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SOLUTION OF THE QUASISTATIONARY STEFAN PROBLEM WITH NEUMANN BOUNDARY CONDITIONS*

Summary. In this paper the analytical solution to the quasistationary two-dimensional Stefan problem is presented for simply connected domain with Neumann boundary conditions. The method uses conformal mapping of the considered domain. Exemplary solutions of this problem are given for unbounded domain, which is the interior of the right angle, and for bounded domain, the boundary of which is an ellipse.

ROZWIĄZANIE QUASI-STACJONARNEGO ZAGADNIENIA STEFANA PRZY WARUNKU BRZEGOWYM DRUGIEGO RODZAJU

Streszczenie. W prezentowanej pracy przedstawiamy analityczne rozwiązanie quasi-stacjonarnego dwuwymiarowego zagadnienia Stefana w obszarze jednospójnym, na którego brzegu zadano warunek brzegowy drugiego rodzaju. Metoda wykorzystuje odwzorowanie kon-

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foremne badanego obszaru. Jako przykłady podano rozwiązanie zadania dla obszaru nieograniczonego, jakim jest wnętrze kąta prostego, i obszaru ograniczonego, którego brzegiem jest elipsa.

1. Introduction

A lot of physical processes are characterized by the Stefan problem. This problem consists in calculating a thermal field and a moving boundary at the same time. The first paper, which refer to moving boundary problems, is G. Lamé's and B. P. Clapeyron's paper [5]. Josef Stefan (1835-1893) described a mathematical model of ice formation in the polar seas and solved a one-dimensional problem in the case of constant Dirichlet boundary conditions [15,16].

For one-dimensional problems, the existence, uniqueness and properties of solutions are well known (see [9,11]) for Dirichlet boundary conditions as well as for Neumann boundary conditions.

In the multi-dimensional case, the existence of classical solution of the one-phase Stefan problem was proved in [4], for two-phase problem in small time interval it was proved in [2,8], and for two-dimensional quasistationary problem in [10].

Another question is how to find this solution?

We can try to use some techniques, such as the heat balance integral, embbeding, isotherm migration and variational methods (see [3]). In some one-dimensional cases it is possible to find an analytical solution. Unfortunately, in general there methods cannot be transfered to multi-dimensional problems. There are also numerical methods, which could be classified into three main groups (see [1]): front tracking methods, front fixing methods and fixed domain methods.

The analytical solution of the quasistationary two-dimensional Stefan problem is presented in this paper for a simply connected domain with Neumann boundary conditions. This solution is a further development of the Neumann boundary conditions case presented in papers [7,12–14].

The discussed method uses conformal mapping of the domain on the consideration another, for which the solution is well known or relativly easy to find. In the case of domains for which an explicit conformal mapping can be given, the solution is given in analytical form. For other simply connected domains, for which the conformal mapping can be given in form $\varphi(z) = \sum_{i=0}^{\infty} c_i z^i$, where $c_i \in \mathbb{C}$, the solution could be as a series, countaining these coefficients. The moving boundary is determined from the heat balance equation on the solid layer, evaluated as a sum of the enthalpy changes in the solid layer and the heat resulting from the phase change.

As examples, this problem is solved for unbounded domain, which is the interior of the right angle, and for bounded domain, the boundary of which is an ellipse.

In Section 2 the mathematical problem is formulated and the method assumptions discussed. In the third section, the analytical solution is presented in a general form. In Sections 4 and 5, as examples, the problem is solved for unbounded domain and for bounded one, as follows.

2. Formulation of the mathematical problem

The scope of the paper ia a solution of the Stefan problem in curvilinear tetragons (Fig. 1). It is assumed that one side of this tetragon is isotherm of freezing front (CD), the opposite side contains cooling segment (AB) and the two remaining sides are adiabates.

This problem is considered in quasistationary approximation, assuming that the temperature distribution in solid layer may be evaluated on the

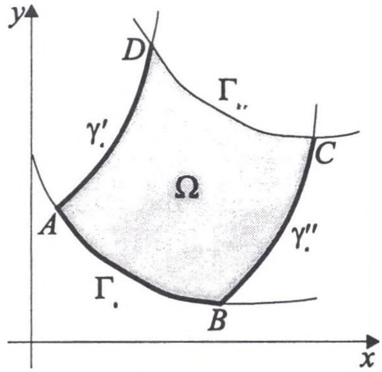

Fig. 1. Solidification area
Rys. 1. Obszar krzepnięcia

basis of the Laplace'a equation:

$$\mathrm{div}\,(\lambda\,\mathrm{grad}\,T) = 0$$

where λ $[W/(m \cdot K)]$ is the coefficient of thermal conductivity and T $[K]$ is temperature.

