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Stefan CZERWIK, Krzysztof DŁUTEK

THE STABILITY OF THE QUADRATIC EQUATION FOR SET-VALUED MAPPINGS*

Summary. In the paper we consider the problem of the Ulam-Hyers and Rassias types of stability of the quadratic functional equation for set-valued functions.

STABILNOŚĆ RÓWNANIA FUNKCJONAŁÓW KWADRATOWYCH W KLASIE MULTIFUNKCJI

Streszczenie. W artykule rozpatrywany jest problem stabilności równania funkcjonałów kwadratowych dla funkcji wielowartościowych.

1. Introduction

The problem of the stability of functional equations has originally been stated by S. M. Ulam in [8]. In the basic paper [4] D. H. Hyers has proved the stability of the linear functional equation.

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Let G_i, $i = 1, 2$ be groups. A function $f : G_1 \to G_2$ satifying the functional equation:

$$f(x+y) + f(x-y) = 2f(x) + 2f(y) \text{ for all } x, y \in G_1 \tag{1}$$

is called the quadratic function.

Some result about the stability of the linear mapping has been presented by T. M. Rassias in [7]. The Hyers-Ulam stability of the quadratic equation (1) has been studied in [2] and the Rassias stability in [3].

In this paper we consider the stability of the quadratic equation for set-valued functions.

2. Preliminary results

Let X be a normed space. Given sets $A, B \subset X$ and a number $t \in \mathbb{R}$ (the set of all real numbers) we define:

$$A + B := \{x \in X; x = a + b, a \in A, b \in B\},$$

$$tA := \{x \in X; x = ta, a \in A\}.$$

Let $CC(X)$ denote the space of all non-empty compact convex subsets of X. Put:

$$d(A, B) := \inf\{t > 0; A \subset B + tK, B \subset A + tK\},$$

where K is the closed unit ball in X and A, B are non-empty closed bounded subset of X. The function d is a metric called the Hausdorff metric induced by the metric of the space X.

The following lemmas will be usefull.

Lemma 1. ([6]) *Let X be a normed space. For any $A, B, C \in CC(X)$ and $t \in \mathbb{R}$, we have:*

$$d(A+C, B+C) = d(A, B), \tag{2}$$

$$d(tA, tB) = |t| d(A, B). \tag{3}$$

Lemma 2. ([1]) *If X is a Banach space, then $CC(X)$ is a complete metric space.*

We define:

$$0^v = \begin{cases} 0 \ , & v \neq 0, \\ 1 \ , & v = 0. \end{cases}$$

Now we shall prove the following:

Lemma 3. *Let E_1, E_2 be two normed spaces. If set-valued functions $F : E_1 \to CC(E_2)$, $G : E_1 \to CC(E_2)$ satisfy the inequality:*

$$d[F(x+y) + F(x-y), G(x) + G(y)] \leqslant \varepsilon + \eta(\|x\|^v + \|y\|^v), x, y \in E_1, \tag{4}$$

where $\varepsilon \geqslant 0, \eta \geqslant 0$ and $v \in \mathbb{R}$ are given numbers, then:

$$d[F(x+y) + F(x-y) + 2F(0), 2F(x) + 2F(y)] \leqslant \varepsilon_1 + 2\eta(\|x\|^v + \|y\|^v), \tag{5}$$

$$d[G(x+y) + G(x-y) + 2G(0), 2G(x) + 2G(y)] \leqslant \varepsilon_2 + \\ + \eta(2\|x\|^v + 2\|y\|^v + \|x+y\|^v + \|x-y\|^v) \tag{6}$$

for all $x, y \in E_1$, where $\varepsilon_1 = 4(\varepsilon + \eta 0^v)$, $\varepsilon_2 = 4\varepsilon + 2\eta 0^v$.

