ZESZYTY NAUKOWE POLITECHNIKI ŚLĄSKIEJ 1999

Seria: MATEMATYKA-FIZYKA z. 84 Nr kol. 1411

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THE ERROR BOUNDS FOR SOME APPROXIMATION PROCESSES IN $L_{\mathbf{P}}(Q)$

Summary. In this paper there are generalized some Schumaker's theorems (see Theorem 2.68 and 2.69 [2]) for the function's two variables in $L_P(Q)$ spaces with mixed norms.

OSZACOWANIE BŁĘDÓW W PEWNYCH PROCESACH APROKSYMACYJNYCH W PRZESTRZENIACH $L_{\mathbf{P}}(Q)$

Streszczenie. W [1] udowodniono równoważność wagowego K -funkcjonału i wagowego modułu gładkości w przestrzeni $L_{\mathbf{P}}(Q)$ z normą mieszaną. W tej notce pokażemy, jak opierając się na tym wyniku w prosty sposób można udowodnić twierdzenia, które są uogólnieniem wyników Schumakera (tw. 2.68 i 2.69 [2]) sformułowanych dla funkcji jednej zmiennej w przestrzeni L_p na funkcje dwóch zmiennych w przestrzeni $L_{\mathbf{P}}(Q)$ z normą mieszaną.

1. Introduction

In [2] L. L. Schumaker showed that for various approximation processes it is often sufficient to establish the bounds for smooth functions only, and the bounds for less-smooth function classes will follow automatically.

In this paper these results are generalized for the case of functions in the spaces $L_{\mathbf{P}}(Q)$, with mixed norms.

NOTATIONS

We shall use the following standard notations and definitions (see [2]):

Let $\mathbf{a} = [a_1, a_2] \in R^2$, $\mathbf{b} = [b_1, b_2] \in R^2$, then we denote;

$\mathbf{a} \otimes \mathbf{b} = [a_1 b_1, a_2 b_2]$,

$\mathbf{1}/\mathbf{a} = [1/a_1, 1/a_2]$,

$(\mathbf{a} \leq \mathbf{b}) \Leftrightarrow (a_i \leq b_i, \forall i)$,

$(\mathbf{a} < \mathbf{b}) \Leftrightarrow (a_i \leq b_i$, and $a_i < b_i$ some $i)$.

Let $\mathbf{r} = [r_1, r_2] \in N^2, \mathbf{i} \in N^2$ then we denote:

$|\mathbf{r}| = r_1 + r_2$,

$\binom{\mathbf{r}}{\mathbf{i}} = \mathbf{r}!/(\mathbf{r} - \mathbf{i})!\mathbf{i}!$,

$\mathbf{r}! = r_1! r_2!$,

$\mathbf{a}^{\mathbf{r}} = a_1^{r_1} a_2^{r_2}$.

DEFINITIONS:

1. Let $A \subset R^2$. We say that A is regular if for some nonnegative α_1, α_2, $\alpha_i \mathbf{e}_i \in A$, $i = 1, 2$ ($\mathbf{e}_i$ unit vector) and if $\mathbf{a} \in A$, then there is no $\mathbf{b} \in A$ such that $\mathbf{a} < \mathbf{b}$.

2. Let:

$$Q = \{\mathbf{x} : \mathbf{x} = (x_1, x_2), x_i \in R, i = 1, 2; |x_i| \leq 1\},$$

$\mathbf{P} = (p_1, p_2)$ and $1 \leq p_1, p_2 < \infty$.

We denote $L_{\mathbf{P}}(Q) = \{f : f$ is a measurable function on Q with $||f||_{\mathbf{P}} < \infty\}$, where:

$$||f||_{\mathbf{P}} = \{\int_{-1}^{1}[\int_{-1}^{1}|f(\mathbf{x})|^{p_1}dx_1]^{p_2/p_1}dx_2\}^{1/p_2}.$$

3. For $\mathbf{r} = [r_1, r_2] \in Z_+^2$ we define the partial derivatives $D^{\mathbf{r}} = D_{x_1}^{r_1}D_{x_2}^{r_2}$, where D_{x_i} stands for the derivative in the ith variable.