It is assumed, that temperature is constant at the freezing front (Γ_{kr}):

$$T\Big|_{\Gamma_{kr}} = T_{kr}.$$

On boundary $\gamma_a^{'}$ and $\gamma_a^{''}$, as adiabates:

$$\frac{\partial T}{\partial n}\Big|_{\gamma_a^{'}} = 0 \qquad \mathrm{i} \qquad \frac{\partial T}{\partial n}\Big|_{\gamma_a^{''}} = 0.$$

Let us consider the case where the heat exchange with the environment determines the Neumann boundary conditions:

$$\lambda\frac{\partial T}{\partial n}\Big|_{\Gamma_0} = q_n.$$

The next important assumption adopted in the paper is the similarity of the isotherm of the freezing front. The final characteristic of this method is used to evaluate the heat balance equation on the solid layer, calculated as the sum of the heat change entalpy in solid layer (Q_e) and heat emited in phase change (Q_{kr}), by the following equation:

$$Q_o = Q_{kr} + Q_e,$$

which could also [7] have the form:

$$\int_0^l \left(\lambda \frac{\partial T}{\partial n}\right)_{\Gamma_0} dl = \varrho \left[L + c\left(T_{kr} - \overline{T}\right)\right] \frac{dS}{dt}, \tag{1}$$

where: L is latent heat, ϱ is mass density, c is specific heat and $\overline{T}$ is mean temperature in the domain, evaluated from the equation:

$$\overline{T} = \frac{1}{S} \iint_{\Omega} T(x,y)\, dx\, dy.$$

3. Solving the problem

Considering domain Ω positioned in plane Oxy. New coordinate system OXY is introduced, associated with the old one by equation:

$$\begin{cases} X = \dfrac{x}{l}, \\ Y = \dfrac{y}{l}. \end{cases}$$

In the new coordinate system, domain Ω corresponds to new domain $\widehat{\Omega}$. The analytic function:

$$\omega = u + iv = f(Z) \tag{2}$$

is conformal mapping of domain $\widehat{\Omega}$ on domain Ω', where the isotherms of the feezing front are paraller to the real axis (Fig. 2).

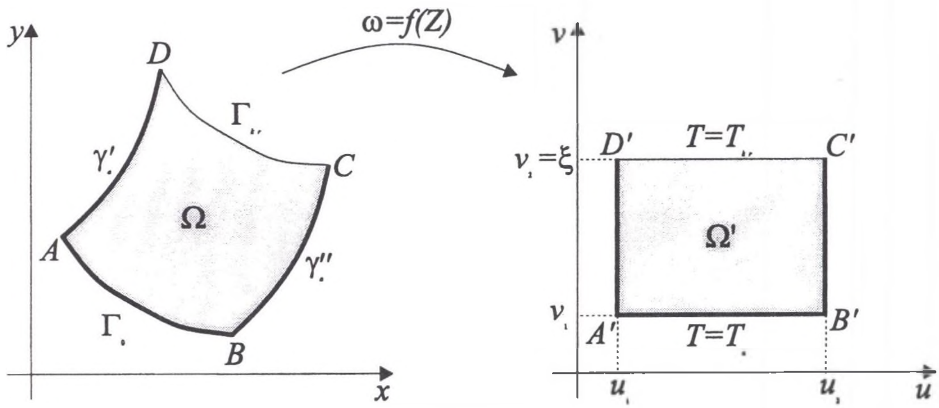

Fig. 2. Transformation of solidification area by means of conformal mapping
Rys. 2. Przekształcenie obszaru krzepnięcia za pomocą odwzorowania konforemnego

The temperature in domain Ω' is given by the linear function [7, 14]:

$$T(v) = T_n + (T_{kr} - T_n)\,\frac{v - v_1}{v_2 - v_1}, \tag{3}$$

where T_n denotes temperature of the surfaces.

