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ERGODITY OF PARTICULAR OSCILLATING PROCESS

Summary. The paper deals with the time-continuous, non-Markov, oscillating process. We are interested in the existing conditions and the form of its ergodic distribution.

ERGODYCZNOŚĆ PEWNEGO PROCESU OSCYLUJĄCEGO

Streszczenie. Tematem pracy jest proces oscylujący z ciągłym czasem. Proces nie jest markowowski. Interesują nas warunki istnienia oraz postać jego rozkładu ergodycznego.

Let $\{\alpha_i\}_{i=1}^{\infty}$ and $\{\kappa_i\}_{i=1}^{\infty}$ be two sequences of random variables. Assume that all these variables be indepedent. Variables α_i have the same distribution V_1. Variables κ_i are non-negative and have the same distribution W_1. Let us create stochastic process $\xi_1(t), t \geqslant 0$, as follows:

$$\xi_1(t) = -\sum_{i=1}^{\vartheta_1(t)} \alpha_i, \quad \vartheta_1(t) = \max\{n \geqslant 0 : \sum_{i=1}^{n} \kappa_i \leqslant t\}, \quad (\sum_{1}^{0} = 0).$$

Denote:

$$\tau_1(y) = \inf\{t : \xi_1(t) < -y\}, \qquad \gamma_1(y) = -\xi_1(\tau_1(y)) - y, \qquad y \geqslant 0.$$

Process $\xi_2(t)$ is defined in similar way. We have another two sequences of random variables $\{\beta_i\}_{i=1}^{\infty}$ i $\{\lambda_i\}_{i=1}^{\infty}$. Assume that all mentioned variables are independent, β_i have the same distribution V_2, λ_i are non-negative and have the same distribution W_2. Similarly for $t \geqslant 0$:

$$\xi_2(t) = \sum_{i=1}^{\vartheta_2(t)} \beta_i, \qquad \vartheta_2(t) = \max\{n \geqslant 0 : \sum_{i=1}^{n} \lambda_i \leqslant t\}.$$

$$\tau_2(x) = \inf\{t : \xi_2(t) > x\}, \qquad \gamma_2(x) = \xi_2(\tau_2(x)) - x, \qquad x \geqslant 0.$$

Our main purpose is to check limit peculiarities of process $\zeta_z(t)$, $z \geqslant 0$, defined as follows:

$$\zeta_z(t) = \begin{cases} \xi_1(t) + z & \text{for } 0 \leqslant t < \tau_1(z), \\ \xi_2(t - \tau_1(z)) - \gamma_1(z) & \text{for } \tau_1(z) \leqslant t < \tau_1(z) + \tau_2(\gamma_1(z)), \\ \zeta_{\gamma_2(\gamma_1(z))}(t - \tau_1(z) - \tau_2(\gamma_1(z))) & \text{for } \tau_1(z) + \tau_2(\gamma_1(z)) \leqslant t. \end{cases}$$

For $z < 0$ the definition is analogous.

In articles [1,4,6] their authors have dealt with oscillating random walk with one switching point. The article [5] deals with more complicated problem – random walk with two switching points. The time-continuous process when switching takes place with one stochastic process with independent increments to another has been described in [2]. In this article we consider the time-continuous process in which switching takes place between two non-Markov processes. The switching point is zero. General idea of examination of the processes of this kind has been taken from [2], but specific character of this process has caused important changes. Here are main theorem.

Theorem 1. *If positive parts of the distributions V_1 and V_2 have non-zero absolutely continuous components and:*

$$0 < E(\alpha_i), \quad 0 < E(\beta_i), \quad E(\kappa_i) < \infty, \quad E(\lambda_i) < \infty,$$

$$E(\alpha_i^3) < \infty, \qquad E(\beta_i^3) < \infty,$$

then for every x there exists a limit:

$$\lim_{t\to\infty} P(\zeta_z(t) < x)$$

and it does not depend on z.

