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Danuta JAMA, Robert CZECH, Barbara JAMA, Michał SZOPA, Roman WITUŁA

Institute of Mathematics
Silesian University of Technology

ON THE POSITIVE SOLUTIONS OF A PARABOLIC EQUATION

Summary. In this paper positive solutions of some weak nonlinear parabolic equations were investigated. Two theorems were proved. One for positive solutions and another includes a prior estimation.

DODATNIE ROZWIĄZANIA RÓWNANIA PARABOLICZNEGO

Streszczenie. Celem pracy jest badanie dodatnich rozwiązań równania parabolicznego z nieliniowością oraz z odpowiednimi warunkami początkowymi i brzegowymi.

The equation is defined in some cylinder $\Omega = \langle 0, T\rangle \times D$, $n+1$ dimensional with axis parallel to t-axis, where: D – bounded, open, connected subset of n-dimensional Euclidean space $\mathbb{R}^n$, $T < \infty$, $\Gamma = \langle 0, T\rangle \times \partial D$ – side boundary

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of cylinder, ∂D – boundary of D, G – down base of the cylinder is considered. We assume the existence and uniqueness of the solution of the initial-boundary problem.

Let us consider the parabolic equation with non-linearity.

$$\frac{\partial u(t,x)}{\partial t} = \sum_{i,j=1}^{n} a_{ij}(t,x)\frac{\partial^2 u(t,x)}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(t,x)\frac{\partial u(t,x)}{\partial x_i} + \\ + c(t,x)\,u(t,x) + g(t,x,u(t,x)) - f(t,x) \qquad \text{for } (t,x) \in \Omega, \quad (1)$$

$$u(0,x) = \varphi_0(x), \qquad \text{for } x \in D, \qquad (2)$$

$$u(t,x) = \psi(t,x), \qquad \text{for } (t,x) \in \Gamma, \qquad (3)$$

where $x = (x_1, x_2, \ldots, x_n)$, $a_{ij}(t,x)$, $b_{ij}(t,x)$, $c(t,x)$, $f(x)$, $\psi(t,x)$, $f(t,x)$, $g(t,x,u(t,x))$, $u(t,x)$ are scalar functions.

Let

$$Lu(t,x) = \sum_{i,j=1}^{n} a_{ij}(t,x)\frac{\partial^2 u(t,x)}{\partial x_i \partial x_j} + \sum_{i,j=1}^{n} b_i(t,x)\frac{\partial u(t,x)}{\partial x_i} + \\ + c(t,x)\,u(t,x) - \frac{\partial u(t,x)}{\partial t}, \quad (4)$$

$$Ku(t,x) = Lu(t,x) + g(t,x,u(t,x)). \qquad (5)$$

System (1)-(3) can be written in the form

$$Ku(t,x) = f(t,x) \qquad \text{for } (t,x) \in \overline{\Omega}, \qquad (6)$$

$$u(0,x) = \varphi_0(x) \qquad \text{for } x \in D, \qquad (7)$$

$$u(t,x) = \psi(t,x) \qquad \text{for } (t,x) \in \Gamma. \qquad (8)$$

Let $f(0,x) = \varphi_0(x)$.

Theorem 1. *If:*

a) function $u(t,x)$ is continuous and has continuous derivatives belonging to the operator L, and

b) for every $u(t,x)$:

w1) $\bigwedge_{(t,x)\in\Omega} g(t,x,u(t,x)) \geqslant 0,$

w2) $\bigvee_{M=const} \bigwedge_{(t,x)\in\Omega} g(t,x,u(t,x)) \leqslant M,$

c) $\bigvee_{M_1=const} \bigwedge_{(t,x)\in\Omega} c(t,x) < M_1$

d) $\bigwedge_{(t,x)\in\Omega} Ku(t,x) \leqslant 0$

e) $\bigwedge_{(t,x)\in\Gamma\cup G} u(t,x) \geqslant 0,$

then $\bigwedge_{(t,x)\in\overline{\Omega}} u(t,x) \geqslant 0.$

Proof. Let us assume that $u(t,x) < 0$ for $(t,x) \in \overline{\Omega}$. If the function $u(t,x)$ would assume a negative minimum in some points belonging to interior of Ω, then there must exist a negative minimum and this minimum must be attained in interior Ω. Next with a maximum principle we have: $Lu\left(t^0,x^0\right) > 0$, where $\left(t^0,x^0\right)$ is a point in which $\min_{(t,x)\in\Omega} u(t,x) = u\left(t^0,x^0\right)$. Because $g(t,x,u(t,x)) \geqslant 0$, then we have $K(t,x) > 0$, and this contradicts the assumption of our theorem. This ends the proof. □

