ZESZYTY NAUKOWE POLITECHNIKI ŚLĄSKIEJ 1999

Seria: MATEMATYKA-FIZYKA z. 84 Nr kol. 1411

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UNIQUE PRODUCT ELEMENTS*

Summary. Any group G that can be given a right order will be a unique product group. What is not known is, if G is a unique product group, can G be given a right order. The purpose of this paper is to study the unique product groups in order to gain more information about the structure of the group and, also, to see what the additional conditions can be added so as to have a right order. Notation used: Let G be a group and let A and B be its subsets. $AB = \{ab \mid a \in A,\ b \in B\}$, $A = \{a \mid a \in A\}$. For singleton set $\{g\} = B$ we write Ag instead $A\{g\}$. For the empty set $\emptyset$ we define $A\emptyset = \emptyset$. Let $F(G)$ denote the semigroup of all finite nonempty subsets of G with multiplication defined above. $|A|$ denotes the cardinality of the set A.

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Streszczenie. Z każdą grupą G można związać półgrupę $F(G)$ złożoną ze skończonych podzbiorów G, z działaniami danymi wzorami: $A \cdot B = \{a \cdot b\ ;\ a \in A,\ b \in B\}$. W półgrupie $F(G)$ badamy najmniejszą przechodnią relację wyznaczoną przez warunek: $A \sim B$, gdy $A{\cdot}C = B{\cdot}C$ dla pewnego $C \in F(G)$. Pokazujemy, że relacja ta jest kongruencją w $F(G)$. Niestety, kongruencja ta nie zawsze jest skracalna.

*Supported by KBN grant No 2 1115 91 01.

W podzbiorach $A \in F(G)$ wyróżniamy element a **upel** spełniający warunek: $A \cdot B \neq (A \backslash \{a\}) \cdot B$ dla każdego $B \in F(G)$. Badamy własności algebraiczne grup **upel**, w których każdy podzbiór zawiera upel element. Klasa takich grup leży pomiędzy grupami uporządkowanymi a **u.p. grupami**.

1. Upel elements

Recall that a group G is said to be a unique product group (u.p.-group) if, given any $A, B \in F(G)$, there exists at least one element $x \in G$ that has a unique representation in the form $x = ab$ with $a \in A$ and $b \in B$ i.e. there exists $a \in A$ such that $AB \neq (A \backslash \{a\})B$.

Definition 1.1. *Given a non-empty subset A of G, a* **unique product element** *for A (* **upel** *) is an element x of A with the property: for any $C \in F(G)$, $(A \backslash \{x\})C \neq AC$. A group G is said to be* **upel group** *if each non-empty finite subset contains an upel element.*

It is clear that for each group G its element x is an upel for $\{x\}$. The condition that C is finite in the definition of upel group appears since if H is a nonidentity subgroup of G then H contains no upel elements.

Theorem 1.2. *Let A be a subset of the group G, $x \in A$ and $B = A \backslash \{x\}$. Then the following conditions are equivalent:*

i) The element x is not an upel for A;

ii) The semigroup generated by $x^{-1}B$ contains e;

iii) The semigroup generated by Bx^{-1} contains e;

iv) The element gx is not upel for gA, for each $g \in G$;

v) There exists $g \in G$, such that gx is not an upel for gA;

vi) If D is a subset of G containing A, then x is not an upel for D;

vii) The element gx is not upel for DA, for each $g \in D \in F(G)$.

Proof. *i)* $\Rightarrow$ *ii)* Suppose for x to be a non-upel for A. By definition $AC = BC$ for some $C \in F(G)$. Choose $c_1 \in C$. Then let $c_2 \in C$ and $b_2 \in B$ be such that $xc_1 = b_2c_2$. Note that $c_1 \neq c_2$ and $c_1 = x^{-1}b_2c_2$. Let $c_3 \in C$ and $b_3 \in B$ be such that $xc_2 = b_3c_3$, $c_2 \neq c_3$ and $c_2 = x^{-1}b_3c_3$. Continue this process to obtain $xc_i = b_{i+1}c_{i+1}$, $c_i \neq c_{i+1}$ and $c_i = x^{-1}b_{i+1}c_{i+1}$. Since C is finite, at some point $c_i = c_j$ where $i < j$. Since $c_k \neq c_{k+1}$ at each stage the number of steps must be at least two. We start the process over with c_i and continue to c_j (i is where the repetition occurred with the following j).

