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STOCHASTIC BOUNDS ON THE TRANSIENT BEHAVIORS OF THE G-NETWORKS

Summary. In this work, we study the transient behaviors of the G-networks which are the extension of the Jackson networks. In fact, the steady-state solution of these networks has a product-form solution, however any analytical solution for their transient behaviors is not known. Following the studies on the Jackson networks, we propose to study the transient behaviors of the G-networks by applying the stochastic comparison approach.

OGRANICZENIA STOCHASTYCZNE STANÓW NIEUSTALONYCH W SIECIACH G

Streszczenie. W artykule badane są stany nieustalone sieci G, stanowiących rozszerzenie sieci Jacksona. Zarówno sieci G, jak i sieci Jacksona posiadają rozwiązanie produktowe w stanie ustalonym, natomiast nieznane jest ich zachowanie w stanie nieustalonym. Zaproponowano analizę stanów nieustalonych w sieciach G poprzez porządkowanie i stochastyczne porównywanie wielowymiarowych łańcuchów Markowa.

1. Introduction

In this paper, we are interested in the transient behavior of the G-networks. These networks are introduced by Gelenbe [2], [3] to generalize the Jackson networks. In the

G-networks, there are two types of customers, positive customers are the usual ones, waiting in the queues or in service, and negative customers destroy the positive ones, and go out of the network. The product-form solution of the stationary behavior of these networks has been proved ([2], [3]). However, there is no analytical method to study their transient behavior. Let us recall here that the transient behaviors of the Jackson networks have been studied by Massey through the stochastic comparison approach [7, 8]. In this work, we analyze the transient behavior of the G-networks with this approach. In other terms, we also stochastically bound the transient behaviors of the G-networks by the models whose transient behaviors are known.

The stochastic ordering applied in this work is the sample-path (strong) stochastic ordering, and it will be denoted by $\leq_{st}$. Massey [9] has studied the $\leq_{st}$ stochastic comparison on multidimensional state spaces with the increasing set formalism. On partially ordered state spaces, there are three stochastic orderings related to the increasing sets: one of them corresponds to the usual sample path comparison ($\leq_{st}$), and there are two weaker orderings corresponding to the comparison of tail and cumulative distribution functions. These three orderings are equivalent to each other, when the state space is totally ordered.

The stochastic comparison methodology is especially useful when one is interested to bound functionals of Markov processes. For instance, in a queuing network of K queues, the state space is represented by a vector $N = (n_1, n_2 \cdots, n_K)$ where n_i is the number of customers in queue i. If we are interested in the total (or partial) sum of components, representing the total (or partial) number of customers in the network, we can bound the functional of the underlying process instead of bounding the process itself.

The proposed bounding models are similar to the models proposed in the case of the Jackson networks. The upper bound in the case of the Jackson networks is provided by a network of independent M/M/1 queues whose transient behavior can be computed through the transient analysis of M/M/1 queues [10]. In the case of the G-networks, we consider queues where services are carried out in batches of 1 or 2 customers. This is indeed a homogeneous general birth and death process where "births" represent the increases by one while "deaths" represent the decreases by one or two. The upper bounding model is constituted of a collection of independent queues with services in batch. The lower bound on the total number of customers in the G-Network is provided by a single queue. Obviously, it is easier to analyze the transient behavior of a generalized birth-death process and a collection of independent such processes than that of a G-network.

In fact, Massey has largely studied the operator-analytic descriptions of Jackson networks and related stochastic dominance results [7, 8, 9]. We are especially interested in the application of the stochastic comparison techniques in the multidimensional case. Our main goal is to give insights in these techniques through the study of the G-networks. There are different ways to demonstrate the proposed stochastic comparison results, and we give the demonstrations which seem us more representative for the application of the techniques presented in this paper.

This paper is organized as follows: first we present briefly the stochastic comparison method and the G-networks. In section 2, we give the bounding models to study the transient behaviors and demonstrate these bounds.

2. Preliminaries

2.1. Stochastic Comparison

The stochastic ordering terminology through increasing sets are included from [9]. Let $\mathcal{E}$ be a denombrable, discrete state space, endowed with a preorder $\preceq$ (reflexive and transitive binary) relation.

Definition 1 *$\Gamma \subset \mathcal{E}$, is an increasing set if and only if $x \in \Gamma$, and $x \preceq y$, then $y \in \Gamma$.*

The following particular increasing sets for a given $x \in \mathcal{E}$ are defined as follows:

$$\{x\}\uparrow = \{y \in \mathcal{E} \mid y \succeq x\} \qquad \{x\}\downarrow = \{y \in \mathcal{E} \mid y \preceq x\}$$

The stochastic orderings $\preceq_{wk}$, $\preceq_{wk^*}$, and $\preceq_{st}$ are then defined respectively through the following families of increasing sets:

$$\phi_{wk}(\mathcal{E}) = \{\{x\}\uparrow,\ x \in \mathcal{E}\}$$

$$\phi_{wk^*}(\mathcal{E}) = \{\mathcal{E} - \{x\}\downarrow \ \mid x \in \mathcal{E}\}$$

$$\phi_{st}(\mathcal{E}) = \{\textit{all increasing sets on } \mathcal{E}\}$$

Definition 2 *Let X (resp. Y) be a random variable taking values in $\mathcal{E}$ defined by a probability vector* p *(resp.* q*), where for $i \in \mathcal{E}$, $p[i] = Prob(X = i)$ (resp. $q[i] = Prob(Y = i)$).*

$X \leq_\phi Y$, $(\preceq_\phi \in \{\preceq_{wk}, \preceq_{wk*}, \preceq_{st}\})$, *if the corresponding probability measures are comparable:*

$$\mathrm{p} \preceq_\phi \mathrm{q} \iff \sum_{x \in \Gamma} p[x] \leq \sum_{x \in \Gamma} q[x], \quad \forall \Gamma \in \phi(\mathcal{E})$$

where $\phi(\mathcal{E})$ is the corresponding increasing set: $\phi(\mathcal{E}) \in \{\phi_{wk}(\mathcal{E}), \phi_{wk*}(\mathcal{E}), \phi_{st}(\mathcal{E})\}$.

