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SYMMETRIC PRESENTATIONS FOR GROUPS

Summary. Let G be a group which has a presentation: $\langle x,\ y \mid \{R_i(x,y),\ i \in I\}\rangle$. This presentation is called *symmetric* if the mapping $x \to y$, $y \to x$ defines an automorphism of the group G. Then G is called symmetrically presented. We investigate finite and infinite groups which have symmetric presentations. We give here some examples of groups with symmetric presentations. In particular we show that all symmetric groups S_n and all alternating groups A_n are symetrically presented.

SYMETRYCZNE PREZENTACJE GRUP

Streszczenie. Niech G będzie grupą posiadającą prezentację:

$$\langle x,\ y \mid \{R_i(x,y),\ i \in I\}\rangle.$$

Mówimy, że prezentacja ta jest *symetryczna,* jeśli odwzorowanie $x \to y$, $y \to x$ można przedłużyć do automorfizmu grupy G, a grupę G nazywamy wtedy symetrycznie prezentowalną. Badamy tu skończone i nieskończone grupy, posiadające symetryczne prezentacje. Podajemy przykłady symetrycznie prezentowalnych grup oraz pokazujemy, że grupy symetryczne S_n i alternujące A_n są symetrycznie prezentowalne. Podajemy również przykład minimalnej grupy skończonej, dwugenerowanej bez symetrycznych prezentacji.

Let G be a group with a presentation:

$$\langle x, y \mid \{R_i(x, y),\ i \in I\}\rangle \tag{1}$$

Definition 1. ([3] or [5]) *We say that the presentation* (1) *is* **symmetric** *if the mapping* $x \to y$, $y \to x$ *defines an automorphism of the group* G

Remark 1. *A presentation (1) is symmetric if and only if every relation* $R(x, y)$ *implies a relation* $R(y, x)$.

The question is:

Question 1. *Which groups (especially finite groups) have symmetric presentations?*

This question was considered in [3]. The necessary and sufficient condition are given for the free product of two cyclic groups to have a symmetric presentation. In this paper, we give some examples of groups with symmetric presentations and we prove that symmetric groups S_n and alternating groups A_n have symmetric presentations. We also prove that the groups of order pq, where p and q are prime, have symmetric presentations.

Lemma 1. *Let* G *be a group with a presentation* $\langle x, y | x^m = y^m = (xy)^n\rangle$ *or* $\langle x, y | x^m = y^m = (xy)^n = 1\rangle$. *Then* G *is symetrically presented.*

Proof. We have to prove that G has relations: $x^m = y^m = (yx)^n$. Indeed it follows from equalities: $(yx)^n = (x^{-1}xyx)^n = x^{-1}(xy)^n x = x^{-1}x^m x = x^m = y^m$. ■

Lemma 2. *Every 2-generator Coxeter group has a symmetric presentation.*

Proof. If G is a 2-generator Coxeter group then G has a presentation:

$$\langle x, y | x^2 = y^2 = (xy)^n = 1 \rangle$$

(see [1]). Then by Lemma 1. G is symetrically presented.∎

Examples of symmetrically presented groups:

1. $S_3 = \langle x, y | x^2 = y^2 = (xy)^3 = 1 \rangle$,

2. $A_4 = \langle x, y | x^3 = y^3 = (xy)^2 = 1 \rangle$
$= \langle x, y | x^3 = y^3 = (xy)^3 = (yx)^3 = 1, x^2 y = y^2 x \rangle$,

3. $S_4 = \langle x, y | x^4 = y^4 = (xy)^3 = (x^{-1}y)^3 = (x^2 y^2)^2 = 1 \rangle$,

4. $Q_8 = \langle x, y | x^2 = y^2 = (xy)^2 \rangle$,

5. Every dihedral group has a symmetric presentation:
$D_n = \langle x, y | x^2 = y^2 = (xy)^n = 1 \rangle$.

Lemma 3. *Every 2-generator abelian group has a symmetric presentation:*

$$\langle x, y | x^r = y^r = 1, x^s = y^s, [x, y] \rangle,$$

where r can be either finite or infinite.