$$\begin{array}{llll}
xc_i = b_{i+1}c_{i+1}, & c_i \neq c_{i+1} & \text{and} & c_i = x^{-1}b_{i+1}c_{i+1}, \\
xc_{i+1} = b_{i+2}c_{i+2}, & c_{i+1} \neq c_{i+2} & \text{and} & c_i = x^{-1}b_{i+1}x^{-1}b_{i+2}c_{i+2}, \\
xc_{j-1} = b_jc_j, & c_{j-1} \neq c_j & \text{and} & c_i = (x^{-1}b_{i+1}x^{-1}b_{i+2}\ldots x^{-1}b_j)c_j.
\end{array}$$

Therefore $e = x^{-1}b_{i+1}x^{-1}b_{i+2}\ldots x^{-1}b_j$ is the nontrivial word with the minimal length expression being $x^{-1}b_{i+1}x^{-1}b_{i+2}$.

ii) $\Rightarrow$ *iii)* Suppose that the semigroup generated by $x^{-1}B$ contains e. Then $e = x^{-1}b_1x^{-1}b_2\ldots x^{-1}b_j$ for some $b_k \in B$. Now $e = b_1x^{-1}b_2\ldots x^{-1}b_jx^{-1}$ is an element of the semigroup generated by Bx^{-1}. Notice the length of the expression is such that $j > 2$.

The proof of implication *iii)* $\Rightarrow$ *ii)* is similar to the above.

ii) $\Rightarrow$ *i)* Suppose that the semigroup generated by $x^{-1}B$ contains e. Then $e = x^{-1}b_1x^{-1}b_2\ldots x^{-1}b_j$ for some $b_k \in B$. Let:

$$C = \{e = x^{-1}b_1x^{-1}b_2\ldots x^{-1}b_j,\ x^{-1}b_2\ldots x^{-1}b_j, \ldots,\ x^{-1}b_j\}.$$

Then $xC \subseteq BC$, so $AC = BC$ and hence x is no upel for A.

i) $\Rightarrow$ *iv)* Let $g \in G$. If C is a set with $AC = BC$ then:

$$gAC = gBC = g(A \setminus \{x\})C = (gA \setminus \{gx\})C.$$

Hence gx is not upel for gA.

Suppose the semigroup generated by $x^{-1}B$ contains e. Since sets $x^{-1}B$ and $(gx)^{-1}gB$ are equal and by *i)* ga is not an upel for gA.

The implication *iv)* $\Rightarrow$ *v)* is clear and the proof of *v)* $\Rightarrow$ *i)* is similar to the above.

ii) $\Rightarrow$ *vi)* If e is an element of the semigroup generated by $x^{-1}B$ so it is an element of the semigroup generated by $x^{-1}(D \setminus \{x\})$. Hence x is not an upel for D.

The implication *vi)* $\Rightarrow$ *i)* is clear.

iv) & *vi)* $\Rightarrow$ *vii)* Let $g \in D \in F(G)$. By *iv)* the element gx is not upel for gA. Since $gA \subseteq DA$ *so* by *vi)* the element gx is not upel for DA.

The implication *vii)* $\Rightarrow$ *i)* is clear.$\blacksquare$

The above theorem immediately yields the following corollaries:

Corollary 1.3. *The upel condition is left right symmetric. Hence for example x is non upel for A if and only if there exists $C \in F(G)$ such that $CA = C(A \setminus \{a\})$.*

Corollary 1.4. *A group G is torsion-free if and only if each its subset of cardinality 2 contains upel element.*

As we will show in Proposition 4.3 there exists a torsion-free group and its subset of cardinality 3 without upel elements.

Now we will describe some group-theoretic properties of the class of upel groups.

Proposition 1.5. *The class of upel groups contains the class of right ordered groups and is contained in the class of u.p. groups.*

Proof. Let A be a nonempty finite subset of a right ordered group G. Then by the proof of Lemma 13.1.7 from [5] the maximal and minimal element of A are upels for A. It is clear that upel groups are u.p.-groups.$\blacksquare$

Lemma 1.6. *Let G be a group. Then the following conditions are equivalent:*

i) G is an upel group;

ii) Every finitely generated subgroup of the group G is an upel group;

iii) Every nonidentity finitely generated subgroup of the group G can be mapped homomorphicaly onto a nonidentity upel group.

Proof. The implications $i) \Rightarrow ii) \Rightarrow iii)$ are obvious.

