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GENERALIZATION OF MAXIMUM PRINCIPLE

Summary. The paper presents maximum principle for weak non-linear parabolic equation defined in $n+1$ dimensional cylinder Ω. Only classical solutions were taken into consideration.

UOGÓLNIENIE ZASADY MAKSIMUM

Streszczenie. Celem pracy jest otrzymanie zasady maksimum dla słabo nieliniowego równania parabolicznego, określonego w $n+1$ wymiarowym walcu Ω. Rozważane są klasyczne rozwiązania rozpatrywanego równania.

The paper presents maximum principle for equation

$$\frac{\partial u(t,x)}{\partial t} = \sum_{i,j=1}^{n} a_{ij}(t,x)\frac{\partial^2 u(t,x)}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(t,x)\frac{\partial u(t,x)}{\partial x_i} + \\ + c(t,x)\,u(t,x) + g(t,x,u(t,x)) \quad (1)$$

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with the assumption for non-linearity

$$\bigwedge_{(t,x)\in\Omega} g\left(t,x,u\left(t,x\right)\right) \geqslant 0 \qquad \text{for every } u\left(t,x\right), \tag{2}$$

$$\bigwedge_{(t,x)\in\Omega} g\left(t,x,u\left(t,x\right)\right) \leqslant M \qquad \text{for every } u\left(t,x\right), \tag{3}$$

where M is some positive constant.

We consider the next equation:

$$Lu\left(t,x\right) + g\left(t,x,u\left(t,x\right)\right) = 0. \tag{4}$$

For operator L in the form

$$Lu\left(t,x\right) = \sum_{i,j=1}^{n} a_{ij}\left(t,x\right)\frac{\partial^2 u\left(t,x\right)}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i\left(t,x\right)\frac{\partial u\left(t,x\right)}{\partial x_i} + c\left(t,x\right)u\left(t,x\right) - \frac{\partial u\left(t,x\right)}{\partial t} \tag{5}$$

defined in some cylinder $\Omega = \langle 0,T\rangle \times D$, where: D – bounded, open, connected subset of n-dimensional Euclidean space $\mathbb{R}^n$, $T < \infty$, $\Gamma = \langle 0,T\rangle \times \partial D$ – side boundary of cylinder, ∂D – boundary of D, $a_{ij}\left(t,x\right)$, $b_{ij}\left(t,x\right)$, $c\left(t,x\right)$, $f\left(x\right)$, $\psi\left(t,x\right)$, $f\left(t,x\right)$, $g\left(t,x,u\left(t,x\right)\right)$, $u\left(t,x\right)$ are scalar functions, $x = \left(x_1,x_2,\ldots,x_n\right)$.

The equation (1) can be written in the form

$$Ku\left(t,x\right) = 0, \tag{6}$$

where

$$Ku\left(t,x\right) = Lu\left(t,x\right) + g\left(t,x,u\left(t,x\right)\right). \tag{7}$$

Theorem 1. *If:*

a) function $u\left(t,x\right)$ is continuous together with its derivatives belonging to the operator L,

b) modules of the coefficients of the operator L are bounded,

c) any acceptable function $u\left(t,x\right)$ fulfills:

$$\bigwedge_{(t,x)\in\Omega} g\left(t,x,u\left(t,x\right)\right) \geqslant 0, \tag{8}$$

$$\bigvee_{M=const} \bigwedge_{(t,x)\in\Omega} g\left(t,x,u\left(t,x\right)\right) \leqslant M, \tag{9}$$

d) any constant isn't a solution of the equation in any subset belonging to Ω,

$$\bigwedge_{(t,x)\in\Omega} c(t,x) < 0, \tag{10}$$

then any continuous solution of equation can't attain a positive maximum in the interior of Ω.

Proof. Let the function $u(t,x)$ be a solution of the equation and in some interior point (t^0, x^0) the function $u(t,x)$ attains a positive maxim. Let $x^0 = 0$. If $x^0 \neq 0$, we can translate the system of coordinates to x^0. So that

$$\max_{(t,x)\in\Omega} u(t,x) = u(t^0, x^0) = m. \tag{11}$$

Assume that there exists a point (t', x') such that $u(t', x') = m_1 < m$. Assume that the point (t', x') belongs to some cylinder

$$Q_1 : \left\{0 \leqslant r < \rho;\ t' \leqslant t \leqslant t^0\right\}$$

where: $t^0 - t'$ is sufficiently small, ρ – radius of the cylinder Q_1, $\rho < 1$ and cylinder Q_1 belongs to Ω. If there aren't such points then $u(t,x) = m$ in the all cylinder Q_1. Considering all such cylinders we have an alternative:

in all cylinder $\langle t, t^0\rangle \times D \quad u(t,x) = m$

or

there exists a cylinder in which there are values of $u(t,x)$ smaller than m.