The area of domain Ω is equal to:

$$S = \iint\limits_{\Omega} dx\,dy = l^2 \iint\limits_{\widehat{\Omega}} dX\,dY = l^2 \iint\limits_{\Omega'} D(u,v)\,du\,dv, \tag{4}$$

where $D(u,v)$ is the Jacobian transformation:

$$\begin{cases} X = X(u,v), \\ Y = Y(u,v). \end{cases}$$

The heat balance equation (1), after it is passed to system Ouv, can be represented in the form [7]:

$$\int\limits_{u_1}^{u_2} \left(\lambda \frac{\partial T}{\partial v}\right)_{v=v_1} du = \varrho \left[L + c\left(T_{kr} - \overline{T}\right)\right] \frac{dS[\xi(t)]}{dt}, \tag{5}$$

where $\xi(t) = v_2(t)$ is the position on the freezing front in moment t. Because:

$$\frac{dS}{dt} = \frac{dS}{d\xi}\frac{d\xi}{dt} = \frac{d\xi}{dt} l^2 \int_{u_1}^{u_2} D(u,\xi)\,du$$

and using constant $\left(\frac{\partial T}{\partial v}\right)_{v=v_1}$, we obtain:

$$\lambda\,(u_2 - u_1)\left(\frac{\partial T}{\partial v}\right)_{v=v_1} = \varrho l^2 \left[L + c\left(T_{kr} - \overline{T}\right)\right] \frac{d\xi}{dt} \int_{u_1}^{u_2} D(u,\xi)\,du. \qquad (6)$$

From equation (3) we have:

$$\frac{\partial T}{\partial v} = \frac{T_{kr} - T_n}{v_2 - v_1}, \qquad \overline{T} = \frac{1}{2}\left(T_{kr} + T_n\right).$$

Substituting this equations into formula (6) and using dimensionless variables:

$$\tau = \frac{\lambda t}{c\varrho l^2}, \quad K = \frac{c\,(T_{kr} - T_o)}{L}, \quad \Theta_n = \frac{T_{kr} - T_n}{T_{kr} - T_o},$$

where T_o – denotes temperature of the environment, after simple rearangement we obtain:

$$\frac{2K\Theta_n}{2 + K\Theta_n}\,d\tau = \frac{\xi - v_1}{u_2 - u_1}\,d\xi \int_{u_1}^{u_2} D(u,\xi)\,du. \qquad (7)$$

The Neumann boundary conditions haven't been used. On the plane Ouv this conditions take the form:

$$\frac{\lambda}{l}\left(\frac{\partial T}{\partial v}\right)_{v=v_1} = q_n. \qquad (8)$$

Considering the derivative of the temperature expression, the dependence defining temperature of the cast surface may be determined:

$$T_n(t) = T_{kr} - \frac{l}{\lambda}\left(\xi(t) - v_1\right) q_n. \qquad (9)$$

Using equations (9) in the expression defining the dimensionless temperature of the cast surface, we have:

$$\Theta_n = \frac{l\, q_n\, (\xi(t) - v_1)}{\lambda\,(T_{kr} - T_o)}. \tag{10}$$

Introducing the obtained formula into heat balance equation (7), we have:

$$\frac{2\,c\,l\,q_n\,(\xi - v_1)}{c\,l\,q_n(\xi - v_1) + 2\,L\,\lambda}\,d\tau = \frac{\xi - v_1}{u_2 - u_1}\,d\xi \int\limits_{u_1}^{u_2} D(u,\xi)\,du.$$

If heat flux q_n doesn't depend on time (is constant), the differential equation with separation variables is obtained. Integrating this equation, a general solution of the discussed problem with the Neumann boundary conditions is found:

$$\tau = \frac{1}{2\,(u_2 - u_1)} \int\limits_{v_1}^{\xi} \left[\xi - v_1 + \frac{2\,L\,\lambda}{c\,l\,q_n}\right] d\xi \int\limits_{u_1}^{u_2} D(u,\xi)\,du. \tag{11}$$

4. Solidification in unbounded domain

As the first example, solidification in unbounded domain is considered, contained between the arms of the right angle. On account of the assumption of the method and constant heat flux, thermal field is symmetrical with regard to the bisector of the right angle. Therefore, it is enough in this discussion to examine only domain Ω (Fig. 3).