$iii) \Rightarrow i)$ Let A be a finite subset of G. Proof by induction on cardinality of A.

a) $|A| = 1$ then x is the upel element of $\{x\}$.

b) Assume that every subset of cardinality less then n contains an upel element. Let $|A| = n$. Since A contains an upel element iff gA contains an upel element for $g \in G$, so we can assume $e \in A$. Let $H = < A >$ be a subgroup generated by A and let $\phi : H \to U$ be a homomorphism onto a non-trivial upel group U. Let $x \in A$ be such that $\phi(x)$ is upel for $\phi(A)$. Set $B = \{a \in A; \phi(a) = \phi(x)\}$. Since $|B| < |A|$, so B contains an upel element b. Since e and $b \in A$ so the group H is generated by the set $b^{-1}A$. Hence without loss of generality we can assume that $b = e$. Now $\phi(b) = e \in U$ is an upel element of $\phi(A)$. We will show that e is an upel for A. Suppose $e = a_1 a_2 \ldots a_t$ for some

$a_i \in A$. Then $e = \phi(a_1)\phi(a_2)\dots\phi(a_t)$. Since $e = \phi(b)$ is an upel for $\phi(A)$ so by Theorem 1.2.ii, $\phi(a_i) = e$, for all i. Now $a_i \in B$ and e is an upel for B so $a_i = e$, for all i. Hence by Theorem 1.2.ii, e is an upel for A.$\blacksquare$

The following theorem shows us that properties of the class of upel groups are similar to properties of classes of right ordered groups and of classes of u.p. groups. (cf. [5]).

Theorem 1.7. *The class of upel group is closed under cxtensions, direct products and coproducts.*

Proof. Let H be a normal subgroup of a group G such that H and G/H are upel groups. Let S be a subgroup of G. If $S \not\subseteq H$ then $S/S \cap H \simeq (S \cdot H)/H$ is nontrivial upel group. If $S \subseteq H$ then S is an upel group. Therefore by Lemma 1.6, G is an upel group.

ii) If H is a nontrivial subgroup of direct product of groups then H can be mapped homomorphicaly onto nontrivial subgroup of some factor. Hence Lemma 1.6 yields the result.

iii) Coproducts. Let us first consider the case of two factors $G * H$. Let $\phi : G * H \to G \times H$ be the homomorphism given by $\phi(g) = G$ for g$\in G$ and $\phi(h) = h$ for $h \in H$. According to ([3] Theorem 3 in Appendix) $\ker\phi$ is a free group. Since free groups are ordered so they are upel. By i) and ii) $G * H$ is an upel group. By induction this result holds for direct coproduct of any finite number of groups. But every finitely generated subgroup of $\Pi_i^* G$ is contained in a coproduct of finitely many factors. Thus by Lemma 1.6 the result is proved.$\blacksquare$

Let G be a right-ordered group and let $A \in F(G)$. As we have shown in Proposition 1.5 the maximal and the minimal element of A are upels. In fact we have:

Theorem 1.8. *Let G be an upel group and let $A \in F(G)$. If $|A| > 2$ then A contains at least two upel elements.*

Proof. Suppose A has only one upel element x. Let us consider the element $y^{-1}z \in A^{-1}A = D$. If $z \neq x$ then $y^{-1}z$ is not upel for D by Theorem 1.2.vii. If $y \neq x$ then $y^{-1}z$ is not upel for D since the upel condition is left right symmetric. If A contains the additional element $t \neq z$ then $z^{-1}z = t^{-1}t$ is not upel for D.∎

2. Relations in the semigroup of subsets

We need the following definition to describe some connections between conditions upel and right-ordered.

Definition 2.1. *Let A and B be some subsets of a group G. We set $A \succ B$ if $A = B$ or $B = A \backslash \{x\}$, where x is a non-upel for A. We say that A is above B if there exists a sequence:*

$$A = A_0 \succ A_1 \succ A_2 \succ \ldots \succ A_k = B$$

of subsets A_i of G. Let $\sim$ be a relation on the set of all subsets of G defined by: $A \sim B$ if there exists subset D of G such that D is above A and B.

Example 2.2. Let G be a right ordered group and let $A, B \in F(G)$ be such that $A \sim B$ then maximal and minimal elements of A are maximal and minimal elements of B. It is sufficient to prove it in the case $A \succ B$. But then $B = A \backslash \{x\}$. Since x is non-upel it can't be maximal nor minimal.

Lemma 2.3. *Let A, B and C are non-empty subsets of a group G. Then:*

i) If $A \succ B$ then $A \cup C \succ B \cup C$;

ii) If A is above B then $A \cup C$ is above $B \cup C$;

iii) If $A \sim B$ then $(A \cup C) \sim (B \cup C)$;

iv) If $A \sim B$ and $A \subseteq C \subseteq B$ then $C \sim B$;

v) If A, B and C are finite and $CA = CB$ or $AC = BC$ then $A \sim B$.