4. Let A be a regular set, we denote:

$$L_{\mathbf{P}}^{A}(Q) = \{f \in L_{\mathbf{P}}(Q) : ||f||_{L_{\mathbf{P}}^{A}(Q)} < \infty\}$$

where $|f|_{L_{\mathbf{P}}^{A}(Q)} = \sum_{\mathbf{r}\in A}|D^{\mathbf{r}}f|_{L_{\mathbf{P}}(Q)}$ and $||f||_{L_{\mathbf{P}}^{A}(Q)} = ||f||_{L_{\mathbf{P}}(Q)} + |f|_{L_{\mathbf{P}}^{A}(Q)}$.

2. The results

Let $f \in L_{\mathbf{P}}(Q)$, and let for $\mathbf{x} \in Q$ be $\mathbf{w}(\mathbf{x}) = [w(x_1), w(x_2)]$, where $w(x_i) = \sqrt{1-x_i^2}$, $i = 1, 2$. For: $\mathbf{r} \in Z_+^2 \setminus \{\mathbf{0}\}$ and $\mathbf{h} = [h_1, h_2] \in R^2$ we denote:

$$\Delta_{\mathbf{h}\otimes\mathbf{w}(\mathbf{x})}^{\mathbf{r}} f(\mathbf{x}) = \sum_{\mathbf{0}\leq\mathbf{i}\leq\mathbf{r}} (-1)^{|\mathbf{r}|-|\mathbf{i}|}\binom{\mathbf{r}}{\mathbf{i}} f(\mathbf{x}+\mathbf{i}\otimes\mathbf{h}\otimes\mathbf{w}(\mathbf{x})).$$

Now, we define the weighted modulus of smoothness:

$$\omega_{A,\mathbf{w}}(f,\mathbf{t})_{\mathbf{P}} = \sum_{\mathbf{r}\in A}\omega_{\mathbf{w}}^{\mathbf{r}}(f,\mathbf{t})_{\mathbf{P}},$$

where A is a set of multi-indices and

$$\omega_{\mathbf{w}}^{\mathbf{r}}(f,\mathbf{t})_{\mathbf{P}} = \sup_{\mathbf{0}<\mathbf{h}\leq\mathbf{t}} \|\ \Delta_{\mathbf{h}\otimes\mathbf{w}}^{\mathbf{r}} f\ \|_{\mathbf{P}},$$

$\mathbf{0} < \mathbf{t} = [t_1, t_2] \in R^2$.

Next, we define the weighted K-funkcjonal as follows:

$$K_{A,\mathbf{w}}(f,\mathbf{t})_{\mathbf{P}} = \inf_{g\in L_{\mathbf{P}}^{A}(Q)}(\|\ f-g\ \|_{\mathbf{P}} + \sum_{\mathbf{r}\in A}\mathbf{t}^{\mathbf{r}}\ \|\ \mathbf{w}^{\mathbf{r}}D^{\mathbf{r}}g\ \|_{\mathbf{P}}).$$

The following theorem was proved in [1]:

Theorem 1. *Let* $f \in L_{\mathbf{P}}(Q)$, $A = \{[r_1, 0], [0, r_2]\}$, $\mathbf{r} = [r_1, r_2] \in N^2$, *then:*

$$M^{-1}\omega_{A,\mathbf{w}}(f,\mathbf{t})_{\mathbf{P}} \le K_{A,\mathbf{w}}(f,\mathbf{t})_{\mathbf{P}} \le M\omega_{A,\mathbf{w}}(f,\mathbf{t})_{\mathbf{P}},$$

for some constants $M > 0$ *and* $\mathbf{0} < \mathbf{t} \le \mathbf{t}_0$, *where* $\mathbf{0} < \mathbf{t}_0 < \mathbf{1}$.

By using this theorem we can prove:

Theorem 2. *Let* $\mathbf{P} = (p_1, p_2)$, $1 \le p_1, p_2 < \infty$, $\mathbf{r} = [r_1, r_2] \in N^2$, *and* $A = \{[r_1, 0], [0, r_2]\}$, *Let* S *be a set of functions in* $L_{\mathbf{P}}(Q)$ *such that for each* $g \in L^A_{\mathbf{P}}(Q)$ *there exists an element* $s_g \in S$ *satisfying:*

$$\| g - s_g \|_{\mathbf{P}} \le C_0 + C_1 \sum_{\mathbf{r}\in A} \mathbf{t}^{\mathbf{r}} \| \mathbf{w}^{\mathbf{r}} D^{\mathbf{r}} g \|_{\mathbf{P}},$$

for some $\mathbf{t} > \mathbf{0}$ *with* C_0, C_1 *constants depending only on* $\mathbf{r}$ *and* $\mathbf{P}$.