Proof of this theorem will be conducted in two stages. First, the existence of this limit will be proven with weaker assumptions and for less complex version of the process $\zeta_z(t)$. Let variables α_i and β_i be non-negative with distributions F_1 and F_2. Then the Theorem 2 is true.

Theorem 2. *If distributions F_1 and F_2 have non-zero absolutely continuous components and:*

$$E(\alpha_i^2) < \infty, \quad E(\beta_i^2) < \infty, \quad E(\kappa_i) < \infty, \quad E(\lambda_i) < \infty,$$

then for every x there exists a limit:

$$\lim_{t\to\infty} P(\zeta_z(t) < x)$$

and it does not depend on z.

Proof of theorems. Theorem 1 will be the corollary from the Theorem 2, which will be proven. Firstly let us make some considerations and prove some lemmas. Let B be the Borel set from $(0,\infty)$. For non-negative y denote $B_y = \{x : x = u + y, u \in B\}$. Let us find distribution $\gamma_1(y)$. We can write equation:

$$P\{\gamma_1(y) \in B\} = F_1(B_y) + \int_0^y P\{\gamma_1(y-u) \in B\} dF_1(u).$$

This renewal equation has solution as follows:

$$P\{\gamma_1(y) \in B\} = F_1(B_y) + \int_0^y F_1(B_{y-u})dH_1(u),$$

where $H_1 = \sum_{n=1}^{\infty} F_1^{n*}$, $F_1^{1*} = F_1$, $F_1^{(n+1)*} = F_1 * F_1^{n*}$. Similarly:

$$P\{\gamma_2(x) \in B\} = F_2(B_x) + \int_0^x F_2(B_{x-u})dH_2(u),$$

Function under last integrate has finite total variation. Distribution F_2 has finite average which we denote m_2, and absolutely continuous component causes that distribution is not lattice. From main renewal theorem we have:

$$\lim_{x\to\infty} P\{\gamma_2(x) \in B\} = \frac{1}{m_2}\int_0^\infty F_2(B_u)du.$$

Denote this limit by $\varphi(B)$. Function $\varphi(B)$ is Borel maesure on $(0,\infty)$ and $\varphi((0,\infty)) = 1$.

Lemma 1. *If non-negative distribution F_2 has absolutely continuous component then for every fixed $0 < \epsilon < 1$ we can find k such that for every positive x and every Borel set A from $(0,\infty)$:*

$$\varphi(A) \leqslant \frac{\epsilon}{2^k} \Longrightarrow P\{\gamma_2(x) \in A\} \leqslant 1 - \frac{\epsilon}{2^k}.$$

Proof of Lemma 1. For convenience we will be using γ, F, H instead of γ_2, F_2, H_2. Assume that for certain ϵ exists sequences x_k, A^k such that:

$$\varphi(A^k) \leqslant \frac{\epsilon}{2^k} \quad i \quad P\{\gamma_2(x_k) \in A^k\} > 1 - \frac{\epsilon}{2^k}.$$

We can establish that x_k has a limit (perhaps ∞). Denote $B^n = \bigcup_{k=n}^{\infty} A^k$, $B = \bigcap_{n=1}^{\infty} B^n$. The following relation is true:

$$\varphi(B^n) \leqslant \sum_{k=n}^{\infty} \varphi(A^k) \leqslant \epsilon \sum_{k=n}^{\infty} \frac{1}{2^k} = \frac{\epsilon}{2^{n-1}},$$

thus:

$$\varphi(B^n) \longrightarrow 0, \quad \varphi(B) = 0.$$

Next for $n \leqslant k$ we have:

$$P\{\gamma(x_k) \in B^n\} \geqslant P\{\gamma(x_k) \in A^k\} > 1 - \frac{\epsilon}{2^k},$$

so for every n $P\{\gamma(x_k) \in B^n\} \longrightarrow 1$ with $k \to \infty$.