Proof. By Lemma 1 and (4) for $x, y \in E_1$ we get the following inequalities:

$$d[F(x+y) + F(x-y) + 2F(0), 2F(x) + 2F(y)] \leqslant$$

$$\leqslant d[F(x+y) + F(x-y) + 2F(0), G(x) + G(y) + 2F(0)] +$$

$$+d[G(x)+G(y)+2F(0),G(x)+G(y)+2G(0)]+$$

$$+d[G(x)+G(y)+2G(0),2F(x)+G(y)+G(0)]+$$

$$+d[2F(x)+G(y)+G(0),2F(x)+2F(y)]\leqslant \varepsilon_1+2\eta(\|x\|^v+\|y\|^v).$$

Analogously,

$$d[G(x+y)+G(x-y)+4G(0),2G(x)+2G(y)+2G(0)]\leqslant$$

$$\leqslant d[G(x+y)+G(x-y)+4G(0),2F(x+y)+G(x-y)+3G(0)]+$$

$$+d[2F(x+y)+G(x-y)+3G(0),2F(x+y)+2F(x-y)+2G(0)]+$$

$$+d[2F(x+y)+2F(x-y)+2G(0),2G(x)+2G(y)+2G(0)]\leqslant$$

$$\leqslant \varepsilon_2+\eta(2\|x\|^v+2\|y\|^v+\|x+y\|^v+\|x-y\|^v),$$

and the proof is completed. ∎

Lemma 4. *Let E_1, E_2 be two normed spaces. If set-valued functions $F, G : E_1 \to CC(E_2)$ satisfy the inequality (4), then:*

$$d[F(2^n x)+(4^n-1)F(0),4^n F(x)]\leqslant$$

$$3^{-1}(4^n-1)\varepsilon_1+4^n\eta(1+a+\ldots+a^{n-1})\|x\|^v, \tag{7}$$

$$d[G(2^n x)+(4^n-1)G(0),4^n G(x)]\leqslant$$

$$\leqslant 3^{-1}(4^n-1)\varepsilon_3+4^n\eta(1+a)(1+a+\ldots+a^{n-1})\|x\|^v, \tag{8}$$

for all $x\in E_1$, $n\in N$ where $a=2^{v-2}$, $\varepsilon_3=\varepsilon_2+\eta 0^v$.

Proof. Putting $x=y$ in (5) we get (7) for $n=1$. Assuming the validity of (7) for some $n\geqslant 1$ we have for $0\neq x\in E_1$:

$$d[F(2^{n+1}x)+(4^{n+1}-1)F(0),4^{n+1}F(x)]\leqslant d[F(2^{n+1}x)+3F(0),4F(2^n x)]+$$

$$+4d[F(2^n x)+(4^n-1)F(0),4^n F(x)]\leqslant$$

$$\leqslant \varepsilon_1 + 4\eta\|2^n x\|^v + 4[3^{-1}(4^n - 1)\varepsilon_1 + 4^n\eta(1 + a + \ldots + a^{n-1})\|x\|^v] =$$
$$= 3^{-1}(4^{n+1} - 1)\varepsilon_1 + 4^{n+1}\eta(1 + a + \ldots + a^n)\|x\|^v,$$

which proves the inequality (7) for all $n \in N$ and all $x \in E_1$ (if $x = 0$, (7) is trivially satisfied). Now we will prove the inequality (8). For $n = 1$ we get it just taking $x = y$ in the inequality (6). Let us assume that (8) is true for some $n \geqslant 1$. Then we have for all $0 \neq x \in E_1$:

$$d[G(2^{n+1}x) + (4^{n+1} - 1)G(0), 4^{n+1}G(x)] \leqslant$$
$$\leqslant d[G(2^{n+1}x) + 3G(0), 4G(2^n x)] + 4d[G(2^n x) + (4^n - 1)G(0), 4^n G(x)] \leqslant$$
$$\leqslant \varepsilon_3 + 4\eta(1+a)\|2^n x\|^v + 4[3^{-1}(4^n - 1)\varepsilon_3 + 4^n\eta(1+a)(1+a+\ldots+a^{n-1})\|x\|^v] =$$
$$= 3^{-1}(4^{n+1} - 1)\varepsilon_3 + 4^{n+1}\eta(1 + a)(1 + a + \ldots + a^n)\|x\|^v.$$

If $x = 0$, then (8) also holds true. This completes the proof of the inequality (8).∎

3. Stability results

In this section we first present the following:

Theorem 1. *Let E_1 be a normed space and E_2 a Banach space. If set-valued functions $F, G : E_1 \to CC(E_2)$ satisfy the inequality (4) with $v < 2$, then there exists exactly one quadratic set-valued function $A : E_1 \to CC(E_2)$ such that:*

$$d[A(x) + F(0), F(x)] \leqslant 3^{-1}\varepsilon_1 + 4(4 - 2^v)^{-1}\eta\|x\|^v, \tag{9}$$

$$d[2A(x) + G(0), G(x)] \leqslant 3^{-1}\varepsilon_3 + (4 + 2^v)(4 - 2^v)^{-1}\eta\|x\|^v \tag{10}$$

for all $x \in E_1$.

