ZESZYTY NAUKOWE POLITECHNIKI ŚLĄSKIEJ 2010
Seria: MATEMATYKA-FIZYKA z. 92 Nr kol. 1829

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# ON THE STABILITY OF DISCRETE-CONTINUOUS SYSTEMS

**Summary**. A stability estimation is established for mixed discrete-continuous system with parabolic operator. One theorem is proved with the help of generalization maximum principle.

# STABILNOŚĆ UKŁADÓW DYSKRETNO-CIĄGŁYCH

**Streszczenie**. W artykule badana jest stabilność dyskretno-ciągłego układu ze sprzężeniem brzegowym, wykorzystuje się uogólnioną zasadę maksimum. Otrzymane zostają warunki stabilności w sensie Lapunova-Mowczana.

We consider the system in the form:

$$\frac{\partial u(t,x)}{\partial t} = \sum_{i,j=1}^{n} a_{ij}(t,x)\frac{\partial^2 u(t,x)}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(t,x)\frac{\partial u(t,x)}{\partial x_i} + c(t,x)\,u(t,x) + g(t,x,u(t,x)) \quad (1)$$

2010 Mathematics Subject Classification: 35B35, 35K55.
Wpłynęło do Redakcji 27.03.2010 r.

for $(t,x) \in \Omega$,

$$u(0,x) = \varphi_0(x) = \eta(x) \cdot |\rho(0)| \qquad \text{for } x \in D, \tag{2}$$

$$u(t,x) = \eta(x) \cdot |\rho(t)| \qquad \text{for } (t,x) \in \Gamma, \tag{3}$$

$$\frac{d\xi(t)}{dt} = f(t,\xi(t)) + \int_D k(x) \frac{\partial u(t,x)}{\partial t} dx \qquad \text{for } t \geqslant 0, \tag{4}$$

where: $D$ – bounded, open, connected subset $n$-dimensional Euclidean space $\mathbb{R}^n$, $T < \infty$, $\Omega = \langle 0,T\rangle \times D$, $\Gamma = \langle 0,T\rangle \times \partial D$ – side boundary of cylinder $\Omega$, $\partial D$ – boundary, $G$ – down base of the cylinder $\Omega$, $\eta(x)$, $\xi(t)$, $f(t)$, $k(x)$, $a_{ij}(t,x)$, $b_i(t,x)$, $c(t,x)$, $g(t,x,u(t,x))$, $u(t,x)$, $\varphi_0(x)$ – scalar functions. By the norms $\|\bullet\|$ for the scalar functions we have understood the maximum from the module of these functions for $(t,x) \in \Omega$.

**Theorem 1.** *If*

*a)* $\bigwedge_{\substack{(t,x)\in\Omega \\ x\in D}} a_{ij}(t,x),\ b_i(t,x),\ g(t,x,u(t,x)),\ \varphi_0(x),\ k(x)$ *are bounded and* $c(t,x) < 0$, $g(t,x,u(t,x)) \geqslant 0$,

*b)* $\bigwedge_{t\geqslant 0} |f(t,\xi(t))| \leqslant \alpha(t)|\xi(t)|$, *and* $\int_0^\infty \alpha(t)\,dt < d < \infty$,

*c)* $\bigwedge_{x\in D} d_1 \leqslant \eta(x) \leqslant d_2$, $d_1, d_2$ *– some positive constants,*

*d)* $\max_{x\in D} |k(x)| \max_{x\in D} |\eta(x)| \cdot |D| < 1$,

*then the solution of the system (1)-(4) is stable in the Lapunov sense.*

*Proof.* Under the assumptions of the theorem the generalized maximum principle is true. The function $u(x,t)$ attains the maximum

A) on the down base of the cylinder – $G$ and then

$$\max_{(t,x)\in\Omega} |u(t,x)| \leqslant \max_{x\in\overline{D}} |\varphi_0(x)|, \tag{5}$$

or

B) on the side boundary of the cylinder – $\Gamma$ and then

$$\max_{(t,x)\in\Omega} |u(t,x)| \leqslant \max_{\substack{x\in\overline{D} \\ t\geqslant 0}} |\eta(x)|\,|\xi(t)|. \tag{6}$$