Assuming that $l = \overline{AB}$, let's introduce new coordinate system OXY associated with the old one by the equations:

$$X = \frac{x}{l}, \qquad Y = \frac{y}{l}.$$

In plane OXY the polar coordinates are introduced:

$$\begin{cases} X = r\cos\varphi, \\ Y = r\sin\varphi, \end{cases}$$

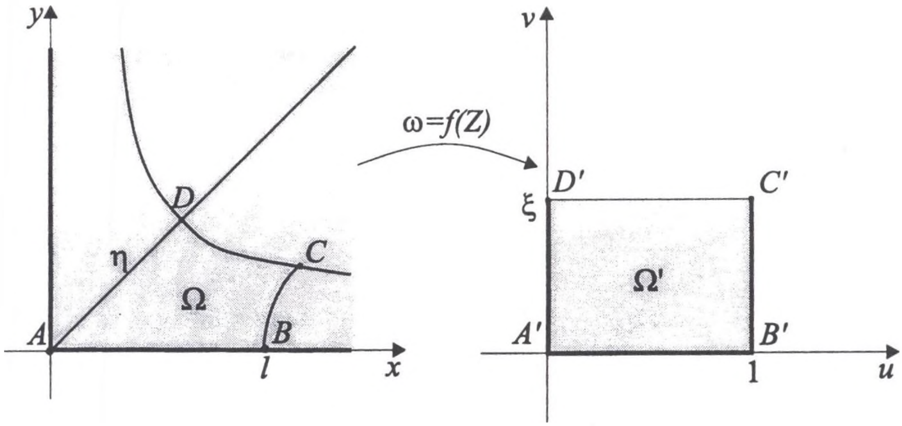

Fig. 3. Solidification area in the form of the right angle and its range through conformal mapping

Rys. 3. Obszar krzepnięcia w postaci wnętrza kąta prostego i jego obraz poprzez odwzorowanie konforemne

and the complex function considered:

$$\omega = f(Z) = Z^2 = r^2 e^{i2\varphi}.$$

This function is analytic and univalent in domain $\mathcal{D} = \{Z : \arg Z \in [0, \frac{\pi}{2}]\}$, therefore, it is conformal mapping in domain Ω. From form function ω, half-time emanating from point $Z = 0$ with slop φ, turns into half-line emanating from point $\omega = 0$ with slop 2φ. Therefore, bisector $\widehat{A}\widehat{D}$ (where $\widehat{P}$ is an image of point P in plane OXY), turns into positive imaginary semi-axis. Positive real semi-axis ($\varphi = 0$) doesn't change. Points $\widehat{A} = (0,0)$ and $\widehat{B} = (1,0)$ are constant points of this conformal mapping. The image of section $\widehat{B}\widehat{C}$ is perpendicular to the real axis and the image of section $\widehat{C}\widehat{D}$, which is the image of the freezing front, is paraller to the real axis.

Because one more (polar) coordinate system is used, in the integral in equation (11) the product of two Jacobians appears in this equation, therefore it takes the form:

$$\tau = \frac{1}{2\,(u_2 - u_1)} \int\limits_{v_1}^{\xi} \left[\xi - v_1 + \frac{2\,L\,\lambda}{c\,l\,q_n}\right] d\xi \int\limits_{u_1}^{u_2} D_1(u,\xi)\, D(u,\xi)\, du. \tag{12}$$

In the discussed example:

$$u_1 = 0, \quad u_2 = 1, \quad v_1 = 0, \quad D_1(r,\varphi) = r \quad \text{i} \quad D(u,v) = \frac{1}{4}\left(u^2 + v^2\right)^{-3/4}.$$

Inserting the above to equation (12) and calculating the integrals, the dependence between the position of the freezing front ($\xi > 0$) in plane Ouv and dimensionless time is obtained:

$$\tau = \frac{1}{8}\left[\sqrt{1+\xi^2} + \frac{4\,L\,\lambda}{c\,l\,q_n}\,\operatorname{arc\,sinh}\xi + \right.$$
$$\left. + \xi\left(\xi + \frac{4\,L\,\lambda}{c\,l\,q_n}\right) \log\left(\frac{1+\sqrt{1+\xi^2}}{\xi}\right)\right]. \tag{13}$$

Next, the dimensionless time corrensponding to parameter ξ may be derived and, remembering about dependence $\tau = \frac{at}{l^2}$, ordinary time.