There are two possibilities: $x_k \to \infty$ or $x_k \to x_0$. In the case when $x_k \to \infty$ we see that $\varphi(B^n) = \lim_{k\to\infty} P\{\gamma(x_k) \in B^n\} = 1$. It is in conflict with $\varphi(B^n) \to 0$.

And now $x_k \to x_0$. Let us write F in form sum of components (absolutely continuous, discrete, singular) $F = F_c + F_d + F_o$. Denote by $P_c\{\gamma(x) \in A\}$ this part of distribution $P\{\gamma(x) \in A\}$ which comes into existence only due to component F_c, i.e.:

$$P_c\{\gamma(x) \in A\} = F_c(A_x) + \int_0^x F_c(A_{x-u})dH_c(u).$$

H_c is renewal function for F_c. Function $P_c\{\gamma(x) \in A\}$ is continuous with respect to x, moreover $P_c\{\gamma(x) \in A\} \leqslant P\{\gamma(x) \in A\}$ and $P_c\{\gamma(x) \in R_+\} = a > 0$. Due to B^n being decreasing we have two possibilities mutually excluding. First one is that for any m $P_c\{\gamma(x_0) \in R_+ \setminus B^m\} = b > 0$. Hence with $k \to \infty$:

$$P\{\gamma(x_k) \in R_+ \setminus B^m\} \geqslant P_c\{\gamma(x_k) \in R_+ \setminus B^m\} \longrightarrow b.$$

It is in conflict with $P\{\gamma(x_k) \in R_+ \setminus B^m\} \longrightarrow 0$. The second possibility is that $P\{\gamma(x_0) \in R_+ \setminus B^n\} = 0$ for every n. We thus infer that $P\{\gamma(x_0) \in B\} = a$ and next that set B has positive Lebesgue measure. But $\varphi(B) = 0$ implicates $\varphi_c(B) = 0$ where:

$$\varphi_c(B) = \frac{1}{m_c} \int_0^\infty F_c(A_u)du.$$

Hence we obtain that B has Lebesgue measure zero. It is impossible.∎

Lemma 2. *Under conditions from Theorem 2:*

$$\sup_{u \geqslant 0} \int_0^\infty zP\{\gamma(u) \in dz\} < \infty.$$

Proof of Lemma 2. Denote $\widehat{H}(t) = H(t) + 1, H(t) \geqslant 0$. Let us fix u and make calculations:

$$\int_0^\infty zP\{\gamma(u) \in dz\} = \int_0^\infty P\{\gamma(u) \geqslant z\}dz =$$

$$= \int_0^\infty \int_0^u (1 - F(z+u-t))d\widehat{H}(t)dz = \int_0^u \int_{u-t}^\infty (1 - F(z))dzd(\widehat{H}(t) - \widehat{H}(u)) =$$

$$= (\widehat{H}(t) - \widehat{H}(u)) \int_{u-t}^\infty (1 - F(z))dz \Big\|_0^u - \int_0^u (\widehat{H}(t) - \widehat{H}(u))(1 - F(u-t))dt =$$

$$= H(u) \int_u^\infty (1 - F(z))dz - \int_0^u \widehat{H}(u-t) - \widehat{H}(u))(1 - F(t))dt =$$

$$= \frac{H(u)}{u} u \int_u^\infty (1 - F(z))dz + \int_0^u (\widehat{H}(u) - \widehat{H}(u-t))(1 - F(t))dt.$$

As you know $\frac{H(u)}{u}$ is bounded. Next $u \int_u^\infty (1 - F(z))dz \leqslant \int_u^\infty z(1 - F(z))dz \leqslant \int_0^\infty z(1 - F(z))dz$ which is finite. And lastly from main renewal theorem $\widehat{H}(u) - \widehat{H}(u-t) \leqslant at + b$ and it implicates that last integrate is finite.∎

Denote by $\nu_0 = 0, \nu_1, \nu_2, \ldots$ the moments when process $\zeta_z(t)$ comes back on positive half-plane. Let us create Markov chain $\chi_n = \zeta_z(\nu_n)$.