Proof. We define (N states for the set of all natural numbers):

$$A_n(x) := 4^{-n}F(2^n x), x \in E_1, n \in N. \tag{11}$$

We shall prove that $\{A_n(x)\}$ is a Cauchy sequence for every $x \in E_1$. In fact, by Lemma 4 and (3) we have for $n > m$ and $0 \neq x \in E_1$:

$$d[A_n(x), A_m(x)] = 4^{-n}d[F(2^n x), 4^{n-m}F(2^m x)] \leqslant$$

$$\leqslant 4^{-n}d[F(2^n x), F(2^n x) + (4^{n-m} - 1)F(0)] + 4^{-n}d[F(2^n x)+$$

$$+(4^{n-m} - 1)F(0), 4^{n-m}F(2^m x)] \leqslant$$

$$\leqslant 4^{-n}d[\{0\}, (4^{n-m} - 1)F(0)] + 4^{-n}[3^{-1}(4^{n-m} - 1)\varepsilon_1+$$

$$+4^{n-m}\eta(1 + a + \ldots + a^{n-m-1})\|2^m x\|^v] \leqslant$$

$$\leqslant 4^{-m}(d[\{0\}, F(0)] + \varepsilon_1) + 2^{m(v-2)}(1 - a)^{-1}\eta\|x\|^v.$$

Hence since $v < 2$ we get the conclusion (for $x = 0$ our statement is obvious). Therefore, by Lemma 2 there exists:

$$A(x) := \lim_{n\to\infty} A_n(x) \text{ for all } x \in E_1. \tag{12}$$

Now we shall check that:

$$2A(x) = \lim_{n\to\infty} 4^{-n}G(2^n x),\ x \in E_1. \tag{13}$$

Indeed, denoting by $\|G(0)\| = d[\{0\}, G(0)]$, we have:

$$d[2A(x), 4^{-n}G(2^n x)] \leqslant d[2A(x), 2^{-2n+1}F(2^n x)]+$$

$$+d[2^{-2n+1}F(2^n x), 4^{-n}G(2^n x) + 4^{-n}G(0)]+$$

$$+d[4^{-n}G(2^n x) + 4^{-n}G(0), 4^{-n}G(2^n x)] \leqslant$$

$$\leqslant 2d[A(x), A_n(x)] + 4^{-n}d[2F(2^n x), G(2^n x) + G(0)] + 4^{-n}\|G(0)\| \leqslant$$

$$2d[A(x), A_n(x)] + 4^{-n}[\varepsilon + \eta(\|2^n x\|^v + 0^v) + \|G(0)\|].$$

Taking into account that $v < 2$, this implies:

$$\lim_{n\to\infty} d[2A(x), 4^{-n}G(2^n x)] = 0,$$

i. e. we have got (13). The function A is a quadratic one. Indeed,

$$d[A_n(x+y) + A_n(x-y), 4^{-n}G(2^n x) + 4^{-n}G(2^n y)] \leqslant$$

$$\leqslant 4^{-n}(\varepsilon + \eta[\|2^n x\|^v + \|2^n y\|^v]).$$

Hence letting $n \to \infty$ in view of the fact $v < 2$ and (13) we obtain the equality:

$$A(x+y) + A(x-y) = 2A(x) + 2A(y) \text{ for all } x, y \in E_1.$$

The estimations (9) and (10) one can establish directly from (7) and (8) respectively. Now we are going to prove that the required function A is unique. To prove it, let us assume that there exist two quadratic set-valued functions $C_i : E_1 \to CC(E_2)$, $i = 1, 2$ such that:

$$d[C_i(x) + F(0), F(x)] \leqslant a_i\varepsilon_i + b_i\|x\|^v, x \in E_1, i = 1, 2,$$

where $a_i \geqslant 0$, $b_i \geqslant 0$, $i = 1, 2$ are real constants.