Proof. *i*) Suppose $A \succ B$. If $A = B$ then $A \cup C \succ B \cup C$. If $B = A \backslash \{x\}$ then $A \cup C = B \cup C$ or $B \cup C = A \cup C \backslash \{x\}$. Since x is not an upel for A so by Theorem 1.2.vi x is not an upel for $A \cup C$. Hence $A \cup C \succ B \cup C$.

ii) Suppose A is above B. Then there exists a sequence:

$$A = A_0 \succ A_1 \succ A_2 \succ \ldots \succ A_k = B$$

of subsets A of G. By *i*):

$$A \cup C = A_0 \cup C \succ A_1 \cup C \succ A_2 \cup C \succ \ldots \succ A_k \cup C = B \cup C.$$

Hence $A \cup C$ is above $B \cup C$.

iii) Suppose $A \sim B$. Then there exists a subset D of the group G such that D is above A and B. By *ii*) $D \cup C$ is above $A \cup C$ and $B \cup C$.

For *iv*) we use *iii*). Now $C = A \cup C \sim B \cup C = B$, thus, $C \sim B$.

v) If $CA = CB$ for some $C \in F(G)$ then $CA = C(A \cup B) = CB$ so we can assume that $B \subseteq A$. Let $A = B \cup \{a_1, a_2, \ldots a_n\}$. Note that:

$$CB = C\left(B \cup \{a_1\}\right) = C\left(B \cup \{a_1, a_2\}\right) = \ldots =$$

$$= C\left(B \cup \{a_1, a_2, \ldots, a_n\}\right) = CA.$$

Each of the sets differ by one non-upel element from the previous, thus we get:

$$A \succ B \cup \{a_1, a_2, \ldots, a_n\} \succ \ldots \succ B \cup \{a_1\} \succ B.$$

Therefore $A \sim B$. Similarly we can prove that $AC = BC$ implies $A \sim B$.■

Corollary 2.4. *Let A and $B_1, B_2, \ldots, B_n$ are subsets of G such that $A \sim B_i$, for $i < n$. Then there exists M above A and all B_i's.*

Proof. The definition immediately follows case $n = 1$. Suppose $n > 1$. By induction there exist subsets M and N of G, such that M is above A and B_i, for $i < n$, and N is above M and B_n. Hence N is above A and all B_i's.■

It is easy to see that if $A, B \subseteq G$, A $\sim$ B and $A \in F(G)$ implies $B \in F(G)$. Hence we will also use $\sim$ for relation on $F(G)$.

Theorem 2.5. *The relation $\sim$ is the equivalence relation generated by $\succ$ on the set of all non-empty subsets of G. Furthermore $\sim$ is a congruence on the semigroup $F(G)$ with multiplication defined in the introduction.*

Proof. It is clear that $A \sim A$ and that $A \sim B$ implies $B \sim A$. Corollary 2.4 yields transitivity. Now we will show that if A, B and $D \in F(G)$ and $A \sim B$ then $AD \sim BD$. It is sufficient to proof that in the case A is above B. Let $A = A_0 \succ A_1 \succ A_2 \succ \ldots \succ A_k = B$. If it is shown that $A_iD \sim A_{i+1}D$, then $AD \sim BD$ will follow by transitivity. By Corollary 1.3 there exists $Y \in F(G)$ such that $YA_i = YA_{i+1}$. Therefore $YA_iD = YA_{i+1}D$. Thus, by lemma 2.3.v) $A_iD \sim A_{i+1}D$. Similarly, $DA \sim DB$.■

Remark. Any semigroup has a minimal cancellative congruence [1, page 14]. Lemma 2.3.v) yields that $\sim$ is contained in the minimal cancellative congruence p^c on $F(G)$. If G is abelian then the congruence $\sim$ is the same as p^c and is given by: $A \sim B$ if $AC = BC$ for some $C \in F(G)$ (see [7]). We will show in Corollary 4.7, there exists a group Γ such that $\sim$ is not cancellative congruence on $F(\Gamma)$.

3. Ordered groups

Let G be a group with a partial right order. It is well known that the set $S = \{g \in G \mid G \geq e\}$ is a subsemigroup of G, such that $S \cap S^{-1} = \{e\}$. Conversely if S is a subsemigroup of G with property $S \cap S^{-1} = \{e\}$, then we can associate a partial right order on G by, $x \leq y$ if $yx^{-1} \in S$. Such semigroup we will call **ordering semigroup**. The ordering semigroup S is a **right order** if the order associated with S is linear which is equivalent to $S \cup S^{-1} = G$.