Using coordinates of point D and D', the thickness (η) of solid phase along bisector AD may be obtained:

$$\eta = l \cdot \sqrt{\xi}. \tag{14}$$

The dependence of solid phase thickness on time (in hours) is presented in Fig. 4. The calculations are carried out for the following values of input data: $c = 670\,[J/(kg \cdot K)]$, $L = 247\,[kJ/kg]$, $\lambda = 30\,[W/(m \cdot K)]$, $q_n = 120\,[kW/m^2]$, $\varrho = 7000\,[kg/m^3]$, $l = 2\,[m]$.

Assuming parameter ξ, the shape of the freezing front can be found (inversing the transformations used above) and the time after which the front assumes this position. The position of the freezing front of studied domain Ω depending on parameter ξ is shown in Fig. 5. The values of the parameters of metal and solidification process are the same as it was in the case of calculating the thickness of solid phase. Parameter ξ is changing from 0.1 (internal curve) to 2.0 (external curve), with step 0.1.

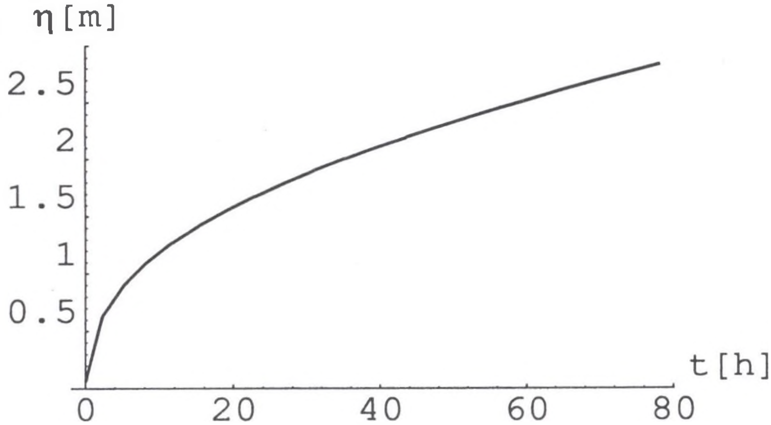

Fig. 4. Dependences of solid phase thickness on time in the case of solidification in unbounded area

Rys. 4. Zależność grubości warstwy zakrzepłej od czasu w przypadku krzepnięcia w obszarze nieograniczonym

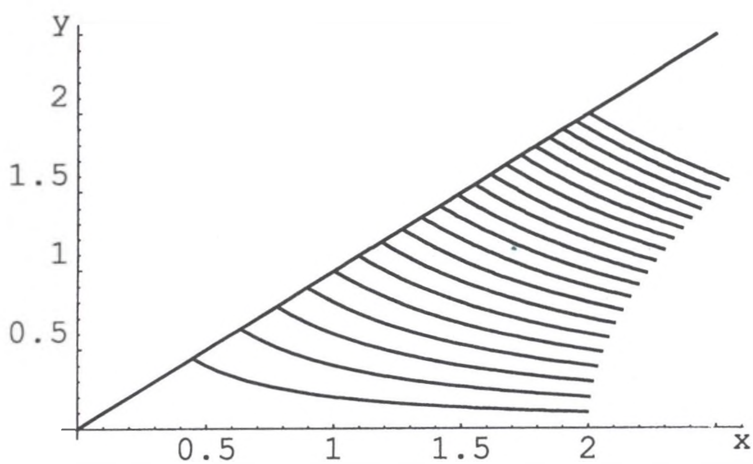

Fig. 5. Some positions of the freezing front in the case of solidification in unbounded area

Rys. 5. Kilka położeń granicy rozdziału faz w przypadku krzepnięcia w obszarze nieograniczonym

5. Solidification in cylindroid

Let us consider the case of the solidification in cylindroid. On account of symmetry, the solidification in one quarter cross-section of solidification area is discussed (Fig. 6).