Proof. It can be deduced from Corollary 3.5.2 from the book [4] that if G is 2-generator, abelian group then G has the presentation: $\langle x, y | x^s = y^r = 1, s|r, [x,y] = 1\rangle$. Because $s|r$ then there exists k such that $r = ks$. We denote by $a = y$ and by $b = xy$, hence $G = \mathrm{gp}(a,b)$ and we have following relations: $a^r = 1$, $b^r = 1$, $a^s = b^s$ and $[a,b] = 1$, as required.∎

The following theorem provides the necessary and sufficient condition for a group to have a symmetric presentation.

Theorem 1. *The 2-generator group G has a symmetric presentation if and only if G has an automorphism α of order two and there exists such an element $g \in G$ that $G = gp(g, g^\alpha)$.*

Proof. The necessary coindition follows directly from the Definition 1. Conversely, if $w(a, a^\alpha) = 1$ then $1 = (w(a, a^\alpha))^\alpha = w(a^\alpha, a)$. The last equality holds because α has order 2.∎

Using the above theorem we can prove that all symmetric groups S_n and all alternating groups A_n have symmetric presentations. First, we want to show some lemmas.

Lemma 4. *Let $a = (1, 2, \ldots, n-1, n)$ and $b = (1, 2, \ldots, n-2, n, n-1)$ be two cycles in S_n. Then $A_n \subseteq \mathrm{gp}(a,b)$.*

Proof. Let us denote by H the group generated by a and b. It is enough to prove that all cycles of the form $(1, 2, k)$ belong to H.
$ab^{-1} = (1, n, n-1) = c \in H$,
$a^{-1}b = (n-2, n-1, n) = d \in H$,
$bcb^{-1} = (1, 2, n-1) \in H$,
$c(1, 2, n-1)^{-1}c^{-1} = (1, 2, n) \in H$,
$d(1, 2, n)d^{-1} = (1, 2, n-2) \in H$.
Now, we show, that if $(1, 2, k) \in H$ then $(1, 2, k-1) \in H$ for $3 < k < n-1$.

$b^{-1}(1,2,k)b = (1,k-1,n-1) \in H$,
$(1,n-1,k-1)\cdot(1,2,k)\cdot(1,k-1,n-1) = (2,k,n-1) \in H$,
$(2,k,n-1)\cdot(1,n-1,k-1)\cdot(2,n-1,k) = (1,2,k-1) \in H$,
and it finishes the proof.∎

Lemma 5. *Let $a = (1,2,\ldots,n-1,n)$ and $b = (1,2,\ldots,n-2,n,n-1)$ be two cycles in S_n then for an even n $S_n = \mathrm{gp}(a,b)$, and for an odd n $A_n = \mathrm{gp}(a,b)$.*

Proof. We know from Lemma 1 that $A_n \subseteq \mathrm{gp}(a,b)$. If n is even then both a and b are odd permutations, so $S_n = \mathrm{gp}(a,b)$. For odd n, a and b are even, hence $A_n = \mathrm{gp}(a,b)$.∎

Lemma 6. *Let $a = (1,2,\ldots,n-1)$ and $b = (1,2,\ldots,n-2,n)$ be two cycles in S_n. Then $A_n \subseteq \mathrm{gp}(a,b)$.*

Proof. Let us denote by H the group generated by a and b. We shall prove that for all $k \in \{3,\ldots,n\}$ cycles $(1,2,k)$ belong to H.
$ab^{-1} = (1,n,n-1) = c \in H$,
$a^{-1}b = (n,n-1,n-2) = d \in H$,
$bcb^{-1} = (1,n-1,2) \in H$, so $(1,2,n-1) \in H$,
$c(1,n-1,2)c^{-1} = (1,2,n) \in H$,
$d^{-1}(1,2,n)d = (1,2,n-2) \in H$.
Now, it is enough to prove that if the cycle $(1,2,k) \in H$ then $(1,2,k-1) \in H$, for $3 < k < n-1$,
$b^{-1}(1,2,k)b = (1,k-1,n) \in H$,
$(1,n,k-1)(1,2,k)(1,k-1,n) = (2,k,n) \in H$,
$(2,k,n)(1,n,k-1)(2,n,k) = (1,2,k-1) \in H$, which was required.∎