Proposition 3.1. *Let S be a subsemigroup of the group G, such that $e \in S$. Then e is an upel of S iff S is ordering semigroup.*

Proof. Suppose s is non-identity element of $S \cap S^{-1}$. Then e is a non-upel for the set $\{e, s, s^{-1}\} \subseteq S$. Hence e is a non-upel for S. If e is a non-upel for S then by Theorem 1.2 $e = s_1 s_2 \ldots s_n$ for some $s_i \in S \backslash \{e\}$. Hence $s_1^{-1} = s_2 \ldots s_n \in S \cap S^{-1}$. ■

Definition 3.2. *Let A be a non-empty subset of a group G. We say that the element x is a* **strong upel** *for A if $a \in A^* = \bigcap \{B \subseteq G \mid B \sim A\}$ i.e. x belongs to the intersection of the equivalence class of A under $\sim$.*

Proposition 3.3. *Let x be an element of a subset A of the group G. If x is a strong upel for A then A is an upel for A.*

Proof. Let a$\in A$, Since $A \succ A\setminus\{x\}$ if and only if x is non-upel for A so each strong upel element is an upel.$_\blacksquare$

Example 3.4. Let A be a subset of a right ordered group $\{G <\}$. Then a minimum and maximum elements of A are strong upels for A.

Example 3.5. Let $G = < x, y \mid y^{-1}xy = x^{-1} >$ and $A = \{e, y^2, x, xy^{-2}\}$. We claim that e is upel for A but e is not strong upel for A. It is easy to see that the group generated by A is commutative. Suppose $e = y^{2m}x^t(xy^{-2})^k$, $m \geq 0, t \geq 0, k \geq 0$. Then $m = k = t = 0$. But since $e = y^{-1}y^2y^{-1}$ so y is not upel for $A \cup \{y\}$ and $A \cup \{y\} \succ A$. Now $e = xyxy^{-2}y$ implies $\{e, y, y^2, x, xy^{-2}\} \succ \{y, y^2, x, xy^{-2}\}$. We obtain $A = \{e, y^2, x, xy^{-2}\} \sim \{y, y^2, x, xy^{-2}\}$. Hence e is not a strong upel for A.

Theorem 3.6. *Let G be a group and let $A \in F(G)$. Then:*

i) If G is torsion free and $|A| = 1$ then the unique element of A is strong upel;

ii) If G is an upel group and $|A| > 2$ then A contains at least two strong upel elements.

Proof. *i*) Suppose $B \in F(G)$ is such that $B \succ A$. Then $B = A$ or $B = \{a, b\} \succ A = \{a\}$ for some non-upel element b. Then Theorem 1.2 implies $e = (b^{-1}a)^n$, for some integer n. Since G is torsion free $a = b$.

ii) Since A is finite there exist sets $B_1, B_2, \ldots, B_n$ related with A such that $A = B_1 \cap B_2 \cap \ldots \cap B_n$. By Corollary 2.4 there exists $M \in F(G)$ above all B_i's. By Theorem 1.8 M contains two upel elements x and y. Now it is sufficient to prove that x and y are upels for each B. Let $M = A_0 \succ A_1 \succ A_2 \succ \ldots \succ A_k = B_j$, for some j. By induction x and y are upels for all A_i and hence for B_j. Therefore $x, y \in A^*$.$_\blacksquare$

Theorem 3.7. *Let G be a torsion free group and S its maximal ordering semigroup. Then the following conditions are equivalent:*

i) S is a right order;

ii) $xy \in S$ implies $x \in S$ or $y \in S$;

iii) $x \notin S$ implies x is an upel for $S \cup \{x\}$;

iv) $x \notin S$ implies x is a strong upel for $S \cup \{x\}$;

v) e is a strong upel for S.

Proof. *i)* $\Rightarrow$ *ii)* Suppose x and y are not in S. Then $x^{-1}, y^{-1} \in S$. We have $(xy)^{-1} = y^{-1}x^{-1} \in S$ which implies $xy \notin S$.

ii) $\Rightarrow$ *iii)* Let x be a non-upel for $S \cup \{x\}$. By Theorem 1.2 there exist $x_1, x_2, \ldots, x_n \in S \setminus \{x\}$ such that $x^{-1}x_1x^{-1}x_2, \ldots, x^{-1}x_n = e$. This equation tells us that e is not an upel for $\{x, x_1, x_2, \ldots, x_n\}$. Since $e \in S$ is a non-upel for S so $x^{-1} \notin S$. Now $xx^{-1} = e \in S$ so by ii) $x \in S$.