It is a simple exercise to verify that for any quadratic function we have the property:

$$C_i(2^n x) = 4^n C_i(x), x \in E_1, n \in N.$$

Now we get for $x \in E_1$:

$$d[C_1(x), C_2(x)] \leqslant d[C_1(x) + F(0), F(x)] + d[F(x), C_2(x) + F(0)] \leqslant$$

$$\leqslant a_1\varepsilon_1 + a_2\varepsilon_2 + (b_1 + b_2)\|x\|^v,$$

and consequently for $0 \neq x \in E_1$:

$$d[C_1(x), C_2(x)] = 4^{-n} d[C_1(2^n x), C_2(2^n x)] \leqslant$$

$$\leqslant 4^{-n}(a_1\varepsilon_1 + a_2\varepsilon_2) + 2^{n(v-2)}(b_1 + b_2)\|x\|^v.$$

Hence $C_1(x) = C_2(x)$ for all $0 \neq x \in E_1$. But $C_1(0) = C_2(0)$, which completes the proof of the theorem.■

Now we state the following:

Lemma 5. *Let E_1, E_2 be normed spaces. If set-valued functions $F, G : E_1 \to CC(E_2)$ satisfy the inequality (4), then:*

$$d[F(x)+(4^n-1)F(0), 4^n F(2^{-n}x)] \leqslant 3^{-1}(4^n-1)\varepsilon_1+\eta(b+\ldots+b^n)\|x\|^v, \quad (14)$$

$$d[G(x) + (4^n - 1)G(0), 4^n G(2^{-n}x)] \leqslant$$
$$\leqslant 3^{-1}(4^n - 1)\varepsilon_3 + \eta(1 + b)(1 + b + \ldots + b^{n-1})\|x\|^v \quad (15)$$

for all $x \in E_1$, where $b = 2^{2-v}$.

Proof. Putting $x = 2^{-n}t$ into (7) and (8) and considering that $a = b^{-1}$ we get directly (14) and (15) respectivelly.■

Considering the case $v > 2$ we can now prove:

Theorem 2. *Let E_1 be a normed space and E_2 a Banach space. Let $F, G : E_1 \to CC(E_2)$ satisfy the inequality:*

$$d[F(x + y) + F(x - y), G(x) + G(y)] \leqslant \eta(\|x\|^v + \|y\|^v), x, y \in E_1 \quad (16)$$

where $\eta \geqslant 0$ and $v > 2$. If, moreover $F(0) = \{0\}$, then there exists exactly one quadratic set-valued function $B : E_1 \to CC(E_2)$ such that:

$$d[B(x), F(x)] \leqslant 4(2^v - 4)^{-1}\eta\|x\|^v, \quad (17)$$

$$d[2B(x), G(x)] \leqslant (2^v + 4)(2^v - 4)^{-1}\eta\|x\|^v \quad (18)$$

for all $x \in E_1$.

Proof. From (16) we get $F(0) = G(0) = \{0\}$. Define:

$$B_n(x) := 4^n F(2 -^n x), x \in E_1, n \in N. \tag{19}$$

In view of Lemma 5 ($\varepsilon_1 = 0$) and the condition $v > 2$ we can prove that for every $x \in E_1$ the sequence $\{B_n(x)\}$ is a Cauchy sequence. Put:

$$B(x) := \lim_{n\to\infty} B_n(x) \ , \ x \in E_1. \tag{20}$$

Then it is easy to verify that:

$$2B(x) = \lim_{n\to\infty} 4^n G(2^{-n}x) \ , \ x \in E_1. \tag{21}$$

Now we can write:

$$d[B_n(x+y) + B_n(x-y), 4^n G(2^{-n}x) + 4^n G(2^{-n}y)] \leqslant$$

$$\leqslant 4^n \eta(\|2^{-n}x\|^v + \|2^{-n}y\|^v) \tag{22}$$

and with respect to (20) and (21) we obtain for all $x, y \in E_1$ (taking limits of both sides as $n \to \infty$):

$$B(x+y) + B(x-y) = 2B(x) + 2B(y),$$

i. e. B is a quadratic function. Taking into account (14) and (15) we get immediately the estimations (17) and (18). To prove the uniqnueness one can proceed similarly as in the proof of Theorem 1.■

In the case $F(0) \neq \{0\}$ we are able to present satisfactory result only for ordinary single-valued functions.