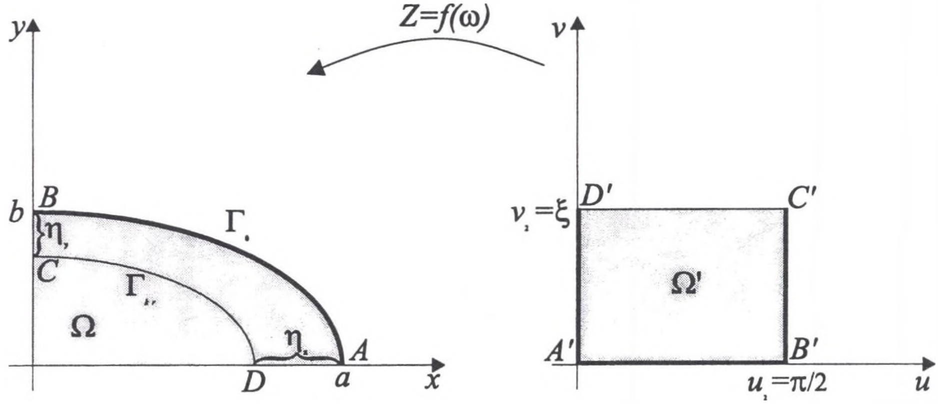

Fig. 6. Solidification area bounded by ellipse and its range through conformal mapping
Rys. 6. Obszar krzepnięcia ograniczony elipsą i jego obraz poprzez odwzorowanie konforemne

The axes of coordinate system Oxy are selected in such a way that they cover are convergent with the axes of the ellipse. The parametric equation of the ellipse is as follows:

$$\begin{cases} x = a\cos u, \\ y = b\sin u, \qquad u \in [0, 2\pi). \end{cases} \tag{15}$$

If new coordinates $X = \frac{x}{l}$, $Y = \frac{y}{l}$ i $Z = X + iY$ are introduced, where $l = \sqrt{a^2 - b^2}$ is the distance between the focus of the ellipse and the origin of the coordinates, the equation of the examined ellipse, takes the form:

$$\begin{cases} X = \frac{a}{l}\cos u, \\ Y = \frac{b}{l}\sin u, \qquad u \in [0, 2\pi), \end{cases} \tag{16}$$

or in a complex form:

$$Z = X + iY = \frac{a}{l}\cos u + i\frac{b}{l}\sin u.$$

Let's introduce the function:

$$Z = X + iY = f(\omega) = \frac{a}{l}\cos\omega + i\frac{b}{l}\sin\omega, \tag{17}$$

where $\omega = u + iv$. This function is conformal mapping, which transforms the zone:

$$P = \left\{\omega = u + iv : \ 0 \leqslant u \leqslant \frac{\pi}{2} \wedge v \geqslant 0\right\}$$

on domain Ω. The Jacobian $D(u, v)$ the transformation of coordinates system OXY on Ouv, is equal:

$$D(u,v) = \frac{1}{4l^2}\left((a+b)^2 e^{-2v} + (a-b)^2 e^{2v} - 2(a^2 - b^2)\cos 2u\right).$$

Because now $u_1 = 0$, $u_2 = \frac{\pi}{2}$ and $v_1 = 0$, the formula determining the dependence between the position of the freezing front in plane Ouv and dimensionless time, takes the form:

$$\tau = \frac{1}{8l^2}\left[(a+b)^2 \frac{1 + 2\,s_q - (1 + 2\,s_q + 2\xi)\,e^{-2\xi}}{4} + \right.$$

$$\left. + (a-b)^2 \frac{1 - 2\,s_q + (-1 + 2\,s_q + 2\xi)\,e^{2\xi}}{4}\right] \tag{18}$$

where

$$s_q = \frac{2\,L\,\lambda}{c\,l\,q_n}.$$

The thickness of the solid phase along axis Ox (η_x) may be calculated using the coordinates of points D and D'. However, the thickness of the solid phase along axis Oy (η_y) can be calculated using the coordinates of points C and C':

$$\begin{cases} \eta_x = a - \frac{1}{2}\left((a+b)e^{-\xi} + (a-b)e^{\xi}\right), \\ \eta_y = b - \frac{1}{2}\left((a+b)e^{-\xi} - (a-b)e^{\xi}\right). \end{cases} \tag{19}$$

Let's attempt to determine, the time of the whole solidification process for this example. The solidification process will be ended when the thickness of the solid phase along axis Oy is equal to small axis of the ellipse ($\eta_y = b$). From formula (19) the following condition may be obtained:

$$\xi = \ln\sqrt{\frac{a+b}{a-b}}. \tag{20}$$

If this condition is introduced to formula (18), the equation determining the dimensionless time of all solidification process is derived.