iii) $\Rightarrow$ *v)* By iii) if M is a subset of G such that $M \succ S$ then $M = S$. Hence if M is above S then $M = S$. Let $A \sim S$. Then S is above A. Let $S = A_0 \succ A_1 \succ A_2 \succ \ldots \succ A_k = A$. By induction e is an upel for all A_i and hence for A. Therefore e is a strong upel for S.

v) $\Rightarrow$ *i)* Since S is an ordering semigroup for G, we need only to show that $S \cup S^{-1} = G$. Let $b \notin S \cup S^{-1}$. Let $S_1 = S < S \cup \{b\} >$ be the subsemigroup of G generated by S and b. Since S is maximal we have that S_1 is not an ordering semigroup for G. Thus we can write $e = b^{-1}y_1b^{-1}y_2, \ldots, b^{-1}y_n$, where $y_i \in S$. This equation tells us that b is not upel for the set $\{b, y_1, y_2, \ldots, y_n\}$. It yields $S \cup \{b\} \succ S$. Using the same procedure for b^{-1}, we can get $\{z_1, z_2, \ldots, z_t\} \subset S$ such that b^{-1} is not upel for $\{b^{-1}, z_1, z_2, \ldots, z_t\}$. It yields $B = S \cup \{b\} \cup \{b^{-1}\} \succ S \cup \{b\} \succ S$. But $B \succ B \setminus \{e\}$ so $S \sim B \setminus \{e\}$ contrary to v).

i) $\Rightarrow$ *iv*) Suppose $\leq$ is a right order associated with S. Let $x \notin S$. Then $x < e$ and x is a minimum element of $S \cup \{x\}$. Hence x is a strong upel for $S \cup \{x\}$.

The implication *iv*) $\Rightarrow$ *iii*) is obvious.■

Theorem 3.8. *Let G be a group and let P be the positive cone of a partial right-order on G. Then the following condition are equivalent:*

i) The partial right-order P can be extended to a right-order on G;

ii) If $A = \{y_1, y_2, \ldots, y_n\} \in F(G)$, then there exist $\varepsilon = \pm 1$ such that the semigroup generated by $B = \{P, y_1^{\varepsilon_1}, y_2^{\varepsilon_2}, \ldots, y_n^{\varepsilon_n}\}$ is an ordering semigroup;

iii) If $A = \{y_1, y_2, \ldots, y_n\} \in F(G)$, then there exist $\varepsilon = \pm 1$ such that e is an upel for $B = \{P, y_1^{\varepsilon_1}, y_2^{\varepsilon_2}, \ldots, y_n^{\varepsilon_n}\}$;

iv) If $A = \{y_1, y_2, \ldots, y_n\} \in F(G)$, then there exist $\varepsilon = \pm 1$ such that e is a strong upel for $B = \{P, y_1^{\varepsilon_1}, y_2^{\varepsilon_2}, \ldots, y_n^{\varepsilon_n}\}$.

Proof. *i*) $\Rightarrow$ *iv*) Let P be the semigroup of non-negative elements of G. Let us choose ε such that $x^{\varepsilon} \in P$. Then e is a minimum element of:

$$B = \{P, y_1^{\varepsilon_1}, y_2^{\varepsilon_2}, \ldots, y_n^{\varepsilon_n}\}.$$

Hence e is a strong upel for B.

The implication *iv*) $\Rightarrow$ *iii*) is obvious.

Proposition 3.1 immediately yields implication *iii*) $\Rightarrow$ *ii*).

The conditions *ii*) and *i*) can be seen to be equivalent by realizing that they are restatements of the conditions given in [4] Theorem 7.6.1.■

Theorem 3.9. *Let G be a torsion-free group. Then the following conditions are equivalent:*

i) For each $A \in F(G)$ every upel for A is a strong upel for A;

ii) For each $A \in F(G)$ every upel for A is a maximal element in some right order on G;

iii) Every partial order in G can be extended to a right order.

Proof. The implication $ii) \Rightarrow i)$ is obvious.

$i) \Rightarrow iii)$ Suppose by contradiction there exists a maximal ordering semigroup S which is not ordering semigroup. Then $S \cap S^{-1} = \{e\}$ and there exists $b \notin S \cup S^{-1}$ such that the semigroups T generated by $S \cup \{b\} \setminus \{e\}$ and generated by $S \cup \{b^{-1}\} \setminus \{e\}$ contain e Hence there exists a finite subset A of S such that $A \cup \{b\} \sim A$ and $A \cup \{b^{-1}\} \sim A$. Let M be above $A \cup \{b\}$ and $A \cup \{b^{-1}\}$. Now $M \sim A$ and e is not upel for M. So e is an upel for A and it is an not strong upel for A.