Lemma 3. *Markov chain χ_n is ergodic and for its stationary measure π_+ the true is that:*

$$\int_0^\infty z\pi_+\{dz\} < \infty.$$

Proof of Lemma 3. Let us write formula for conditional distribution for chain χ_n:

$$P\{\chi_{i+1} \in B|\chi_i = x\} = \int_0^\infty P\{\gamma_1(x) \in du\}P\{\gamma_2(u) \in B\} \equiv Q(x, B).$$

Function Q complies with Doeblin condition when the measure occuring in this condition is φ which was denoted earlier. The true is:

$$Q(x, B) = \int_0^\infty P\{\gamma_1 \in du\}P\{\gamma_2(u) \in B\} \leqslant 1 - \frac{\epsilon}{2^k} = \delta$$

for certain natural k and every $\epsilon \in (0; 1)$ if $\varphi(B) \leqslant \frac{\epsilon}{2^k}$. We yet have to prove the existence of the average. For stationary distribution we have $\pi_+(A) = \int_0^\infty Q(u, A)\pi_+\{du\}$ hence, in accordance with Lemma 2:

$$\int_0^\infty z\pi_+\{dz\} = \int_0^\infty\int_0^\infty\int_0^\infty zP\{\gamma_2(\nu) \in dz\}P\{\gamma_1(u) \in d\nu\}\pi_+\{du\} \leqslant$$

$$\leqslant \sup_{\nu\geqslant 0}\int_0^\infty zP\{\gamma_2(\nu) \in dz\} \leqslant \infty. \blacksquare$$

Proof of Theorem 2. Let us create the process η_z as follows: $\eta_z(t) = \zeta_z(\nu_i)$, $\nu_i \leqslant t < \nu_{i+1}$. The process η_z is a semi-Markovian one with imbeded Markov chain χ_i. Denote by $Q(t, x, B)$ transition function of this process. Let be $Q_1(t, x, B) = Q(t, x, B)$ and:

$$Q_{n+1}(t, x, B) = \int_0^t\int_0^\infty Q(ds, x, dy)Q_n(t - s, y, B),\ n \geqslant 1,$$

$$R(ds,z,dy)=\sum_{n=1}^{\infty}Q_n(ds,z,dy).$$

Assume also that $x\geqslant 0$. Then:

$$P\{\zeta_z(t)\geqslant x\}=P\{\xi_1(t)+z\geqslant x\}+\int_0^t\int_0^{\infty}R(ds,z,dy)P\{\xi_1(t-s)+y\geqslant x\}.$$

We will use now more general version of the main renewal theorem (see [7]):

Theorem 3. *If transition function $Q(t,x,B)$ complies with the three following conditions:*

1) Function $Q(x,B)=Q(\infty,x,B)$ complies with Doeblin condition,

2) $m_+=\int_0^{\infty}x\pi_+\{dx\}<\infty$ where π_+ is stationary distribution for $Q(x,B)$,

3) $Q(t,x,B)$ is not lattice,

then for any non-negative function $g(t,y),t\geqslant 0,y\geqslant 0$, immedietely integrated in Riemann sense with respect to t for any y such that $\int_0^{\infty}\pi_+\{dy\}\int_0^{\infty}g(t,y)dt<\infty$, true is:

$$\lim_{t\to\infty}\int_0^t\int_0^{\infty}R(ds,z,dy)g(t-s,y)=\frac{1}{m_+}\int_0^{\infty}\pi_+\{dy\}\int_0^{\infty}g(t,y)dt.$$