In fact $\preceq_{st}$ is the well-known sample-path ordering, while $\preceq_{wk}$ corresponds to the comparison of the tail distributions and $\preceq_{wk*}$ to the comparison of cumulative distributions. $\preceq_{wk}$, and $\preceq_{wk^*}$ are weaker than the strong stochastic ordering $\preceq_{st}$, in the sense that $\preceq_{st}$ implies both of them. Moreover, in the case of the totally ordered spaces, they are equivalent to each other.

Definition 3

$$X \preceq_{wk} Y \iff Pr(X \succeq x) \leq Pr(Y \succeq x), \ \forall x \in \mathcal{E}$$

$$X \preceq_{wk^*} Y \iff Pr(X \preceq x) \geq Pr(Y \preceq x), \ \forall x \in \mathcal{E}$$

In the case the random variables are not taking values on the same space, it is possible to compare the images of random variables on a common space.

Definition 4 *Let X (resp. Y) be a random variable taking values in $\mathcal{F}$ (resp. $\mathcal{G}$) defined by a probability vector* p *(resp.* q*). We define two many-to-one applications, α, β. Let $\alpha : \mathcal{F} \to \mathcal{E}$ and $\beta : \mathcal{G} \to \mathcal{E}$. $\alpha(X) \leq_\phi \beta(Y)$, $(\preceq_\phi \in \{\preceq_{wk}, \preceq_{wk*}, \preceq_{st}\})$, if the corresponding probability measures are comparable:*

$$\alpha(\mathrm{p}) \preceq_\phi \beta(\mathrm{q}) \iff \sum_{x|\alpha(x) \in \Gamma} p[x] \leq \sum_{x|\beta(x) \in \Gamma} q[x], \quad \forall \Gamma \in \phi(\mathcal{E})$$

where $\phi(\mathcal{E})$ is the corresponding increasing set: $\phi(\mathcal{E}) \in \{\phi_{wk}(\mathcal{E}), \phi_{wk*}(\mathcal{E}), \phi_{st}(\mathcal{E})\}$.

We are interested in the stochastic comparison of Markov processes. In this work, the stochastic comparison of Markov processes $\{X(t),\ t \geq 0\}$, and $\{Y(t),\ t \geq 0\}$ taking values on the same state space $\mathcal{E}$ is defined as the comparison of the corresponding random variables at each instant:

Definition 5 *We say that Markov process $\{X(t),\ t \geq 0\}$ is less than Markov process $\{Y(t),\ t \geq 0\}$ in the sense of $\preceq_\phi$, that will be noted by $\{X(t),\ t \geq 0\} \preceq_\phi \{Y(t),\ t \geq 0\}$, if*

$$X(t) \preceq_\phi Y(t), \ \forall t \geq 0$$

We consider time-homogeneous Markov processes, and give the definition of the stochastic monotonicity and the comparison of the infinitesimal- generators. Let Q^X (resp. Q^Y) be the infinitesimal generator of $\{X(t),\ t \geq 0\}$ (resp. $\{Y(t),\ t \geq 0\}$).

Definition 6 *Markov process $\{X(t),\ t \geq 0\}$ is said to be stochastically monotone (in the sense of $\preceq_\phi$) if for all probability vectors* $\mathbf{p}$ *and* $\mathbf{q}$ *in* $\mathcal{E}$, *we have*

$$\mathbf{p} \preceq_\phi \mathbf{q} \quad \textit{implies that} \quad \mathbf{p}\ exp(t\ Q^X) \preceq_\phi \mathbf{q}\ exp(t\ Q^X)$$

In the case of time-homogeneous Markov chains, the comparison of Markov chains can be defined in means of the monotonicity and the comparison of the corresponding infinitesimal-generators.

Definition 7 *We denote by $Q^X(x,\ *)$ the row corresponding to state $x \in \mathcal{E}$ of generator Q^X representing the transition rates from state x to each state of $\mathcal{E}$.*

$$Q^X \preceq_\phi Q^Y \Longleftrightarrow Q^X(x, *) \preceq_\phi Q^Y(x,\ *), \quad \forall x \in \epsilon$$

where

$$Q^X(x, *) \preceq_\phi Q^Y(x,\ *) \Longleftrightarrow \sum_{y \in \Gamma} Q^X(x,\ y) \leq \sum_{y \in \Gamma} Q^Y(x,\ y), \ \forall \Gamma \in \phi(\epsilon)$$

The sufficient conditions to compare Markov processes are given as follows (theorem 3.4 of [9]):

Theorem 1 *If the following conditions are satisfied*

1) $X(0) \preceq_\phi Y(0)$,

2) $\{X(t),\ t \geq 0\}$ or $\{Y(t),\ t \geq 0\}$ is $\preceq_\phi$ monotone,

3) $Q^X \preceq_\phi Q^Y$

then $\{X(t)\ t \geq 0\} \preceq_\phi \{Y(t),\ t \geq 0\}$.

2.2. G-Networks

In the last years, the G-networks proposed by Gelenbe has been extended by different authors (see the book [11]). We consider here G-Network with n queues which is the extension of the Jackson networks with "positive" and "negative" customers. Positive customers have the same behavior as customers in the Jackson networks: they are waiting in the queue or they are in service. Negative customers delete positive customers, and go out.

For each queue i, we have the following parameters:

1) λ_i^-: is the external Poisson arrival rate of negative customers. The effect of this arrival is to destroy one positive customer in queue i.