Let A)be true, then inequality (5) is fulfilled. Integrating in the interval $\langle 0,T\rangle$ and taking the norm of both sides we have under the assumption b) the inequality

$$|\xi(t)| \leqslant |\varphi_0(x)| + \int_0^t \alpha(\tau)\,|\xi(\tau)|\,d\tau + \\ + \int_D |k(t)|\,|u(t,x)|\,dx + \int_D |k(t)|\,|\varphi_0(x)|\,dx, \quad (7)$$

so

$$|\xi(t)| \leqslant |\varphi_0(x)| + 2\max_{x\in\overline{D}} |k(t)| \max_{x\in\overline{D}} |\varphi_0(x)| \cdot |D| + \int_0^t \alpha(\tau)\,|\xi(\tau)|\,d\tau \quad (8)$$

with (8) and from Gronwall-Belmann inequality we have

$$|\xi(t)| \leqslant \left[|\xi(0)| + 2\max_{x\in\overline{D}} |k(t)| \max_{x\in\overline{D}} |\varphi_0(x)| \cdot |D|\right] \exp \int_0^t \alpha(\tau)d\tau, \quad (9)$$

so that from (9) we have

$$\|\xi(\tau)\| \leqslant \max|\varphi_0(x)| \left[\frac{1}{d_1} + 2\max_{x\in D}|k(x)| \cdot |D|\right] d, \quad (10)$$

putting

$$m = \left[\frac{1}{d_1} + 2\max_{x\in D}|k(x)| \cdot |D|\right] d, \quad (11)$$

we have

$$\|\xi(\tau)\| \leqslant m\,\|\varphi_0(x)\|\,; \quad (12)$$

Taking the norm $\|(u|\xi)\|$ in the form

$$\|(u|\xi)\| = \|u(t,x)\| + \|\xi(t)\|\,, \quad (13)$$

where: $\|u(t,x)\| = \max_{(t,x)\in\Omega} |u(t,x)|\,,\ \ \|\xi(t)\| = \max_{t\in\langle 0,T\rangle} |\xi(t)|$ from inequality (5) we have

$$\|(u|\xi)\| \leqslant \|\varphi_0(x)\| + m\,\|\varphi_0(x)\| \quad (14)$$

where: $\|\varphi_0(x)\| = \max_{x\in D} |\varphi_0(x)|$ Putting $\overline{m} = 1 + m$ we have

$$\|(u|\xi)\| \leqslant \overline{m}\,\|\varphi_0(x)\|\,, \quad (15)$$

and we have Theorem if A) is fulfilled.

If B) is fulfilled that is if inequality (6) is fulfilled analogically as in A) we may get inequality in the form

$$|\xi(t)| \leqslant |\varphi_0(x)| + \int_0^t \alpha(\tau)|\xi(\tau)|\,d\tau + \max_{x\in\overline{D}}|k(x)|\max_{x\in\overline{D}}|\eta(x)|\,|\xi(t)|\cdot|D| + \\ + \max_{x\in\overline{D}}|k(x)|\max_{x\in\overline{D}}|\varphi_0(x)|\cdot|D|, \quad (16)$$

so

$$|\xi(t)|\left[1-\max_{x\in\overline{D}}|k(x)|\max_{x\in\overline{D}}|\eta(x)|\cdot|D|\right] \leqslant \\ \leqslant |\xi(0)| + \max_{x\in\overline{D}}|k(x)|\max_{x\in\overline{D}}|\varphi_0(x)|\cdot|D| + \int_0^t \alpha(\tau)|\xi(\tau)|\,d\tau. \quad (17)$$

Under the assumption d) we have

$$1-\max_{x\in\overline{D}}|k(x)|\max_{x\in\overline{D}}|\eta(x)|\cdot|D| = C > 0. \quad (18)$$