Proof. We will prove, that Theorem 3 implicates Theorem 2. Three conditions, which are mentioned earlier, are complied. It is a consequence of assumptions and considerations made above. We will prove that function $P\{\xi_1(t)+y\geqslant x\}$ is immediately integrated in Riemann sense. This function is decreasing, thus to be ordinary integrated is sufficient. We have:

$$\int_0^{\infty}P\{\xi_1(t)+y\geqslant x\}dt\leqslant\int_0^{\infty}P\{\xi_1(t)+y\geqslant 0\}dt\leqslant\int_0^{\infty}P\{\tau_1(y)\geqslant t\}dt=$$

$$= E(\tau_1(y)) = E(\kappa_i)(H_1(y)+1) \leqslant cy + d,$$

what arises from Wald identity and propetries of renewal function. This and Lemma 3 implicates also that:

$$\frac{1}{m_+}\int_0^\infty \pi_+\{dy\}\int_0^\infty P\{\xi_1(t)+y \geqslant x\}dt < \infty.$$

All of conditions of Theorem 3 are complied. Then, because of $P\{\xi_1(t)+z \geqslant x\} \to 0$ with $t \to \infty$, we obtain:

$$\lim_{t\to\infty} P\{\zeta_z(t) \geqslant x\} = \frac{1}{m_+}\int_0^\infty \pi_+\{dy\}\int_0^\infty P\{\xi_1(t)+y \geqslant x\}dt.$$

Theorem 2 for non-negative x is proven. We can prove existence of the limit for negative x in the same way by taking Markov chain in points where process comes back to negative half-plane.∎

Proof of Theorem 1. We will assymilate process $\zeta_z(t)$ to less comlex version which was consider in Theorem 2. For that purpose we will use ladder random variables. For random walk generated by $\{\alpha_i\}$ first ladder height we denote by A_1, and first ladder index by I_1. Ladder height number k can be pictured in form $A_1 + \ldots + A_k$ independent random variables with distribution like A_1. Ladder index number k can be pictured in form $I_1 + \ldots + I_k$ independent random variables with distribution like I_1. For random walk generated by $\{\beta_i\}$ we denote in similar way B_i and J_i. Denote $K_1 = \sum_{m=1}^{I_1} \kappa_m$, $K_2 = \sum_{m=I_1+1}^{I_1+I_2} \kappa_m$, etc. $i = 1, 2, 3, \ldots$. Similarly: $\Lambda_1 = \sum_{m=1}^{J_1} \lambda_m$, $\Lambda_2 = \sum_{m=J_1+1}^{J_1+J_2} \lambda_m$, etc. In our situation sequences A_i, K_i, B_i, Λ_i are the sequences of indepedent random variables with the same distributions which we denote by F_1, G_1, F_2, G_2. Moreover true is $E(A_i^2) < \infty$, $E(B_i^2) < \infty$, what arises from existence of the third moments for α_i, β_i, and $E(K_i) < \infty$,

$E(\Lambda_i) < \infty$. It is clear that distributions F_1, F_2 have absolutely continuous components. Therefore in identicall way we introduce Markov chain and prove Lemmas 1, 2, 3. Similarly we introduce process $\eta_z(t)$, transition function $Q(t, x, B)$. Here for $x \geqslant 0$ we have:

$$P\{\zeta_z(t) \geqslant x\} = P\{\xi_1(t) + z \geqslant x, \min_{u<t} \xi_1(u) + z \geqslant 0\}+$$

$$+\int_0^t \int_0^\infty R(ds, z, dy) P\{\xi_1(t-s) + y \geqslant x, \min_{u<t-s} \xi_1(u) + y \geqslant 0\}.$$