Lemma 7. *Let* $a = (1, 2, \ldots, n-1)$, $b = (1, 2, \ldots, n-2, n)$ *be cycles in* S_n. *Then for even* n, $A_n = gp(a, b)$, *and for odd* n, $S_n = gp(a, b)$.

Proof. We know from the Lemma 3, that $A_n \subseteq \mathrm{gp}(a, b)$. If n is even then cycles a and b are both even, so $A_n = \mathrm{gp}(a, b)$. For n odd the cycles a and b are odd, hence $S_n = \mathrm{gp}(a, b)$. ∎

Theorem 2. *Symmetric groups* S_n *and alternating groups* A_n *have symmetric presentations.*

Proof. We use Theorem 1, Lemma 2 and Lemma 4. It follows from Lemmas 2 and 4 that there exist an element $a \in S_n$ and the automorphism α of order 2, such that:

$$S_n = \mathrm{gp}(a, a^{\alpha}),$$

where α is the inner automorphism that maps every x into $x^{(n,n-1)}$ (α changes n and $n-1$). The element a depends on n, for even n, $a = (1, 2, \ldots, n)$, and for odd n, $a = (1, 2, \ldots, n-1)$. ∎

Symmetric presentations of alternating groups A_n were found by H. S. M. Coxeter ([2]):

$$A_{2m+1} = \langle x, y | x^{2m+1} = y^{2m+1} = (xy)^m, (x^{-j}y^j)^2 = 1, 2 \le j \le m \rangle,$$

$$A_{2m} = \langle x, y | x^{2m-1} = y^{2m-1} = (xy)^m, (x^{-j}y^{-1}xy^j)^2 = 1, 2 \le j \le m-1 \rangle.$$

Theorem 3. *If the G is a finite non-abelian group of order pq, where p and q are primes, then G has symmetric presentation.*

Proof. Let us assume that $p < q$. It is not difficult to show that G has a presentation: $\langle x, y | x^p = y^q = 1, xy^k = yx \rangle$, where $k^p \equiv 1 \bmod q$ and $k \neq 1$. We consider the mapping:

$$\alpha : \begin{cases} x \to a = y^{-1}x \\ y \to b = y^{-1} \end{cases}$$

The group G is generated by elements x and x^α. By Theorem 1, it is enough to prove that α defines the automorphism of order 2 of G. To prove that α is the automorphism of G we use the Lemma 3.3 from [4] and we show that a and b satisfy the same relations as x and y do. It is clear that $a^p = b^q = 1$. We prove now that $ab^k = ba$, indeed $ab^k = y^{-1}xy^{-k} = (xy^{-k})y^{-1} = y^{-1}xy^{-1} = ba$. Clearly α has order 2.∎

Let $u = x$ and $v = y^{-1}x$ then $G = \mathrm{gp}(u, v)$ and $x = u$, $y = uv^{-1}$. Elements u and v satisfy following relations: $u^p = v^p = (uv^{-1})^q = 1$ and $(uv^{-1})^k = v^{-1}u$. Hence G has the following symmetric presentation:

$$\langle x, y | x^p = y^p = (xy^{-1})^q = 1, (xy^{-1})^k = y^{-1}x \rangle,$$

where $k^p \equiv 1 \bmod q$, and $k \neq 1$.

Now, we show that there exist 2-generator groups without symmetric presentations. Theorem of O. Macedońska and D. Solitar [3] shows that the free product of cyclic groups of order r and s, where $0 \leq s \leq r$, has a symmetric presentation if and only if $r = s$ or if r or s is odd. So we get examples of infinite groups without symmetric presentations. In next theorem we show the example of a finite group without symmetric presentations.

