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ON CANCELLATIVE CONGRUENCES FOR SEMIGROUPS*

Summary. Given a semigroup identity $u = v$. We describe a smallest normal subgroup N in a free group F, such that F/N contains a relatively free cancellative semigroup which satisfies the identity $u = v$.

O KONGRUENCJACH SKRACALNYCH DLA PÓŁGRUP

Streszczenie. Niech $u = v$ oznacza tożsamość półgrupową. Rozpatrujemy najmniejszą podgrupę normalną N w grupie wolnej F, taką że F/N zawiera relatywnie wolną półgrupę skracalną, spełniającą tożsamość $u = v$.

Let F be a free group and $\mathcal{F}$ be a free semigroup ($\mathcal{F} \ni 1$), both generated by the same set $M = \{x, y, z, \ldots\}$. A **semigroup identity** of a group G (or a semigroup S) is a nontrivial identity of the form $u = v$ where $u, v \in \mathcal{F}$, which holds under every substitution of generators by elements from G (elements from S).

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Research concerning semigroup identities was initiated by A. I. Mal'cev in early fifties and is still continued (see e.g. [6, 7, 10, 11, 13–15, 18]).

One of the open questions concerning semigroup identities in a group G and a subsemigroup S generating G is whether an identity can hold in S without holding in G [1]. Partial answers to this question are given in [2,5,9]. We give here some technical result which relates a cancellative congruence in $\mathcal{F}$, providing an identity in a quotient semigroup, and a normal subgroup it defines in F.

A congruence in $\mathcal{F}$, providing the identity $u = v$ has to contain (u, v) and the set which is the smallest invariant (under endomorphisms of $\mathcal{F}$) reflexive, and symmetric closure of (u, v). We denote this set by $(u, v)^{irs}$. To extend $(u, v)^{irs}$ to a transitive relation we define a step.

Two words $a, b \in \mathcal{F}$ are called **connected by** $(u, v)^{irs}$**-step** if $a = c_1 s c_2$, $b = c_1 t c_2$, and $(s, t) \in (u, v)^{irs}$.

It is clear that if $(s, t) \in (u, v)^{irs}$, then $(a, b) \in (u, v)^{irs}$. However if $a = c_1 s c_2 \to c_1 t c_2 = d_1 s' d_2 \to d_1 t' d_2 = b$, where each arrow denotes $(u, v)^{irs}$-step, then it is not necessary that $(a, b) \in (u, v)^{irs}$.

A congruence in $\mathcal{F}$, providing the identity $u = v$ in the quotient semigroup without necessity to be cancellative is described in [4]. Namely, two words are congruent if and only if they are connected by a finite sequence of $(u, v)^{irs}$-steps.

In [3] the smallest cancellative congruence containing a given relation is described as an infinite sum of relations.

In [8] there is given the following convenient description (conjectured by J. Krempa) of the smallest cancellative congruence $(u, v)^{\#}$ providing the semigroup identity $u = v$ in the quotient semigroup.

Theorem 1. *Two words a and b are $(u, v)^{\#}$-congruent if and only if for some $p, q \in \mathcal{F} \cup \emptyset$, the words paq and pbq are connected by a finite sequence of $(u, v)^{irs}$-steps.*

We consider a normal subgroup $N \subseteq F$ such that the natural map $F \to F/N$ restricted to the free semigroup $\mathcal{F}$ gives $\mathcal{F} \to \mathcal{F}/(u,v)^{\#}$. By using Theorem 1 we can describe N as follows.

Theorem 2. *The smallest normal subgroup N, such that the natural map $F \to F/N$ restricted to $\mathcal{F}$ gives $\mathcal{F} \to \mathcal{F}/(u,v)^{\#}$, is normally generated by images of $u^{-1}v$ under all endomorphisms which map the generating set M into $\mathcal{F}$.*

Proof. We denote:

$$\mathcal{N} = \{a^{-1}b \mid (a,b) \in (u,v)^{\#}\}.$$

By ([3], 12.8), $\mathcal{F}/(u,v)^{\#}$ can be embedded into a group F/N, where N is normally generated by the set $\mathcal{N}$.