2) λ_i^+: is the external Poisson arrival rate of positive customers. The effect of this arrival is to increase by one the number of positive customers in queue i.

3) μ_i: is the mean exponential service rate. After a service, a positive customer can

- depart from the network with the probability d_i
- go to queue j as a positive customer with probability P_{ij}^+
- go to queue j as a negative customer with probability P_{ij}^-.

Thus for each queue i, we have the following relation:

$$\sum_{j=1}^{n} P_{ij}^+ + \sum_{j=1}^{n} P_{ij}^- + d_i = 1$$

It has been proved by Gelenbe [2], and [3] that stationary distributions of the G-networks have product form solutions. Let $\Pi(x_1, .., x_n)$ be the stationary distribution where x_i is the number of positive customers in queue i. If the following system of equations has a solution such that for each i: $0 < q_i < 1$:

$$q_i = \frac{\sum_{j=1}^{n} P_{ji}^+ \mu_j q_j + \lambda_i^+}{\mu_i + \sum_{j=1}^{n} P_{ji}^- \mu_j q_j + \lambda_i^-}$$

the stationary distribution has the following product-form solution:

$$\Pi(x_1, .., x_n) = (1 - q_i) \prod_{i=1}^{n} [q_i]^{x_i}$$

However, their transient behaviors are difficile to carry out. We propose to compare stochastically G-Networks with systems whose transient distributions are known.

3. Bounds on the transient behaviors of the G-networks

Let us recall here that in the case of the Jackson networks, the bounding models are given by means of M/M/1 queues which correspond to homogeneous birth-death processes where λ represents the birth rate while μ represents the death rate. The upper bounding model consists in n independent M/M/1 processes, if there are n queues in the corresponding Jackson network. Obviously, the rates of these independent queues are computed by taking into account the parameters of the underlying network. On the other hand, the number of the customers are lower bounded by a single M/M/1 queue with rates computed from the underlying network.

The bounding models to study the transient behaviors of the G-networks will be constructed in a similar manner. In the case of the G-networks, the construction element will be birth-death processes where deaths may occur by one or two, rather than simple birth-death processes. The upper bounding model is also constructed by a collection of independent such birth-death processes, while the lower bound is studied through a single birth-death process.

In the sequel, the bounding models will be formally defined. The demonstration of the stochastic comparison results are based in theorem 1. In fact, the homogeneous generalized birth-death processes, where the skips are not limited to one step, are monotone in the sense of the sample-path ordering [12]. In [7], Massey has studied the monotonicity of the birth-death processes corresponding to M/M/1 queues with an operator-theoric approach. Moreover, he has established through the same approach, the generator of a multidimensional birth-death processes (births and deaths occur only for a component at once). The $\preceq_{wk}$ monotonicity of the upper bounding model for the Jackson network which is a collection of independent M/M/1 queues has been also proved through this approach [7]. The $\preceq_{wk}$ monotonicity of the collection of independent birth-death processes where deaths are by one or two can be proved in a similar manner, by including an operator to shift two times to the left to the operators which are defined in [7]. Here, we do not give the demonstration of the monotonicity. By applying theorem 1, we establish the comparison results by proving the comparison of the corresponding generators.

3.1. Upper Bounding Model

The state of a G-network with n queues can be represented by a state vector $x \in N^n$, where x_i is the number of customers in queue i, $1 \leq i \leq n$:

$$x = (x_1, \dots, x_i, \dots, x_n)$$

We consider the usual component-wise ordering ($\preceq$) to compare vectors on N^n:

$$x \preceq y \Longleftrightarrow x_i \leq y_i, \quad 1 \leq i \leq n$$

The upper bounding model consists of n independent birth-death processes where deaths may occur by one or two. queues. For each queue i, these rates are computed from the parameters of the ith queue of the corresponding G-network. Let us first give the intuition to compute these rates.

The customer increasing rate in queue i, which means the transition rate from x_i to x_{i+1} :

$$\lambda_i^+ + \sum_{j \neq i} \mu_j P_{ji}^+ \mathbb{1}_{\{x_j > 0\}}$$

The customer decreasing rate by 1 corresponding to the transition rate from x_i to x_{i-1} :

$$\lambda_i^- \mathbb{1}_{\{x_i > 0\}} + \mu_i \mathbb{1}_{\{x_i > 0\}} (1 - P_{ii}^+ - \mathbb{1}_{\{x_i > 1\}} P_{ii}^-) + \sum_{j \neq i} \mu_j P_{ji}^- \mathbb{1}_{\{x_j > 0\}}$$

The customer decreasing rate by 2 corresponding to the transition rate from x_i to x_{i-2} :

$$\mu_i \mathbb{1}_{\{x_i > 1\}} P_{ii}^-$$

Since we construct an upper bound on the underlying G-network, in the upper bounding model the increasing rate must be greater while the decreasing rate must be less than the rates of the underlying G-network. On the other hand, the increasing and decreasing rates in the G-networks are described through the indicator functions. Thus we must replace these values by their extreme values to define the rates in the bounding model.

1) The increasing rate, $\lambda_i^+ + \sum_{j \neq i} \mu_j P_{ji}^+ \mathbb{1}_{\{x_j > 0\}}$ reaches its maximal value when $\mathbb{1}_{\{x_j > 0\}} = 1, \forall j \neq i$. Hence the maximal increasing rate by assuming all queues non-empty is

$$\lambda_i^+ + \sum_{j \neq i} \mu_j P_{ji}^+ \tag{1}$$

2) The decreasing rate by 1, reaches its minimal value when $\mathbb{1}_{\{x_j>0\}} = 0, \forall j \neq i$. The minimal decreasing rate by assuming all other queues empty is

$$\lambda_i^- \mathbb{1}_{\{x_i>0\}} + \mu_i \mathbb{1}_{\{x_i>0\}}(1 - P_{ii}^+ - \mathbb{1}_{\{x_i>1\}} P_{ii}^-) \tag{2}$$

3) The decreasing rate by 2 is

$$\mu_i \mathbb{1}_{\{x_i>1\}} P_{ii}^- \tag{3}$$

Let us now demonstrate formally the stochastic comparison. We denote by:

1) $\{X(t),\ t \geq 0\}$ the Markov process which represents the evolution of the G-Network system, with n queues. Q^X represents the infinitesimal generator of this process.