The dependence of solid phase thickness on time (in hours) along axis Ox and Oy (η_x and η_y) is presented in Fig. 7. The calculations are achieved for the following values of input data: $c = 670\,[J/(kg \cdot K)]$, $L = 247\,[kJ/kg]$, $\lambda = 30\,[W/(m \cdot K)]$, $\varrho = 7000\,[kg/m^3]$, $q_n = 100\,[kW/m^2]$, $a = 1\,[m]$ $b = 0.8\,[m]$.

The value of parameter ξ, which corresponds to the end of solidification process is equal to 1.09861. However, the coresponding time of the whole process is equal $6.68\,[h]$.

The position of the freezing front of the studied domain, in depending on parameter ξ is shown in Fig. 8. The values of parameters of metal and solidification process are the same as it was in the case of calculating the thickness of the solid phase.

Parameter ξ is changing from 0 (boundary of casting mould, external curve) to 1.0 (internal curve), with step 0.1.

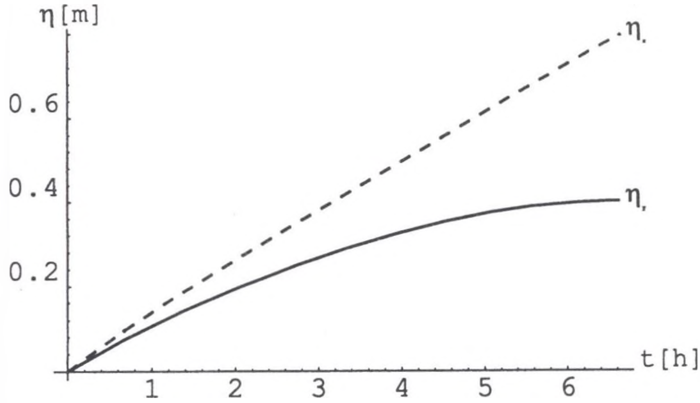

Fig. 7. Dependence of the solid phase thickness on time in the case of solidification in cylindroid (solid line — thickness of solid phase along axis Oy, dots line — along axis Ox)

Rys. 7. Zależność grubości warstwy zakrzepłej od czasu w przypadku krzepnięcia w walcu eliptycznym (linia ciągła - grubość warstwy zakrzepłej wzdłuż osi Oy, linia przerywana - wzdłuż osi Ox)

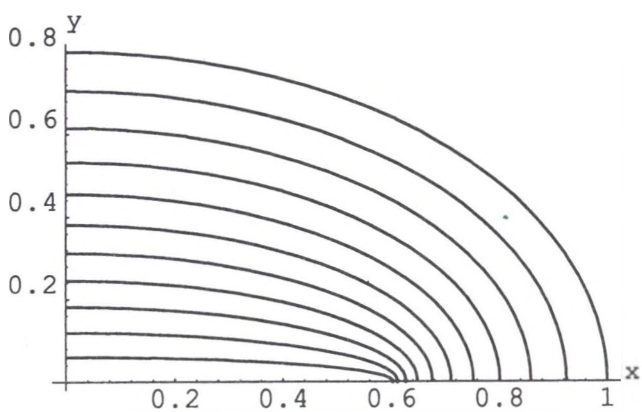

Fig. 8. Some positions of the freezing front in the case of solidification in cylindroid

Rys. 8. Kilka położeń granicy rozdziału faz w przypadku krzepnięcia w walcu eliptycznym

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Streszczenie

W prezentowanej pracy przedstawiamy analityczne rozwiązanie quasi-stacjonarnego dwuwymiarowego zagadnienia Stefana w obszarze jednospójnym, na którego brzegu zadano warunek brzegowy drugiego rodzaju. Metoda wykorzystuje odwzorowanie konforemne badanego obszaru na inny, dla którego rozwiązanie zadania nie przedstawia trudności. Położenie granicy rozdziału faz jest wyznaczone z równania bilansu ciepła w warstwie stałej, obliczanego jako suma ciepła zmiany entalpii warstwy stałej i ciepła wydzielanego przy przejściu fazowym. Jako przykłady podano rozwiązanie zadania dla obszaru nieograniczonego, jakim jest wnętrze kąta prostego, i obszaru ograniczonego, którego brzegiem jest elipsa.