As has been done before we use Theorem 3. All three conditions, mentioned there, are complied. Similarly:

$$\int_0^\infty P\{\xi_1(t) + y \geqslant x, \min_{u<t} \xi_1(t) + y \geqslant 0\} dt \leqslant \int_0^\infty P\{\tau_1(y) \geqslant t\} dt =$$

$$= E(\tau_1(y)) = E(K_i)(H_1(y) + 1) \leqslant cy + d,$$

$$\frac{1}{m_+} \int_0^\infty \pi_+\{dy\} \int_0^\infty P\{\xi_1(t) + y \geqslant x, \min_{u<t} \xi_1(t) + y \geqslant 0\} dt < \infty,$$

and function $P\{\xi_1(t) + y \geqslant x, \min_{u<t} \xi_1(t) + y \geqslant 0\}$ is immediately integrated in Riemann sense. Thus:

$$\lim_{t \to \infty} P\{\zeta_z(t) \geqslant x\} =$$

$$= \frac{1}{m_+} \int_0^\infty \pi_+\{dy\} \int_0^\infty P\{\xi_1(t) + y \geqslant x, \min_{u<t} \xi_1(t) + y \geqslant 0\} dt.$$

We finish the proof for negative x like for the less complex process. ∎

References

1. А. А. Боровков, *Предельные распределения осциллирующего случайного блуждания,* Теор. Вероятностей и её Применения **25** (1980), 663-665.

2. N. S. Bratijchuk, D. V. Gusak, *Ergodic distribution of an oscillating process with independent increments,* Ukrain. Math. J. **38** (1986), 465-471.

3. W. Feller, *Wstęp do rachunku prawdopodobieństwa,* tom 2, PWN, Warszawa 1969.

4. J. H. Kemperman, *The oscillating random walk,* Stochastic Proc. Appl. **2** (1974), 1-29.

5. V. I. Lotov, *On oscillating random walks,* Sib. Math. J. **37** (1996), 764-774.

6. Б. А. Рогозин, С. Г. Фосс, *Возвратность осциллирующего случайного блуждания,* Теор. Вероятностей и её Применения **23** (1978), 161-168.

7. В. М. Шуренков, *К теории марковского восстановления,* Теор. Вероятностей и её Применения **29** (1984), 248-263.

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Streszczenie

Błądzeniem przypadkowym nazywamy ciąg postaci $\zeta_n = \alpha_1 + \alpha_2 + \ldots + \alpha_n$, gdzie $\{\alpha_1, \alpha_2, \ldots\}$ jest ciągiem niezależnych zmiennych losowych o jednakowym rozkładzie. Błądzenie przypadkowe jest łańcuchem Markowa o dobrze poznanych własnościach. W [1, 4, 6] był rozpatrywany nowy rodzaj błądzenia przypadkowego, w którym rozkład skoku w chwili n zależy od znaku ζ_n. Mamy wtedy $\zeta_{n+1} = \zeta_n + \alpha_i$, jeżeli $\zeta_n > 0$, oraz $\zeta_{n+1} = \zeta_n + \beta_j$, jeżeli $\zeta_n \leqslant 0$, gdzie $\{\alpha_1, \alpha_2, \ldots\}$, $\{\beta_1, \beta_2, \ldots\}$ są niezależnymi ciągami niezależnych zmiennych losowych o jednakowym rozkładzie w każdym ciągu. Taki rodzaj błądzenia nazywamy oscylującym błądzeniem przypadkowym. W tym artykule wprowadzamy analog z ciągłym czasem oscylującego błądzenia przypadkowego. Skoki procesu $\zeta(t)$ mają miejsce w momentach odnowy $t_1, t_2, \ldots$, gdy $\zeta(t)$ jest w dodatnej półpłaszczyźnie, lub w momentach odnowy $s_1, s_2, \ldots$, gdy $\zeta(t)$ jest w ujemnej półpłaszczyźnie. Procesy odnowy $t_1, t_2, \ldots, s_1, s_2, \ldots$ są generowane przez dwa ciągi nieujemnych niezależnych zmiennych losowych $\{\kappa_i\}$, $\{\lambda_i\}$. W artykule znajdziemy warunki wystarczające do ergodyczności procesu $\zeta(t)$ oraz wzór określający jego rozkład ergodyczny.