \ldots, z_i) = P_i(x, y, z_1, \ldots, z_{i-1}) z_i P_i(y, x, z_1, \ldots, z_{i-1}).$$