And putting $C_1 = \frac{1}{C}$ we get

$$|\xi(t)| \leqslant C_1\left[|\xi(0)| + \max_{x\in\overline{D}}|k(x)|\max_{x\in\overline{D}}|\varphi_0(x)|\cdot|D|\right] + C_1\int_0^t \alpha(\tau)|\xi(\tau)|\,d\tau. \quad (19)$$

From Gronwall-Bellman inequality we have

$$|\xi(t)| \leqslant C_1\left[|\xi(0)| + \max_{x\in\overline{D}}|k(x)|\max_{x\in\overline{D}}|\varphi_0(x)|\cdot|D|\right]\exp C_1\int_0^t \alpha(\tau)d\tau, \quad (20)$$

and from b) there exists a constant $M$ such that

$$\exp C_1\int_0^t \alpha(\tau)d\tau < M. \quad (21)$$

Putting $C_1\cdot M = C_2$ we have

$$|\xi(t)| \leqslant C_2\left[|\xi(0)| + \max_{x\in\overline{D}}|k(x)|\max_{x\in\overline{D}}|\varphi_0(x)|\cdot|D|\right]. \quad (22)$$

From (22) and assumptions of our Theorem we have

$$\|\xi(t)\| \leqslant C_2 \max_{x\in D}|\varphi_0(x)|\left[\frac{1}{d_1} + \max_{x\in D}|k(x)|\cdot|D|\right], \quad (23)$$

putting

$$m_1 = C_2 \left[ \frac{1}{d_1} + \max_{x \in D} |k(x)| \cdot |D| \right], \tag{24}$$

we have

$$\|\xi(t)\| \leqslant m_1 \|\varphi_0(x)\|. \tag{25}$$

From inequalities (6), (25) putting $m_2 = m_1 d_2$ we have

$$\|u(t,x)\| \leqslant m_2 \|\varphi_0(x)\|. \tag{26}$$

Using the norm given by (13), from inequalities (25), (26) we have

$$\|(u|\xi)\| \leqslant \overline{\overline{m}} \|\varphi_0(x)\| \text{ for } \overline{\overline{m}} = (m_1 + m_2), \tag{27}$$

so the proof in the case B) is ended. □

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## Streszczenie

W pracy zbadana została stabilność dyskretno-ciągłego układu ze sprzężeniem brzegowym w postaci:

$$\frac{\partial u(t,x)}{\partial t} = \sum_{i,j=1}^{n} a_{ij}(t,x)\frac{\partial^2 u(t,x)}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(t,x)\frac{\partial u(t,x)}{\partial x_i} + \\ + c(t,x)\,u(t,x) + g(t,x,u(t,x))$$

dla $(t,x) \in \Omega$,

$$u(0,x) = \varphi_0(x) = \eta(x)\cdot|\rho(0)| \qquad \text{dla } x \in D,$$

$$u(t,x) = \eta(x)\cdot|\rho(t)| \qquad \text{dla } (t,x) \in \Gamma,$$

$$\frac{d\xi(t)}{dt} = f(t,\xi(t)) + \int_D k(x)\frac{\partial u(t,x)}{\partial t}dx \qquad \text{dla } t \geqslant 0,$$

gdzie: $D$ – ograniczony, otwarty, spójny podzbiorów $n$-wymiarowej przestrzeni Euklidesowej $\mathbb{R}^n$; $T < \infty$ $\Omega = \langle 0,T\rangle \times D$, $\Gamma = \langle 0,T\rangle \times \partial D$ – boczna powierzchnia walca $\Omega$; $\partial D$ – brzeg obszaru $D$; $G$ – dolna podstawa walca $\Omega$; $\eta(x)$, $\xi(t)$, $f(t)$, $k(x)$, $a_{ij}(t,x)$, $b_i(t,x)$, $c(t,x)$, $g(t,x,u(t,x))$, $u(t,x)$, $\varphi_0(x)$ – funkcje skalarne. Wykorzystując uogólnioną zasadę maksimum z pracy [5], otrzymuje się warunki stabilności w sensie Lapunowa-Mowczana.