2) $\{Y(t),\ t \geq 0\}$ the Markov process which represents the evolution of the network of n independent birth-death processes with arrival and service rates given in equations 1, 2,3. Q^Y represents the infinitesimal generator of this process.

We assume that at the beginning tail distribution of both systems are comparable, and demonstrate that this order is preserved each time.

Theorem 2 *If* $X(0) \preceq_{wk} Y(0)$, *then*

$$\{X(t),\ t > 0\} \preceq_{wk} \{Y(t),\ t > 0\}$$

Proof: We use here the monotonicity of the bounding model. Therefore, the stochastic comparison of the processes can be established through theorem 1 by demonstrating the comparison of the corresponding generators. In the case of the Jackson networks, the comparison of generators has been established analytically [6, 8]. We apply here the increasing set approach to prove this comparison. Let us remark that we need to define increasing sets of $\phi_{wk}(N^n)$. However since the state space is infinite, the number of increasing sets of $\phi_{wk}(N^n)$ is also infinite. We propose to define a methodology in order to define a finite number of increasing sets, which are necessary to compare the underlying processes. The main idea is to define each increasing set through events occuring in the system, since transitions occur due to the events. We give now all the events that occur in the systems. Let E_l the set of events which occur in queue l:

$$E_l = \{ev_l,\ ev_{l^+},\ ev_{l^-},\ ev_{l^{--}},\ ev_{kl^+},\ ev_{kl^-}\}$$

The impacts of these events in queue l are as follows:

1) event ev_l: the number of customers of queue l does not change.

2) event ev_l+: the number of customers of queue l increases by 1, corresponding to an arrival.

3) event ev_l-: the number of customers of queue l decreases by 1, following a service.

4) event ev_l--: the number of customers of queue l decreases by 2, following a service.

5) event $ev_{kl}+$: the number of customers of queue k decreases by 1, and the number of customers of file l increases by 1. A customer of queue k joins queue l.

6) event $ev_{kl}+$: the number of customers of queue l and k decrease by 1: a negative customer leaves queue k, deletes a positive customer of queue l, and disappears.

With each of these events, we associate an increasing set of $\phi_{wk}(I\!N^n)$, defined from a state $x \in I\!N^n$. Let e_i be a vector in $I\!N^n$, which all the components equal to 0, except the ith one which equals to 1:

$$\begin{aligned}
\Gamma_{ev_l} &= \{x\}\uparrow \\
\Gamma_{ev_l+} &= \{x+e_l\}\uparrow \\
\Gamma_{ev_l-} &= \{x-e_l, x_l>0\}\uparrow \\
\Gamma_{ev_l--} &= \{x-e_l-e_l, x_l>1\}\uparrow \\
\Gamma_{ev_{kl}+} &= \{x-e_k+e_l, k\neq l x_k>0\}\uparrow \\
\Gamma_{ev_{kl}-} &= \{x-e_k-e_l, k\neq l, x_k>0 and x_l>0\}\uparrow
\end{aligned}$$

To compare the corresponding generators in the sense of the $\preceq_{wk}$, we must compare for each state x through each increasing set belonging to $\phi_{wk}(I\!N^n)$. However, transition rates are non-null only for the increasing sets defined above, so it is sufficient to compare only through them. Therefore the generators are comparable, if

$$\forall\Gamma\in A, \forall x\in I\!N^n, \sum_{y\in\Gamma} Q^X(x,y) \leq \sum_{y\in\Gamma} Q^Y(x,y)$$

where $A = \{\Gamma_{ev_l}, \Gamma_{ev_l+}, \Gamma_{ev_l-}, \Gamma_{ev_l--}\Gamma_{ev_{kl}+}, \Gamma_{ev_{kl}-}\} \subset \phi_{wk}(I\!N^n)$

1) Increasing set Γ_{ev_l}:

$$\begin{aligned}
\sum_{y\in\Gamma_{ev_l}} Q^X(x,y) &= Q^X(x,x)+\sum_{i=1}^{n} Q^X(x,x+e_i)\\
&= -\sum_{i=1}^{n}\lambda_i^+ - \sum_{i=1}^{n}\mu_i \mathbb{1}_{\{x_i>0\}}(1-P_{ii}^+) - \sum_{i=1}^{n}\lambda_i^- \mathbb{1}_{\{x_i>0\}} + \sum_{i=1}^{n}\lambda_i^+\\
&= -\sum_{i=1}^{n}\mu_i \mathbb{1}_{\{x_i>0\}}(1-P_{ii}^+) - \sum_{i=1}^{n}\lambda_i^- \mathbb{1}_{\{x_i>0\}}
\end{aligned}$$

$$\begin{aligned}
\sum_{y\in\Gamma_{ev_l}} Q^Y(x,y) &= Q^Y(x,x)+\sum_{i=1}^{n} Q^Y(x,x+e_i)\\
&= -\sum_{i=1}^{n}\lambda_i^+ - \sum_{i=1}^{n}\sum_{j\neq i}\mu_j P_{ji}^+ - \sum_{i=1}^{n}\mu_i \mathbb{1}_{\{x_i>0\}}(1-P_{ii}^+)\\
&- \sum_{i=1}^{n}\lambda_i^- \mathbb{1}_{\{x_i>0\}} + \sum_{i=1}^{n}\lambda_i^+ + \sum_{i=1}^{n}\sum_{j\neq i}\mu_j P_{ji}^+\\
&= -\sum_{i=1}^{n}\mu_i \mathbb{1}_{\{x_i>0\}}(1-P_{ii}^+) - \sum_{i=1}^{n}\lambda_i^- \mathbb{1}_{\{x_i>0\}}\\
&\Longrightarrow \sum_{y\in\Gamma_{ev_l}} Q^X(x,y) = \sum_{y\in\Gamma_{ev_l}} Q^Y(x,y)
\end{aligned}$$