Theorem 3. *Let E_1 be a normed space and E_2 a Banach space. Let $F, G : E_1 \to E_2$ satisfy the inequality:*

$$\|F(x+y) + F(x-y) - G(x) - G(y)\| \leqslant \eta(\|x\|^v + \|y\|^v), x, y \in E_1 \quad (23)$$

where $\eta \geqslant 0$ and $v > 2$. Then there exists exactly one quadratic function $p : E_1 \to E_2$ such that:

$$\|p(x) + F(0) - F(x)\| \leqslant 4(2^v - 4)^{-1}\eta\|x\|^v, \quad (24)$$

$$\|2p(x) + G(0) - G(x)\| \leqslant (2^v + 4)(2^v - 4)^{-1}\eta\|x\|^v \quad (25)$$

for all $x \in E_1$.

If, moreover, F is measurable (i.e. $F^{-1}(U)$ is Borel set in E_1 for every set U open in E_2) or $\mathbb{R} \ni t \to F(tx)$ is continuous for every fixed $x \in E_1$, then p has the property:

$$p(tx) = t^2 p(x), x \in E_1, t \in \mathbb{R}. \quad (26)$$

Proof. Let us denote $c := F(0) = G(0)$ and $f(x) := F(x) - c$, $g(x) := G(x) - c$ for $x \in E_1$. From (5) we get:

$$\|f(x+y) + f(x-y) - 2f(x) - 2f(y)\| \leqslant 2\eta(\|x\|^v + \|y\|^v), x \in E_1.$$

Therefore, as in Theorem 2 we can define:

$$p(x) := \lim_{n\to\infty} 4^n f(2^{-n}x) \ , \ x \in E_1,$$

and p is a quadratic function satisfying the condition:

$$2p(x) = q(x) := \lim_{n\to\infty} 4^n g(2^{-n}x) \ , \ x \in E_1.$$

Hence applying (17) and (18) we get:

$$\|p(x) - f(x)\| \leqslant 4(2^v - 4)^{-1}\eta\|x\|^v,$$

$$\|2p(x) - g(x)\| \leqslant (2^v + 4)(2^v - 4)^{-1}\eta\|x\|^v$$

for all $x \in E_1$, i.e. the conditions (23) and (24). The proof of the uniqueness runs similarly to that one presented for Theorem 1. Now let L be a continuous linear functional on the space E_2. Let $\phi : \mathbb{R} \to \mathbb{R}$ be given by:

$$\phi(t) = L[p(tx)], x \in E, t \in \mathbb{R}.$$

Then ϕ is a quadratic function and as the pointwise limit of the sequence:

$$\phi_n(t) = 4^n L[f(2^{-n}tx)], t \in \mathbb{R}, n \in N, x \in E_1$$

is measurable and hence (see [5]) has the form $\phi(t) = t^2\phi(1)$, for $t \in \mathbb{R}$. Therefore:

$$L[p(tx)] = \phi(t) = t^2\phi(1) = L[t^2p(x)],$$

which implies the condition (25). This ends the proof.∎

Remark. *If $v = 2$, our theorems are not true even for ordinary functions (for suitable example see [3]).*

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Stefan Czerwik
Krzysztof Dłutek
Institute of Mathematics
Silesian Technical University
Kaszubska 23
44-100 Gliwice

Streszczenie

Niech A, B będą przestrzeniami unormowanymi oraz $CC(B)$ będzie zbiorem zwartych, wypukłych i niepustych podzbiorów B. Funkcję $F : A \to CC(B)$ będziemy nazywać kwadratową, jeśli spełnia równanie (1) zwane równaniem funcjonałów kwadratowych:

$$F(x+y) + F(x-y) = 2F(x) + 2F(y). \tag{1}$$

W pracy podano warunki stabilności w sensie Hyersa-Ulama oraz Rassiasa równania funkcjonałów kwadratowych dla multifunkcji F.