Let us consider the first member of the alternative. If $t^0 = T$, $t^1 = 0$, then

$$\bigwedge_{(t,x)\in\Omega} u(t,x) = m,$$

where $\Omega = \langle 0, T\rangle \times D$, in the order case the proof can be finished in the finite number of steps using the same reasoning in $\langle t^2, t^1\rangle \times D$, $\langle t^3, t^2\rangle \times D, \ldots$, where $t^0 - t^1 = t^1 - t^2 = \ldots$. Then the constant m is a solution of the equation (1) in Ω, and this contradicts the assumption. Let us consider the second member of the alternative and let such cylinder Q_1 exists. We take the auxiliary function:

$$w(t,x) = m - (m - m_1)\left(\rho^2 - r^2\right) e^{-\alpha(t-t_1)} - u(t,x), \tag{12}$$

where: ρ – radius of the base cylinder Q_1, α – some positive constant,

$$r^2 = \sum_{i=1}^{n} x_i^2, \tag{13}$$

$$
\begin{aligned}
Lw(t,x) &= L\left[m-(m-m_1)\left(\rho^2-r^2\right)e^{-\alpha\left(t-t'\right)}-u(t,x)\right]= \\
&= \sum_{i,j=1}^{n} a_{ij}\frac{\partial^2}{\partial x_i\partial x_j}\left[m-(m-m_1)\left(\rho^2-r^2\right)e^{-\alpha\left(t-t'\right)}-u(t,x)\right]+ \\
&+ \sum_{i=1}^{n} b_i(t,x)\frac{\partial}{\partial x_i}\left[m-(m-m_1)\left(\rho^2-r^2\right)e^{-\alpha\left(t-t'\right)}-u(t,x)\right]+ \\
&+ c(t,x)\left[m-(m-m_1)\left(\rho^2-r^2\right)e^{-\alpha\left(t-t'\right)}-u(t,x)\right]+ \\
&- \frac{\partial}{\partial t}\left[m-(m-m_1)\left(\rho^2-r^2\right)e^{-\alpha\left(t-t'\right)}-u(t,x)\right], \qquad (14)
\end{aligned}
$$

so that

$$
\begin{aligned}
Lw(t,x) &= 2(m-m_1)e^{-\alpha\left(t-t'\right)}\sum_{i=1}^{n} a_{ii} - \sum_{i,j=1}^{n} a_{ij}(t,x)\frac{\partial^2}{\partial x_i\partial x_j}u(t,x)+ \\
&+ 2(m-m_1)e^{-\alpha\left(t-t'\right)}\sum_{i=1}^{n} b_i(t,x)x_i - \sum_{i=1}^{n} b_i(t,x)\frac{\partial}{\partial x_i}u(t,x)+ \\
&- (m-m_1)e^{-\alpha\left(t-t'\right)}\left(\rho^2-r^2\right)c(t,x) - c(t,x)u(t,x) + c(t,x)m+ \\
&- \alpha(m-m_1)\left(\rho^2-r^2\right)e^{-\alpha\left(t-t'\right)} + \frac{\partial}{\partial t}u(t,x) \qquad (15)
\end{aligned}
$$

and

$$
\begin{aligned}
Lw(t,x) &= c(t,x)m - Lu(t,x) + \\
&- (m-m_1)e^{-\alpha\left(t-t'\right)}\left[-2\sum_{i=1}^{n} a_{ii} - 2\sum_{i=1}^{n} b_i(t,x)x_i+\right. \\
&\left. + \left(\rho^2-r^2\right)c(t,x) + \alpha\left(\rho^2-r^2\right)\right]. \qquad (16)
\end{aligned}
$$

Because $c(t,x)m<0$, so that

$$
Lw(t,x) \leqslant -Lu(t,x) - (m-m_1)e^{-\alpha\left(t-t'\right)}\left[-h(t,x)+\alpha\left(\rho^2-r^2\right)\right], \qquad (17)
$$

where

$$
h(t,x) = 2\sum_{i=1}^{n} a_{ij}(t,x) + 2\sum b_i(t,x)x_i - \left(\rho^2-r^2\right)c(t,x), \qquad (18)
$$

and $|h(t,x)|$ is bounded. It can be written

$$
\begin{aligned}
Kw(t,x) &= Lw(t,x) + g(t,x,w) \leqslant \\
&\leqslant -(m-m_1)e^{-\alpha\left(t-t'\right)}\left\{-h(t,x)+\alpha\left(\rho^2-r^2\right)\right\} - Lu(t,x) - g(t,x,u) + \\
&+ g(t,x,u) + g(t,x,w), \qquad (19)
\end{aligned}
$$

$$Kw\left(t,x\right)\leqslant-\left(m-m_{1}\right)e^{-\alpha\left(t-t'\right)}\left\{-h\left(t,x\right)+\alpha\left(\rho^{2}-r^{2}\right)\right\}+ \\ -Ku\left(t,x\right)+2M. \quad (20)$$