2) Increasing set $\Gamma_{ev_{l+}}$:

$$\sum_{y\in\Gamma_{e_{l+}}} Q^X(x,y) = Q^X(x,x+e_l) = \lambda_l^+$$

$$\sum_{y\in\Gamma_{e_{l+}}} Q^Y(x,y) = Q^Y(x,x+e_l) = \lambda_l^+ + \sum_{j\neq l}\mu_j P_{jl}^+$$

$$\Longrightarrow \sum_{y\in\Gamma_{e_{l+}}} Q^X(x,y) \leq \sum_{y\in\Gamma_{e_{l+}}} Q^Y(x,y)$$

3) Increasing set Γ_{ev_l-} :

$$\begin{aligned}
\textstyle\sum_{y\in\Gamma_{e_l-}} Q^X(x,y) &= Q^X(x,x-e_l) + Q^X(x,x) + \textstyle\sum_{i\neq l} Q^X(x,x-e_l+e_i) \\
&\quad + \textstyle\sum_i Q^X(x,x+e_i) \\
&= -\textstyle\sum_{i\neq l}\mu_i \mathbb{1}_{\{x_i>0\}}(1-P_{ii}^+) - \textstyle\sum_{i\neq l}\lambda_i^- \mathbb{1}_{\{x_i>0\}} \\
&\quad - \textstyle\sum_{j\neq l}\mu_l P_{lj}^- \mathbb{1}_{\{x_j>0\}} - \mu_l P_{ll}^- \mathbb{1}_{\{x_l>1\}}
\end{aligned}$$

$$\begin{aligned}
\textstyle\sum_{y\in\Gamma_{e_l-}} Q^Y(x,y) &= Q^Y(x,x-e_l) + Q^Y(x,x) + \textstyle\sum_i Q^Y(x,x+e_i) \\
&= -\textstyle\sum_{i\neq l}\mu_i \mathbb{1}_{\{x_i>0\}}(1-P_{ii}^+) - \textstyle\sum_{i\neq l}\lambda_i^- \mathbb{1}_{\{x_i>0\}} - \mu_l P_{ll}^- \mathbb{1}_{\{x_l>1\}}
\end{aligned}$$

$$\Longrightarrow \sum_{y\in\Gamma_{e_l-}} Q^X(x,y) \leq \sum_{y\in\Gamma_{e_l-}} Q^Y(x,y)$$

4) Increasing set $\Gamma_{ev_{l--}}$:

$$\begin{aligned}
\textstyle\sum_{y\in\Gamma_{e_{l--}}} Q^X(x,y) &= Q^X(x,x-e_l-e_l) + Q^X(x,x-e_l) + Q^X(x,x) \\
&\quad + \textstyle\sum_{i\neq l} Q^X(x,x-e_l+e_i) + \textstyle\sum_i Q^X(x,x+e_i) \\
&= -\textstyle\sum_{i\neq l}\mu_i \mathbb{1}_{\{x_i>0\}}(1-P_{ii}^+) - \textstyle\sum_{i\neq l}\lambda_i^- \mathbb{1}_{\{x_i>0\}} - \textstyle\sum_{j\neq l}\mu_l P_{lj}^- \mathbb{1}_{\{x_j>0\}}
\end{aligned}$$

$$\begin{aligned}
\textstyle\sum_{y\in\Gamma_{e_{l--}}} Q^Y(x,y) &= Q^X(x,x-e_l-e_l) + Q^Y(x,x-e_l) + Q^Y(x,x) + \textstyle\sum_i Q^Y(x,x+e_i) \\
&= -\textstyle\sum_{i\neq l}\mu_i \mathbb{1}_{\{x_i>0\}}(1-P_{ii}^+) - \textstyle\sum_{i\neq l}\lambda_i^- \mathbb{1}_{\{x_i>0\}}
\end{aligned}$$

$$\Longrightarrow \sum_{y\in\Gamma_{e_{l--}}} Q^X(x,\ y) \leq \sum_{y\in\Gamma_{e_{l--}}} Q^Y(x,\ y)$$

5) Increasing set $\Gamma_{ev_{kl+}}$:

$$\textstyle\sum_{y\in\Gamma_{e_{kl+}}} Q^X(x,y) = Q^X(x,x+e_l) + Q^X(x,x-e_k+e_l) = \lambda_l^+ + \mu_k P_{kl}^+$$

$$\textstyle\sum_{y\in\Gamma_{e_{kl+}}} Q^Y(x,y) = Q^Y(x,x+e_l) = \lambda_l^+ + \textstyle\sum_{i\neq l}\mu_i P_{il}^+$$

$$\Longrightarrow \sum_{y\in\Gamma_{e_{kl+}}} Q^X(x,\,y) \le \sum_{y\in\Gamma_{e_{kl+}}} Q^Y(x,\,y)$$