$iii) \Rightarrow ii)$ Suppose that the group G satisfies condition $iii)$ and let x be an upel element of the subset A of G. By Theorem 1.2 the subsemigroup generated by Ax^{-1} is an ordering semigroup. We extend it to the right order P and we define this order by:

$$x \leq y \Longleftrightarrow xy^{-1} \in P.$$

Let $b \in A$, then $bx^{-1} \in P$ so $b \leq x$. Hence x is the maximal element of A. ∎

The class of groups satisfying condition $iv)$ is investigated in [4] chapter 7.6. For example Rhemtulla ([4] Corollary 7.6.5) has shown that this class contains all torsion-free locally nilpotent groups.

4. Examples

In this section we illustrate theory of unique product on the example of the well known group:

$$\Gamma = < x, y \mid x^{-1}y^2x = y^{-2},\ y^{-1}x^2y = x^{-2} > .$$

Lemma 4.1. *([5] Lemma 13.3.3) Let H be a subgroup of Γ generated by $\{x^2, y^2, (xy)^2\}$. Then H is normal free abelian subgroup of Γ of rank 3 with Γ/H the Klein four group. Furthermore Γ is a torsion free group but not right orderable.*

Promislow has found a subset S of Γ of cardinality 14 such that all multiplicities in SS are larger then 1. It shows that G is not u.p. group (see [6]).

Lemma 4.2. *Let $\{e, a, b, c\}$ be a coset representatives of H in Γ. Then:*

$$a^{-1}b^2a = b^{-2}.$$

Proof. Let $b = xh$, $h \in H$. Then $b^2 = x^{2i}$ for some integer i. Similarly $(yh)^2 = y^{2j}$ and $(xyh)^2 = (xy)^{2k}$. Since all cases are similar we will give the proof only for the case $a = xyg$, $g \in H$, and $b = xh$. Now $a^{-1}b^2a = g^{-1}y^{-1}x^{-1}x^{2i}xyg = x^{-2i} = b^{-2}$.■

Proposition 4.3. *The group Γ is not upel. Furthermore the set $\{e, x, y\}$ contains no upel element.*

Proof. First we show that y is not upel for $\{e, x, y\}$. By Theorem 1.2 it is sufficient to show that the subsemigroup generated by y and $y^{-1}x$ contains e. But:

$$y^{-1}y^{-1}xy^{-1}y^{-1}xy^{-1}xy^{-1}y^{-1}xy^{-1} = x^2yxy^{-2}xy = x^2yx^2y^{-1} = e.$$

Similary x is non-upel since x and y play symmetric role. Furthermore e is non-upel since $e = xyyxyxxy$.∎

For describing properties of $\sim$ in Γ we will use the transfer map. Pick coset representatives $\{x_1, x_2, x_3, x_4\}$ for H in Γ. If g is in Γ, we construct $4 * 4$ monomial matrix by placing $x_i g x_j^{-1}$ in the i^{th} row and j^{th} column if that element is in H and by placing 0 there otherwise. This induces an injective homomorphism from Γ to the group of matrices with precisely one nonzero entry, an element of H, in each row and column. The transfer map $T : G \to H$ is gotten by taking the product of the nonzero entries of the „monomial" representation. T is homomorphism and is independent of coset representatives (see [2]).

Lemma 4.4. *(cf [2] Theorem 23) Let $T : \Gamma \to H$ be the transfer map. Then $T(G) = \{e\}$.*

Proof. We will show that generators of Γ belong to the kernel of the transfer map. Indeed using for coset representatives $\{e, x, y, xy\}$ we obtain:

$$T(x) = exx^{-1} \cdot xxe \cdot yx(xy)^{-1} \cdot (xy)xy^{-1} =$$
$$= xxyxy^{-1}x^{-1}xyxy^{-1} = x^2yx^2y^{-1} = e$$

and:

$$T(y) = eyy^{-1} \cdot xy(xy)^{-1} \cdot yye \cdot (xy)yx^{-1} = e.$$∎

Lemma 4.5. *Let $A = \{e, x, y, xy\}$ be coset representatives for H in Γ. Let $B = AA^{-1}$. If $D \in F(G)$, is such that $B \subset D$ then $B \sim D$.*

Proof. Let $D = B \cup \{d_1, d_2, \ldots, d_n\}$. By Lemma 4.4, for each $d \in D$ the element $T(d^{-1}) = e$ belongs to the semigroup generated by B and d. According to Theorem 1.2 we get $B \sim B \cup \{d\}$. Continuing this process we obtain:

$$B \sim B \cup \{d_1\} \sim B \cup \{d_1, d_2\} \sim \ldots \sim B \cup \{d_1, d_2, \ldots, d_n\} = D.$$∎

Theorem 4.6. *The relation* $\sim$ *in the semigroup* $F(\Gamma)$ *has the following description:* $A \sim D$ *if and only if* $|A| \geq 2$ *and* $|D| \geq 2$ *or* $|A| = 1$ *and* $A = D$.