Let us select some cylinder Q_2 co-axis with Q_1 belonging to the interior of cylinder Q_1 and such that the distance between their boundaries is constant and equals some sufficient small positive number and let $\left(t^0,x^0\right)\in Q_2$. For sufficiently large α and

$$\bigwedge_{(t,x)\in Q_2} -h\left(t,x\right)+\alpha\left(\rho^{2}-r^{2}\right)$$

is positive and can be sufficiently large. Let $t-t'$ be sufficiently small and α large enough that

$$\left(m-m_{1}\right)e^{-\alpha\left(t-t'\right)}\left\{-h\left(t,x\right)+\alpha\left(\rho^{2}-r^{2}\right)\right\}>2M, \quad (21)$$

from (11)-(12)

$$Kw\left(t,x\right)\leqslant0 \qquad \text{for every } \left(t,x\right)\in Q_{2}. \quad (22)$$

On the boundary side of cylinder Q_1 we have:

$$w\left(t,x\right)=m-u\left(t,x\right)\geqslant0. \quad (23)$$

Equality can't be fulfilled because then constant is a solution of the investigates equation, what contradicts the assumption d). So that

$$w\left(t,x\right)>0. \quad (24)$$

Let on the down base of $Q_1 u\left(t,x\right)\leqslant m_1$. Then for $t=t^1$ we have:

$$w\left(t,x\right)=m-\left(m-m_{1}\right)\left(\rho^{2}-r^{2}\right)-u\left(t,x\right)>m-\left(m-m_{1}\right)-m=0 \quad (25)$$

and

$$\bigwedge_{(t,x)\in\partial Q_1} w\left(t,x\right)>0, \quad (26)$$

from (22) and (26) we have: on the boundary Q_1, $w\left(t,x\right)>0$ and $Kw\left(t,x\right)\leqslant0$ in the interior of the cylinder Q_2. Because the function $w\left(t,x\right)$ is continuous in the cylinder Ω and positive on the boundary of $Q_1\subset\Omega$, so $w\left(t,x\right)$ is positive in some neighborhood of the boundary of the cylinder Q_1. If we take out the cylinder Q_2 sufficiently near Q_1, we can say that the function $w\left(t,x\right)$ is non-negative on the boundary of Q_1. From Theorem 1 from the paper [4] we have:

$$\bigwedge_{(t,x)\in Q_2} w\left(t,x\right)\geqslant0, \quad (27)$$

from (12) and (27) we have

$$\bigwedge_{(t,x)\in Q_2} u(t,x) \leqslant m - (m - m_1)\left(\rho^2 - r^2\right) e^{-\alpha\left(t-t'\right)}. \tag{28}$$

Because $\left(t^0, x^0\right) \in Q_1$, $\left(t^0, x^0\right) \in Q_2$ so from (26) we have

$$u\left(t^0, x^0\right) < m \tag{29}$$

and this contradicts the assumption that $u\left(t^0, x^0\right) = m$. So $u(t,x) = m$ in all Q_2, and that contradicts assumption d). This ends the proof. □

References

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Streszczenie

Celem pracy było otrzymanie zasady maksimum dla słabo nieliniowego równania parabolicznego w postaci:

$$\frac{\partial u(t,x)}{\partial t} = \sum_{i,j=1}^{n} a_{ij}(t,x)\frac{\partial^2 u(t,x)}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(t,x)\frac{\partial u(t,x)}{\partial x_i} + \\ + c(t,x)\,u(t,x) + g(t,x,u(t,x)),$$

określonego w $n+1$ wymiarowym walcu Ω. Rozważane były klasyczne rozwiązania powyższego równania. Głównym rezultatem pracy jest:

Twierdzenie. Jeżeli funkcja $u(t,x)$ jest klasycznym rozwiązaniem badanego równania i:

a) $\bigwedge\limits_{(t,x)\in\Omega} g(t,x,u(t,x)) \geqslant 0$ dla każdego $u(t,x)$,

b) $\bigwedge\limits_{(t,x)\in\Omega} g(t,x,u(t,x)) \leqslant M$, M – pewna stała dodatnia,

c) $\bigwedge\limits_{(t,x)\in\Omega} c(t,x) < 0$,

d) współczynniki $a_{ij}(t,x)$, $b_i(t,x)$, $c(t,x)$ są ograniczone co do modułu dla $(t,x) \in \Omega$,

e) stała nie jest rozwiązaniem równania w dowolnym obszarze zawartym w Ω,

to dowolne rozwiązanie badanego równania nie może przyjmować dodatniego maksimum wewnątrz Ω.