6) Increasing set $\Gamma_{ev_{kl-}}$:

$$\begin{aligned}
\textstyle\sum_{y\in\Gamma_{e_{kl-}}} Q^X(x,y) &= Q^X(x,x-e_k-e_l)+Q^X(x,x-e_k)+Q^X(x,x-e_l)+Q^X(x,x)\\
&\quad +\textstyle\sum_{i\neq l} Q^X(x,x-e_l+e_i)+\sum_{i\neq k} Q^X(x,x-e_k+e_i)\\
&\quad +\textstyle\sum_i Q^X(x,x+e_i)\\
&= -\textstyle\sum_{i\neq k,i\neq l}\mu_i\,\mathbb{1}_{\{x_i>0\}}(1-P^-_{ii}-\sum_{j\neq k,j\neq l}\mu_l P^-_{lj}\,\mathbb{1}_{\{x_j>0\}})\\
&\quad -\textstyle\sum_{j\neq l,j\neq k}\mu_k P^-_{kj}\mathbb{1}_{\{x_j>0\}}-\mu_l P^-_{ll}\,\mathbb{1}_{\{x_l>1\}}\\
&\quad -\mu_k P^-_{kk}\mathbb{1}_{\{x_k>1\}}-\textstyle\sum_{i\neq l,i\neq k}\lambda^-_i\,\mathbb{1}_{\{x_i>0\}}
\end{aligned}$$

$$\begin{aligned}
\textstyle\sum_{y\in\Gamma_{e_{kl-}}} Q^Y(x,y) &= Q^Y(x,x-e_k)+Q^Y(x,x-e_l)+Q^Y(x,x)+\textstyle\sum_i Q^Y(x,x+e_i)\\
&= -\textstyle\sum_{i\neq l,i\neq k}\mu_i\mathbb{1}_{\{x_i>0\}}(1-P^+_{ii}-\sum_{i\neq l,i\neq k}\lambda^-_i\,\mathbb{1}_{\{x_i>0\}})
\end{aligned}$$

$$\Longrightarrow \sum_{y\in\Gamma_{e_{kl-}}} Q^X(x,y) \le \sum_{y\in\Gamma_{e_{kl-}}} Q^Y(x,y)$$

Since these inequalities are satisfied for all of the defined increasing sets, we have the comparison of the generators. Therefore from theorem 1, by considering the monotonicity of Q^Y, if $X(0) \preceq_{wk} Y(0)$, then the stochastic order is preserved all the time:

$X(t) \preceq_{wk} Y(t),\ \forall t \ge 0$

Remark 1 *The stochastic order $\preceq_{wk}$ between $\{X(t),\ t\ge 0\}$ and $\{Y(t),\ t\ge 0\}$ can not be extended to the sample-path ordering ($\preceq_{st}$). We will show this through the following increasing set:*

$\Gamma_{st} = \{y \mid y \succeq x,\ or\ y \succeq x-e_i+e_j,\ \forall i\neq j,\}$

which belongs to $\phi_{st}(\mathcal{E})$, but not to $\phi_{wk}(\mathcal{E})$. The transition rate of the G-Network to Γ_{st}, is equal to:

$$\begin{aligned}\sum_{y\in\Gamma_{st}} Q^X(x,\ y) &= \sum_{i=1}^n \sum_{j\neq i} Q^X(x, x-e_i+e_j) + Q^X(x,\ x) + \sum_{i=1}^n Q^X(x,\ x+e_i) \\ &= -\sum_{i=1}^n \lambda_i^- \mathbb{1}_{\{x_i>0\}} - \sum_{i=1}^n \mu_i \mathbb{1}_{\{x_i>0\}}(1-P_{ii}^+) + \sum_{i=1}^n \mu_i \mathbb{1}_{\{x_i>0\}} \sum_{j\neq i} P_{ij}^+ \\ &= \sum_{i=1}^n \lambda_i^- \mathbb{1}_{\{x_i>0\}} - \sum_{i=1}^n \mu_i \mathbb{1}_{\{x_i>0\}}(1-\sum_{j=1}^n P_{ij}^+)\end{aligned}$$

And in the upper bound it is:

$$\begin{aligned}\sum_{y\in\Gamma_{st}} Q^Y(x,\ y) &= Q^Y(x,\ x) + \sum_{i=1}^n Q^Y(x,\ x+e_i) \\ &= -\sum_{i=1}^n \lambda_i^- \mathbb{1}_{\{x_i>0\}} - \sum_{i=1}^n \mu_i \mathbb{1}_{\{x_i>0\}}(1-P_{ii}^+)\end{aligned}$$

Since $(1-\sum_{j=1}^n P_{ij}^+) \leq (1-P_{ii}^+)$,

$$\sum_{y\in\Gamma_{st}} Q^X(x,\ y) > \sum_{y\in\Gamma_{st}} Q^Y(x,\ y).$$

This inequality contradicts with the other ones corresponding to the weak ordering, so we deduce that the strong stochastic ordering $\preceq_{st}$ cannot exist between $\{X(t),\ t\geq 0\}$ and $\{Y(t),\ t\geq 0\}$.

3.2. Lower Bounding Model

In this section, we bound from down the total number of customers in the G-Networks. First let us explain briefly the evolution of the total number of customers in a G-network:

- it increases by one due to a positive customer arrival,
- it decreases by one when a customer is served and it leaves the network, or a negative customer coming from a queue joins an empty queue, or finally when a negative customer coming from outside destroys a positive customer.
- it decreases by two when a negative customer coming from a queue joins a non-empty queue, and destroys a positive customer.

As in the previous section, we denote by $\{X(t), t \geq 0\}$ the underlying G-network. The lower bounding model is one queue where the service is by batch of one or two, and the number of customers changes as follows:

- it increases by one with a rate λ.
- it decreases by one with a rate $\mu\, p$, and by two with a rate $\mu\,(1-p)$.