Then there exists a constant C_2 *depending only on* $\mathbf{r}$ *and* $\mathbf{P}$ *such that for* $f \in L_{\mathbf{P}}(Q)$, *there exists* $s_f \in S$ *satisfying:*

$$\| f - s_f \|_{\mathbf{P}} \le C_0 + C_2\omega_{A,\mathbf{w}}(f,\mathbf{t})_{\mathbf{P}}.$$

Proof. Let $f \in L_{\mathbf{P}}(Q)$, then for any $g \in L^A_{\mathbf{P}}(Q)$:

$$\| f - s_g \|_{\mathbf{P}} \le \| f - g \|_{\mathbf{P}} + \| g - s_g \|_{\mathbf{P}} \le$$

$$\le \| f - g \|_{\mathbf{P}} + C_0 + C_1 \sum_{\mathbf{r}\in A} \mathbf{t}^{\mathbf{r}} \| \mathbf{w}^{\mathbf{r}} D^{\mathbf{r}} g \|_{\mathbf{P}} \le$$

$$\le C_0 + \max(C_1, 1)(\| f - g \|_{\mathbf{P}} + \sum_{\mathbf{r}\in A} \mathbf{t}^{\mathbf{r}} \| \mathbf{w}^{\mathbf{r}} D^{\mathbf{r}} g \|_{\mathbf{P}}).$$

Since the K-functional is defined as an infimum, if we vary g in $L^A_P(Q)$, then we can find some $g^* \in L^A_{\mathbf{P}}(Q)$ such that:

$$\| f - g^* \|_{\mathbf{P}} + \sum_{\mathbf{r}\in A} \mathbf{t}^{\mathbf{r}} \| \mathbf{w}^{\mathbf{r}} D^{\mathbf{r}} g^* \|_{\mathbf{P}} \le 2K_{A,\mathbf{w}}(f,\mathbf{t})_{\mathbf{P}}.$$

This implies:

$$\| f - s_{g^*} \|_{\mathbf{P}} \le C_0 + 2\max(C_1, 1)K_{A,\mathbf{w}}(f,\mathbf{t})_{\mathbf{P}}.$$

Hence, by Theorem 1. we get:

$$\| f - s_{g^*} \|_{\mathbf{P}} \leq C_0 + C_2 \omega_{A,\mathbf{w}}(f, \mathbf{t})_{\mathbf{P}}. \blacksquare$$

Similarly, we can prove:

Theorem 3. *Let* $\mathbf{P} = (p_1, p_2)$, $1 \leq p_1, p_2 < \infty$, $\mathbf{r} = [r_1, r_2] \in N^2$, *and* $A = \{[r_1, 0], [0, r_2]\}$.

Let L *be a bounded linear operator mapping* $L_{\mathbf{P}}(Q)$ *into itself.*

Suppose that for $\mathbf{t} > \mathbf{0}$ *and all* $g \in L^A_{\mathbf{P}}(Q)$:

$$\| g - Lg \|_{\mathbf{P}} \leq C_0 + C_1 \sum_{\mathbf{r} \in A} \mathbf{t}^{\mathbf{r}} \| \mathbf{w}^{\mathbf{r}} D^{\mathbf{r}} g \|_{\mathbf{P}},$$

where C_0, C_1 *depending only on* $\mathbf{r}$ *and* $\mathbf{P}$.

Then there exists a constant C_2 *depending on* $\mathbf{r}$, $\mathbf{P}$ *and* $||L||$ *such that for all* $f \in L_{\mathbf{P}}(Q)$:

$$\| f - Lf \|_{\mathbf{P}} \leq C_0 + C_2 \omega_{A,\mathbf{w}}(f, \mathbf{t})_{\mathbf{P}}.$$