Let N_1 denote the normal subgroup generated by images of $u^{-1}v$ under all endomorphisms which map the generating set M into $\mathcal{F}$. clearly $N_1 \subseteq N$. To show the opposite we note first that N_1 contains all $a^{-1}b$ for $(a,b) \in (u,v)^{irs}$.

Now, let $(a,b) \in (u,v)^{\#}$, then by Theorem 1 for some $p,q \in \mathcal{F}$ the words paq and pbq are connected by a finite number of $(u,v)^{irs}$–steps. Such a step changes a word of the form su_1t into the word sv_1t where $(u_1,v_1) \in (u,v)^{irs}$. We note here that $(su_1t)(t^{-1}\,u_1^{-1}v_1\,t) = sv_1t$, that is the step means a multiplication by a conjugate $(u_1^{-1}v_1)^t \in N_1$. So we get $paq\,\xi = pbq$, where ξ is a product of conjugates in N_1. This gives $\xi = (a^{-1}b)^q \in N_1$ and hence $a^{-1}b \in N_1$, which implies $N \subseteq N_1$ as required.∎

It follows from the last sentence in ([3], section 12.3) that according to [12] the subgroup generated by the set $\mathcal{N}$ is always normal. Following [12] we prove now that it is not true.

Theorem 3. *If the subgroup generated by the set $\mathcal{N}$ is normal (equal to N) then the semigroup $\mathcal{F}/(u,v)^{\#}$ is a group of finite exponent.*

Proof. Let us assume that $gp(\mathcal{N}) = N$. The word $u^{-1}v$ is in $\mathcal{N}$, however its conjugate vu^{-1} is not in $\mathcal{N}$. By assumption vu^{-1} can be obtained as a product of elements in $\mathcal{N}$. We show that this is impossible unless $\mathcal{F}/(u,v)^{\#}$ satisfies an identity $x^n = 1$.

So we assume that vu^{-1} is in N:

$$vu^{-1} = a_1^{-1}b_1 \cdot a_2^{-1}b_2 \dots a_k^{-1}b_k,$$

is an equality in F, where each small letter is a word in $\mathcal{F}$. We find a minimal i such that the word $W_i := a_1^{-1}b_1 \cdot a_2^{-1}b_2 \dots a_i^{-1}b_i$ has a reduced form vT_1, but $W_{i-1} := a_1^{-1}b_1 \cdot a_2^{-1}b_2 \dots a_{i-1}^{-1}b_{i-1}$ does not have a reduced form vT_2. Then $W_{i-1} = W_i(a_i^{-1}b_i)^{-1} = vT_1b_i^{-1}a_i$. Since W_{i-1} does not begin with v we conclude that T_1 cancels with b_i^{-1} and hence $T_1 \in \mathcal{F}$. This implies that the word $c := vT_1$ is in $\mathcal{F}$ and in $gp(\mathcal{N}) = N$ simultaneously. So $(c, 1)$ is in $(u,v)^{\#}$ and hence $(x^n, 1)$ is in $(u,v)^{\#}$ for n equal to the length of c. This implies that $\mathcal{F}/(u,v)^{\#}$ is a group of finite exponent. ∎

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Streszczenie

Niech F oznacza grupę wolną, a $\mathcal{F}$ półgrupę wolną – obydwie generowane przez zbiór $M = \{x, y, z, \ldots\}$. Tożsamością półgrupową w grupie G nazywa się tożsamość typu $u = v$, gdzie u, v są a elementami $\mathcal{F}$. Jest to zależność, która jest spełniona w G przy podstawieniu dowolnych elementów z G zamiast $x, y, z, \ldots$. Przy każdym homomorphizmie naturalnym $F \to F/N$ półgrupa $\mathcal{F} \subseteq F$ jest odwzorowywana w półgrupę $S = \mathcal{F}/\rho$, gdzie ρ oznacza kongruencję zależną od N. Praca dotyczy określenia minimalnego dzielnika normalnego N, takiego że S jest relatywnie wolną półgrupą skracalną z daną tożsamością $u = v$. Pokazano, że N jest generowany jako dzielnik normalny obrazami słowa $u^{-1}v$ przy wszystkich endomorphizmach dodatnich, czyli odwzorowujących zbiór M w półgrupę $\mathcal{F}$. Pokazano również, że ostatnie zdanie w ([3], rozdział 12.3) nie jest prawdziwe.