Theorem 4. *Let G be a group with the presentation*

$$\langle x, y | x^2 = 1, y^x = y^3 \rangle.$$

Then G is the metacyclic group of order 16 and G has no symmetric presentations. Morover, G is the smallest group with this property.

Proof. The subgroup of G generated by y, is normal and the quotient group $G/\langle y \rangle$ is cyclic and has order 2. So G is the metacyclic group [2] and every element of G can be written in the form $x^k y^l$, $k \in \{0, 1\}$. To find the order of G it is enough to find the order of y. From the relation $y^3 = y^x$ we get $y^9 = x^{-1} y^3 x = x^{-2} y x^2 = y$ and hence $y^8 = 1$. It means that G has the order 16. Now, we show that G has no symmetric presentations. We consider all pairs a, b of elements which have the same orders. Every element of G has order 2, 4 or 8. Elements: y^4, x, xy^2, xy^4, xy^6 have order 2, $y^2, y^6, xy, xy^3, xy^5, xy^7$ have order 4, and y, y^3, y^5, y^7 have order 8. If two elements a, b have order 2 then the subgroup generated by a and b is contained in $K = \langle x, y^2 \rangle$, and K has 8 elements, so $K \neq G$. The subgroup generated by two elements of order 4 is contained in $H = \langle xy, y^2 \rangle$ and H has order 8. Indeed we have $y^6 \in H$, $xy^3 = xy \cdot y^2 \in H$, $xy^5 = xy^3 \cdot y^2 \in H$, $xy^7 = xy^5 \cdot y^2 \in H$. We denote $a = xy, b = y^2$ then a and b satisfy relation $a^4 = b^4 = 1, a^2 = b^2, a^{-1} b a = b^{-1}$, hence H is isomorphic to the quaternion group Q_8, which has 8 elements. It means thet every pair of elements of order 4 generates a proper subgroup of G. Every element of order 8 is contained in $\langle y \rangle$, so every pair of elements of order 8 does not generate G. This group is the smallest group without symmetric presentations because it follows from the table 1 in [2] that every group of order less then 16 has the presentation of the form $\langle x, y | x^2 = y^2 = (xy)^m \rangle$ or $\langle x, y | x^2 = y^2 = (xy)^m = 1 \rangle$ and both types of presentations are symmetric. ∎

References

1. N. Bourbaki, *Elements de Mathematique, Groupes et algebres de Lie,* chap. IV, Hermann, Paris 1968.

2. H. S. M. Coxeter, W. O. J. Moser, *Generators and relations for discrete groups,* Springer-Verlag, Berlin 1972.

3. O. Macedońska, D. Solitar, *On binary σ-invariant words in a group,* Contemporary Math. **169** (1994), 431-449.

4. W. Magnus, A. Karrass, D. Solitar, *Combinatorial group theory,* Wiley Interscience Publishers, New York 1966.

5. E. F. Robertson, C. Campbell, *Symmetric presentations,* de Gruyter, Berlin 1989.

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Streszczenie

Niech G będzie grupą posiadającą prezentację:

$$\langle x,\ y \mid \{R_i(x,y),\ i \in I\}\rangle.$$

Prezentacja ta jest *symetryczna,* jeśli odwzorowanie $x \to y$, $y \to x$ można przedłużyć do automorfizmu grupy G, a grupę G nazywamy wtedy symetrycznie prezentowalną. W pracy tej pokazujemy, że następujące grupy posiadają symetryczne prezentacje:

- grupa kwaternionów Q_8,

- wszystkie dwugenerowane grupy Coxetera,

- wszystkie dwugenerowane grupy abelowe,

- grupy symetryczne S_n i grupy alternujące A_n.

Podajemy również przykłady dwugenerowanych grup, które nie posiadają symetrycznych prezentacji. Minimalną grupą o tej własności jest grupa, która ma prezentację $\langle x,\ y|\ x^2 = 1, y^x = y^3\rangle$.