Proof. Since G is torsion free prove of the case $|A| = 1$ is the same as in Theorem 2.5. Let $|A| \geq 2$. It is sufficient to prove that $A \sim B$, where B is the set from Lemma 4.5.

Step 1: $A = \{e, a\}$, $a \notin H$. Analysis similar to that in Lemma 4.2 and according to Lemma 4.5 we get $A \sim \{e, a, b\}$ for some $b \notin (H \cup Ha)$. Continuing this process we obtain $A \sim \{e, a, b\} \sim \{e, a, b, c\}$, where $\{e, a, b, c\}$ is a coset representatives of H in Γ. Now after few steps we get $A \sim B$.

Step 2: $A = \{e, h\}$, $h = x^{2t}y^{2k}(xy)^{2i} \in H$ and suppose $t > 0$. Then equality $e = (x^{-1}h)^2 x^{2-4t}$ implies $A \sim \{e, h, x\}$. Now $A \sim \{e, h, x\} \sim B$ follows by the same methods we used to prove step 1.

Step 3: General case. Let $|A| \geq 2$. Then there exist $g \in \Gamma$ such that $\{e, a\} \subseteq Ag$. Now, by steps 1 and 2, $\{e, \Gamma\} \sim B \sim B \cup Bg$. This gives $\{g^{-1}, ag^{-1}\} \sim Bg^{-1} \cup B$. We thus get:

$$A = \{g^{-1}, ag^{-1}\} \cup A \sim Bg^{-1} \cup B \cup A \sim B. \blacksquare$$

Corollary 4.7. $\sim$ *is not a cancellative congruence on* $F(\Gamma)$. *Furthermore every cancellative congruence on* $F(\Gamma)$ *is universal.*

Proof. Let r be a cancellative congruence on $F(\Gamma)$. Then for $A, B \in F(\Gamma)$, $|C| \geq 2$, then $|AC| \geq 2$ and $|BC| \geq 2$. Now $AC\ r\ BC$ implies ArB, hence r is universal. $\blacksquare$

Corollary 4.8. $\Gamma \cup \{0\}$ *and* $F(\Gamma)/\sim$ *are isomorphic semigroups.*

Comments. Let G be a group generated by two elements x and y. Theorem 3.8 yields that if two words $w(x, y)$ and $v(x, y)$ are relations in G

then G can not be right ordered. If in the group G two words (x, y) and $v'(y, yx)$ are relations then e and y are non-upels for the set $A = \{e, x, y\}$. Hence, by Theorem 1.8, G is not upel group. We know that the group G given on generators x and y with defining relations:

$$xyx^2y^2x^3y^3 = x^{-1}yx^{-2}y^2x^{-3}y^3 = e,$$

can not be right ordered but it is a candidate for an upel group.

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Streszczenie

Z każdą grupą G można związać półgrupę $F(G)$ złożoną ze skończonych podzbiorów G, z działaniami danymi wzorami:

$$A \cdot B = \{a \cdot b\ ;\ a \in A,\ b \in B\}.$$

W półgrupie $F(G)$ badamy najmniejszą przechodnią relację wyznaczoną przez warunek: $A \sim B$, gdy $A \cdot C = B \cdot C$ dla pewnego $C \in F(G)$. Pokazujemy, że relacja ta jest kongruencją w $F(G)$. Niestety, kongruencja ta nie zawsze jest skracalna - podajemy przykład grupy, w której półgrupa ilorazowa $F(G)/\sim$ jest izomorficzna z półgrupą G z dołączonym zerem. W podzbiorach $A \in F(G)$ wyróżniamy element a **upel** spełniający warunek: $A \cdot B \neq (A \setminus \{a\}) \cdot B$ dla każdego $B \in F(G)$. Pokazujemy, że klasa grup **upel**, w których każdy podzbiór zawiera **upel** element, leży pomiędzy grupami uporządkowanymi a **u.p. grupami**. Klasa ta jest zamknięta na iloczyny podproste i rozszerzenia. Pokazujemy ponadto, że każdy podzbiór upel grupy zawiera co najmniej dwa upel elementy.