Thus, the lower bounding model $\{Y(t), t \geq 0\}$ takes values on N. On the other hand, $\{X(t), t \geq 0\}$ takes values on N^n. In fact, we demonstrate that the sum of the number of customers of the underlying G-network is bounded from down by $\{Y(t), t \geq 0\}$, for some values of λ, μ and p. First, we define the sum function $S : N^n \rightarrow$ as follows:

$$x = (x_1, \ldots, x_n) \rightarrow S(x) = \sum_{i=1}^{n} x_i$$

Obviously, the comparison is established on N, which is totally ordered. Hence the comparison will be in the sense of $\leq_{st}$.

Theorem 3

$$Y(t) \geq 0 \leq_{st} S(X(t)), \quad \forall t$$

if

$$\lambda \leq \sum_{i=1}^{n} \lambda_i^+$$

$$\mu \geq \sum_{i=1}^{n} \mu_i d_i + \sum_{i=1}^{n} \lambda_i^- + \sum_{i=1}^{n} \sum_{j=1}^{n} \mu_i P_{ij}^-$$

$$p \leq 1 - \frac{\sum_{i=1}^{n} \sum_{j=1}^{n} P_{ij}^-}{\mu}$$

Proof First of all, let us remark that we apply here the same proof approach as the bounding model which is based on the increasing sets. In fact, in this case which consists in establishing $\leq_{st}$ on N, it would be possible to apply the related coupling techniques[12]. Since we are especially interested in studying the increasing set approach, we demonstrate the lower bounding model through this approach.

The lower bounding model which is a generalized homogeneous birth death process is $\leq_{st}$ monotone[12]. By applying theorem 1, we must compare the corresponding infinitesimal generators. However the processes are not defined on the same state space, we compare the lower bounding process with the sum of the components of the considered G-network. As in the upper bounding case, we define increasing sets through events occuring in the system. Let E be the set of events:

$$E = \{ev_0,\ ev_{+1},\ ev_{-1},\ ev_{-2}\}$$

The impacts of these events on the sum of the number of customers are as follows:

1) event ev_0: the sum does not change.

2) event ev_{+1}: the sum increases by 1.

3) event ev_{-1}: the sum decreases by 1.

4) event ev_{-2}: the sum decreases by 2.

From a state $x' \in I\!N$, we define these increasing sets.

- $\Gamma_{ev_0} = \{x'\}\uparrow$
- $\Gamma_{ev_{+1}} = \{x' + 1\}\uparrow$
- $\Gamma_{ev_{-1}} = \{x' - 1, x' > 0\}\uparrow$
- $\Gamma_{ev_{-2}} = \{x' - 2, x' > 1\}\uparrow$

The comparison of the images of generators are established (definition 2.1) by demonstrating for all increasing sets $\forall\Gamma = \{\Gamma_{ev_0},\ \Gamma_{ev_{+1}},\ \Gamma_{ev_{-1}}, \Gamma_{ev_{-2}}\}$

$$\sum_{y\in\Gamma} Q^Y(x',\ y) \leq \sum_{z|S(z)\in\Gamma} Q^X(x,\ z) \tag{4}$$

$$forall\ \ x \in I\!N^n,\ \ forall\ \ x' \in I\!N \ \text{ such that } x' = S(x)$$

First we give the transition rates for the considered G-Network:

$$\sum_{z|S(z)=S(x)+1} Q^X(x,z) = \sum_{i=1}^n \lambda_i^+$$

$$\sum_{z|S(z)=S(x)} Q^X(x,z) = -\sum_{i=1}^n \lambda_i^+ - \sum_{i=1}^n \mu_i \mathbb{1}_{\{x_i>0\}}(1-\sum_{j=1}^n P_{ij}^+) - \sum_{i=1}^n \lambda_i^- \mathbb{1}_{\{x_i>0\}}$$

$$\sum_{z|S(z)=S(x)-1} Q^X(x,z) = \sum_{i=1}^n \mu_i d_i \mathbb{1}_{\{x_i>0\}} + \sum_{i=1}^n \sum_{j\neq i} \mu_i P_{ij}^- \mathbb{1}_{\{x_i>0\}} \mathbb{1}_{\{x_j=0\}}$$

$$+ \sum_{i=1}^n \lambda_i^- \mathbb{1}_{\{x_i>0\}} + \sum_{i=1}^n \mu_i P_{ii}^{-1} \mathbb{1}_{\{x_i=1\}}$$

$$\sum_{z|S(z)=S(x)-2} Q^X(x,z) = \sum_{i=1}^n \sum_{j\neq i} \mu_i P_{ij}^- \mathbb{1}_{\{x_i>0\}} \mathbb{1}_{\{x_j>0\}} + \sum_{i=1}^n \mu_i P_{ii}^{-1} \mathbb{1}_{\{x_i=1\}}$$

We now give the inequalities for each increasing set as in the upper bounding model.

1) Increasing set $\Gamma_{ev_{+1}}$:

$$\sum_{z|S(z)\in\Gamma_{ev_{+1}}} Q^X(x,z) = \sum_{z|S(z)=S(x)+1} Q^X(x,z)$$

$$= \sum_{i=1}^n \lambda_i^+$$

$$\sum_{y\in\Gamma_{ev_{+1}}} Q^Y(x',y) = \lambda$$

2) Increasing set Γ_{ev_0}:

$$\sum_{z|S(z)\in\Gamma_{ev_0}} Q^X(x,z) = \sum_{z|S(z)=S(x)} Q^X(x,z) + \sum_{z|S(z)=S(x)+1} Q^X(x,z)$$

$$= -\sum_{i=1}^n \mu_i \mathbb{1}_{\{x_i>0\}}(1-\sum_{j=1}^n P_{ij}^+) - \sum_{i=1}^n \lambda_i^- \mathbb{1}_{\{x_i>0\}}$$

$$\sum_{y\in\Gamma_{ev_0}} Q^Y(x',y) = -\mu$$

3) Increasing set $\Gamma_{ev_{-1}}$:

$$\sum_{z|S(z)\in\Gamma_{ev_{-1}}} Q^X(x,z) = \sum_{z|S(z)=S(x)-1} Q^X(x,z) + \sum_{z|S(z)=S(x)} Q^X(x,z) + \sum_{z|S(z)=S(x)+1} Q^X(x,z)$$

$$= -\sum_{i=1}^{n} \mu_i(\mathbb{1}_{\{x_i>0\}} \sum_{j\neq i} P^-_{ij} \mathbb{1}_{\{x_j>0\}} + \mathbb{1}_{\{x_i>1\}} P^-_{ii})$$

$$\sum_{y\in\Gamma_{ev_{-1}}} Q^Y(x',y) = -\mu(1-p)$$

4) Increasing set $\Gamma_{ev_{-2}}$:

$$\sum_{z|S(z)\in\Gamma_{ev_{-2}}} Q^X(x,z) = \sum_{z|S(z)=S(x)-2} + \sum_{z|S(z)=S(x)-1} Q^X(x,z) + \sum_{z|S(z)=S(x)} Q^X(x,z) + \sum_{z|S(z)=S(x)+1} Q^X(x,z) = 0$$

$$\sum_{y\in\Gamma_{ev_{-2}}} Q^Y(x',y) = 0$$

Inequalities 4 are summarized as follows:

$$\lambda \leq \sum_{i=1}^{n} \lambda_i^+ \tag{5}$$

$$\mu \geq \sum_{i=1}^{n} \mu_i \mathbb{1}_{\{x_i>0\}}(1-\sum_{j=1}^{n} P^+_{ij}) + \sum_{i=1}^{n} \lambda_i^- \mathbb{1}_{\{x_i>0\}} \tag{6}$$

$$\mu\,(1-p) \geq \sum_{i=1}^{n} \mu_i \mathbb{1}_{\{x_i>0\}} \sum_{j=1}^{n} P^-_{ij} \mathbb{1}_{\{x_j>0\}} \tag{7}$$

Obviously, these inequalities must be satisfied for all possible values. Therefore for equations 6, 7, we take the indicator functions equal to 1: $\mathbb{1}_{\{x_i>0\}} = \mathbb{1}_{\{x_j>0\}} = 1$. Thus, equations 5, 6 correspond respectively the conditions on λ et μ. We can rewrite equation 7 to give condition on p:

$$p \leq 1 - \frac{\sum_{i=1}^{n}\sum_{j=1}^{n} \mu_i P^-_{ij}}{\mu}$$

Moreover, the following condition must be satisfied to have $0 \leq p \leq 1$:

$$\mu \geq \sum_{i=1}^{n}\sum_{j=1}^{n} \mu_i P^-_{ij}$$

In fact, by replacing $1 - \sum_{j=1}^{n} P_{ij}^{+}$ by $\sum_{j=1}^{n} P_{ij}^{-} + d_i$ in inequality 6:

$$\mu \geq \sum_{i=1}^{n} \mu_i d_i + \sum_{i=1}^{n} \lambda_i^{-} + \sum_{i=1}^{n}\sum_{j=1}^{n} \mu_i P_{ij}^{-} \geq \sum_{i=1}^{n}\sum_{j=1}^{n} \mu_i P_{ij}^{-}$$

thus it completes the proof.

4. Conclusion

In this paper, we are especially interested in the stochastic comparison of multidimensional Markov chains through the increasing set approach. The stationary distribution of both the Jackson networks and the G-networks has a product-form solution. However their transient behaviors are hard to study. Massey has proposed to study the transient behaviors of the Jackson networks through the stochastic comparison approach. In fact, bounding models whose transient behaviors are easier to study are proposed and the stochastic comparison results are established. In this work, following this idea, we study the transient behaviors of the G-networks by applying the stochastic comparison approach. The main difficulty of the stochastic comparison approach comes from the multidimensional state space. We are especially interested in the increasing set methodology and demonstrate the stochastic comparison results by this methodology.

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Streszczenie

W modelowaniu systemów informatycznych szczególną rolę odgrywają łańcuchy Markowa. Stany modelowanego obiektu (np. sieci komputerowej lub jej fragmentu) są odwzorowane przez stany odpowiedniego łańcucha Markowa. Rozwiązując równania łączące prawdopodobieństwa stanów tego łańcucha uzyskujemy prawdopodobieństwa stanów badanego obiektu. Sieć Jacksona to markowowski model sieci stanowisk obsługi reprezentujących elementy systemu informatycznego. Wymagany jest wykładniczy rozkład czasów obsługi i poissonowskie strumienie zgłoszeń. Sieć G to zaproponowane przez E. Gelenbego uogólnienie sieci Jacksona, w której krążą klienci pozytywni, obsługiwani w stanowiskach i klienci negatywni, niszczący przy spotkaniu klientów pozytywnych. Formalizm ten pozwala opisać różne uwarunkowania synchronizacyjne występujące w badanym systemie.

Rozkład stacjonarny dla sieci Jacksona i sieci G ma formę iloczynową, to znaczy prawdopodobieństwo stanu całej sieci wyraża się iloczynem prawdopodobieństw stanów poszczególnych stanowisk w sieci. Mało natomiast wiadomo o prawdopodobieństwach stanów obu sieci w stanie nieustalonym, gdy prawdopodobieństwa stanów zależą od czasu.

W modelowaniu systemów informatycznych, których obciążenie zmienia się nieusta nie, modelowanie stanów nieustalonych jest bardzo ważne. W artykule próbuje się osz cować prawdopodobieństwa stanów nieustalonych w sieciach G poprzez stochastyczn porównywanie wielowymiarowych łańcuchów Markowa, wprowadzając formalizm porzą kowania zbiorów. Jest to rozszerzenie rezultatów Masseya uzyskanych dla sieci Jackson