Proof. For all $f \in L_{\mathbf{P}}(Q)$ and $g \in L^A_{\mathbf{P}}(Q)$ by the definition of the K-functional we have:

$$\| f - Lf \|_{\mathbf{P}} \leq \| f - g \|_{\mathbf{P}} + \| g - Lg \|_{\mathbf{P}} + \| Lf - Lg \|_{\mathbf{P}} \leq$$

$$\leq C_0 + \max(C_1, 1)(\| L \| + 1)(\| f - g \|_{\mathbf{P}} + \sum_{\mathbf{r} \in A} \mathbf{t}^{\mathbf{r}} \| \mathbf{w}^{\mathbf{r}} D^{\mathbf{r}} g \|_{\mathbf{P}} \leq$$

$$\leq C_0 + 2\max(C_1, 1)(\| L \| + 1) K_{A,\mathbf{w}}(f, \mathbf{t})_{\mathbf{P}} \leq$$

$$\leq C_0 + C K_{A,\mathbf{w}}(f, \mathbf{t})_{\mathbf{P}} \leq C_0 + C_2 \omega_{A,\mathbf{w}}(f, \mathbf{t})_{\mathbf{P}},$$

which completes the proof. $\blacksquare$

References

1. G. Kozłowska, *A relationship between the K-functional and the modulu of smoothness in $L_{\mathbf{P}}(Q)$,* Commentationes Math. **35** (1995), 171-182.

2. L. L. Schumaker, *Spline functions: Basic Theory,* Wiley, New York 1981

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Streszczenie

Niech $f \in L_{\mathbf{P}}(Q)$, gdzie:

$$Q = \{\mathbf{x} : \mathbf{x} = (x_1, x_2), x_i \in R, i = 1, 2; |x_i| \leq 1\},$$

$\mathbf{P} = (p_1, p_2)$ and $1 \leq p_1, p_2 < \infty$.

Zdefiniujmy jak w [1] wagowy moduł gładkości:

$$\omega_{A,\mathbf{w}}(f, \mathbf{t})_{\mathbf{P}} = \sum_{\mathbf{r} \in A} \omega_{\mathbf{w}}^{\mathbf{r}}(f, \mathbf{t})_{\mathbf{P}},$$

W tej pracy dowodzimy następujących dwu twierdzeń:

Twierdzenie 2. *Niech S będzie zbiorem funkcji z $L_{\mathbf{P}}(Q)$ takim, że dla każdego $g \in L_{\mathbf{P}}^{A}(Q)$ istnieje $s_g \in S$ dla którego:*

$$\| g - s_g \|_{\mathbf{P}} \leq C_0 + C_1 \sum_{\mathbf{r} \in A} \mathbf{t}^{\mathbf{r}} \| \mathbf{w}^{\mathbf{r}} D^{\mathbf{r}} g \|_{\mathbf{P}},$$

gdzie $\mathbf{t} > \mathbf{0}$, *oraz* C_0 *i* C_1 *są stałymi zależnymi od* $\mathbf{r}$ *i* $\mathbf{P}$. *Wtedy dla każdego* $f \in L_{\mathbf{P}}(Q)$, *istnieje* $s_f \in S$, *takie że:*

$$\| f - s_f \|_{\mathbf{P}} \leq C_0 + C_2 \omega_{A,\mathbf{w}}(f, \mathbf{t})_{\mathbf{P}},$$

gdzie: C_2 *jest stałą zależną od* $\mathbf{r}$ *and* $\mathbf{P}$.

Twierdzenie 3. *Niech* L *będzie ograniczonym operatorem liniowym odwzorowującym przestrzeń* $L_{\mathbf{P}}(Q)$ *w siebie. Załóżmy, że dla każdego* $g \in L_{\mathbf{P}}^{A}(Q)$ *i* $\mathbf{t} > \mathbf{0}$:

$$\| g - Lg \|_{\mathbf{P}} \leq C_0 + C_1 \sum_{\mathbf{r} \in A} \mathbf{t}^{\mathbf{r}} \| \mathbf{w}^{\mathbf{r}} D^{\mathbf{r}} g \|_{\mathbf{P}},$$

gdzie C_0 *i* C_1 *zależą od* $\mathbf{r}$ *i* $\mathbf{P}$. *Wtedy dla każdego* $f \in L_{\mathbf{P}}(Q)$:

$$\| f - Lf \|_{\mathbf{P}} \leq C_0 + C_2 \omega_{A,\mathbf{w}}(f, \mathbf{t})_{\mathbf{P}},$$

gdzie C_2 *jest stałą zależną od* $\mathbf{r}$, $\mathbf{P}$ *i* $||L||$.