Putting

$$u(t,x) = v(t,x)\, e^{M_1 t}, \tag{9}$$

we have

$$Ku(t,x) = e^{M_1 t} \sum_{i,j=1}^{n} a_{ij}(t,x) \frac{\partial^2 v(t,x)}{\partial x_i \partial x_j} + e^{M_1 t} \sum_{i=1}^{n} b_i(t,x) \frac{\partial v(t,x)}{\partial x_i} +$$

$$+ (c(t,x) - M_1)\, v(t,x)\, e^{M_1 t} - e^{M_1 t} \frac{\partial v(t,x)}{\partial t} + g\left(t,x,v(t,x)\, e^{M_1 t}\right) \tag{10}$$

and

$$g\left(t,x,v(t,x)\, e^{M_1 t}\right) = \overline{g}(t,x,v(t,x)) \geqslant 0, \tag{11}$$

$$\overline{g}(t,x,v(t,x)) \leqslant M, \tag{12}$$

$$c(t,x) - M_1 \leqslant M_2 < 0. \tag{13}$$

So that without limitation of generality we can assume

$$M_1 < 0, \tag{14}$$

may be accepted.

Theorem 2. *If:*

a) assumptions a), b), e) of Theorem 1 are fulfilled,

b) $\bigwedge_{(t,x)\in\Omega} |f(t,x)| \leqslant m$

$c(t,x) \leqslant -C_0 < 0,$

c) $\bigwedge_{(t,x)\in\Gamma} |\Psi(t,x)| < m_1$

$\bigwedge_{x\in\Omega} |\varphi_0(x)| < m_2$ *where* m, C_0, m_1, m_2 *– some positive constants,*

so $\bigwedge_{(t,x)\in\Omega} |u(t,x)| \leqslant S$ *where* $S = \max\left\{m_1, m_2, \frac{2M+m}{C_0}\right\}$.

Proof. Let us put a function in the form

$$w_{\pm}(t,x) = S \pm u(t,x). \tag{15}$$

A) Functions $w_{\pm}(t,x)$ are non-negative for $(t,x) \in \Gamma \cup G$, if $S \geqslant m_1$ and $S \geqslant m_2$:

$$Kw_{\pm}(t,x) = g(t,x,S \pm u(t,x)) \pm Lu(t,x) + Sc(t,x). \tag{16}$$

B) After some transformations we have

$$Kw_{+}(t,x) = g(t,x,S+u(t,x)) - g(t,x,u(t,x)) + g(t,x,u(t,x)) + \\ + Lu(t,x) + Sc(t,x) \tag{17}$$

and from our assumptions we have

$$Kw_{+}(t,x) \leqslant m + 2M - SC_0. \tag{18}$$

For

$$S \geqslant \frac{2M+m}{C_0}, \tag{19}$$

we have

$$Kw_{+}(t,x) \leqslant 0. \tag{20}$$

C) Analogically

$$Kw_{-}(t,x) = g(t,x,S-u(t,x)) + g(t,x,u(t,x)) - g(t,x,u(t,x)) - \\ - Lu(t,x) + Sc(t,x). \tag{21}$$

From above we have

$$Kw_{-}(t,x) \leqslant m + 2M - SC_0 \tag{22}$$

and also for

$$S \geqslant \frac{2M+m}{C_0}, \tag{23}$$

$$Kw_{-}(t,x) \leqslant 0.$$

From Theorem 1 and (15), A), B), C) we have

$$|u(t,x)| \leqslant S, \tag{24}$$

where

$$S = \max\left\{m_1, m_2, \frac{2M+m}{C_0}\right\}. \tag{25}$$

This ends the proof. □

References

1. Ames W.F.: *Non-linear partial differential equations.* Academic Press, New York 1967.
2. Murray J.D.: *Lectures on nonlinear differential equation models in biology.* Claredon Press, Oxford 1978.
3. Pazy A.: *Asymptotic expansions of solutions of ordinary differential equations in Hilbert spaces.* Arch. Rat. Mech. And Anal. **24** (1967), 193–218.
4. Rudin W.: *Functional analysis.* McGraw-Hill, New York 1973.
5. Szarski J.: *Differential inequalities.* PWN, Warszawa 1967.

Streszczenie

Celem pracy było zbadanie rozwiązań dodatnich równania parabolicznego z nieliniowością w postaci:

$$\frac{\partial u(t,x)}{\partial t} = \sum_{i,j=1}^{n} a_{ij}(t,x)\frac{\partial^2 u(t,x)}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(t,x)\frac{\partial u(t,x)}{\partial x_i} + c(t,x)\,u(t,x) + g(t,x,u(t,x)) - f(t,x),$$

z odpowiednimi warunkami początkowymi i brzegowymi. Ustalono warunki gwarantujące nieujemne rozwiązanie powyższego równania i ograniczoność rozwiązań. Rezultaty zawarte zostały w dwóch twierdzeniach.