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STABILITY OF SOLUTIONS OF MIXED INTEGRAL-DIFFERENTIAL SYSTEMS

Summary. Stability of some mixed integral-differential system with parabolic operator are investigated by generalization of maximum principle.

STABILNOŚĆ ROZWIĄZAŃ UKŁADÓW MIESZANYCH RÓŻNICZKOWO-CAŁKOWYCH

Streszczenie. W artykule badana jest stabilność układu mieszanego, złożonego z parabolicznego równania rzędu drugiego i układu równań różniczkowo-całkowych. Wykorzystując uogólnioną zasadę maksimum, ustalono warunki dla stabilności rozwiązań w sensie Lapunova-Mowczana.

We consider the system of equations defined in the cylinder $\Omega = \langle 0, T\rangle \times D$ $n+1$ dimensional, with the axis parallel to t – axis, where: D – bounded, open, connected subset n-dimensional Euclidean space $\mathbb{R}^n$, $T < \infty$, $\Omega = \langle 0, T\rangle \times D$,

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$\Gamma = \langle 0, T \rangle \times \partial D$ – side boundary of cylinder Ω, ∂D – boundary, G – the base of the cylinder lying in the hyperplane $t = 0$, $x = [x_1, x_2, \ldots, x_n]$. The norm $\|.\|$ for the vector functions is the Euclidean norm, for scalar functions is the maximum norm. Let us consider the mixed system with nonlinearity in the form:

$$\frac{\partial u(t,x)}{\partial t} = \sum_{i,j=1}^{n} a_{ij}(t,x) \frac{\partial^2 u(t,x)}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(t,x) \frac{\partial u(t,x)}{\partial x_i} + \\ + c(t,x)\, u(t,x) + g(t,x,u(t,x)) \qquad \text{for } (t,x) \in \Omega, \tag{1}$$

$$u(0,x) = \varphi_0(x) = \eta(x) \cdot |\xi(0)| \qquad \text{for } x \in D, \tag{2}$$

$$u(t,x) = \eta(x) \cdot |\xi(t)| \qquad \text{for } (t,x) \in \Gamma, \tag{3}$$

$$\frac{d\xi(t)}{dt} = (A + B(t))\,\xi(t) + f(t,\xi(t)) + \int_D h(t,x)\, u(t,x) dx, \tag{4}$$

where: A – constant matrix, $B(t)$ – function matrix, $\xi(t)$, $f(t,\xi(t))$, $h(t,x)$ – vector functions, $c(t,x)$, $g(t,x,u(t,x))$, $u(t,x)$, $\eta(t,x)$ – scalar functions, $a_{ij}(t,x)$, $b_i(t,x)$, $c(t,x)$ – coefficients of the equation – scalar functions.

Let

$$Lu(t,x) = \sum_{i,j=1}^{n} a_{ij}(t,x) \frac{\partial^2 u(t,x)}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(t,x) \frac{\partial u(t,x)}{\partial x_i} + \\ + c(t,x)\, u(t,x) - \frac{\partial u(t,x)}{\partial t}, \tag{5}$$

system (1)-(4) can be written in the form

$$Lu(t,x) + g(t,x,u(t,x)) = 0, \tag{6}$$

$$u(0,x) = \varphi_0(x) = \eta(x) \cdot |\xi(0)| \qquad \text{for } x \in D, \tag{7}$$

$$u(t,x) = \eta(x) \cdot |\xi(t)| \qquad \text{for } (t,x) \in \Gamma, \tag{8}$$

$$\frac{d\xi(t)}{dt} = (A + B(t))\,\xi(t) + f(t,\xi(t)) + \int_D h(t,x)\, u(t,x) dx. \tag{9}$$

Only the classical solutions are taken into consideration. Using the generalization of maxim principle from paper [6] the next theorem can be established.

Theorem 1. *If*

a) coefficients of the parabolic operator L *are bounded by module for* $(t,x) \in \Omega$, *and* $c(t,x) < 0$ *for* $(t,x) \in \Omega$,

b) $g(t,x,u(t,x)) \geqslant 0$, $g(t,x,u(t,x)) \leqslant M$ *for* $(t,x) \in \Omega$, M *– some positive constant,*

c) any constant isn't a solution of equation (6),

d) $0 \leqslant \eta(x) \leqslant d$ *for* $x \in D$, d *– a positive constant,*

e) $\|h(t,x)\|$, $\|B(t)\|$ *are bounded for* $(t,x) \in \Omega$, $t \geqslant 0$,

f) $\|f(t,\xi(t))\| \leqslant \overline{c}\,\|\xi(t)\|$ *for* $t \geqslant 0$, $\overline{c}$ *– some positive constant,*

g) the Cauchy matrix $X(t,\tau)$ *generated by* A *fulfills the estimation*

$$\|X(t,\tau)\| \leqslant C \exp\left[-\gamma(t-\tau)\right], \qquad t \geqslant \tau \geqslant 0,$$

γ, C *– positive constants and*

$$\gamma \geqslant M_1 C$$

where

$$\|B(t)\| + \overline{c} + \|\eta(x)\| \int_D \|h(t,x)\| dx \leqslant M_1 \qquad \text{for } t \geqslant 0$$

Then solution of the system (1)-(4) is stable in Mowczan's sense.

Proof. Using our assumptions and results of the papers [5,6] we have that the solution of equation (1) is non-negative inside Ω and attains the maximum on the boundary Ω.

1^0 If the maximum is attained on the down base of the cylinder Ω, then

$$\|u(t,x)\| \leqslant \|u(0,x)\| \leqslant \|\eta(x) \cdot |\xi(0)|\| \leqslant \|\eta(x)\| \, \|\xi(0)\| \tag{10}$$

2^0 If the maximum is attained on the side boundary then

$$\|u(t,x)\| \leqslant \|\eta(x) \cdot |\xi(t)|\| \leqslant \|\eta(x)\| \, \|\xi(t)\| \tag{11}$$

Let 1^0 be fulfilled so inequality (10) is fulfilled. Using the operator $X(t,\tau)$ to (4) we have

$$\xi(t) = X(t,0)\,\xi(0) + \int_0^t X(t,\tau)\,B(\tau)\xi(\tau)\,d\tau + \int_0^t X(t,\tau)\,f(\tau,\xi(\tau))\,d\tau + \\ + \int_0^t X(t,\tau)\left[\int_D h(t,\tau)\,u(\tau,x)\,dx\right]d\tau. \quad (12)$$

Taking the norm and using the assumption g) we have

$$\|\xi(t)\| \leqslant C\exp(-\gamma t)\,\|\xi(0)\| + C\int_0^t \exp[-\gamma(t-\tau)]\,\|B(\tau)\|\,\|\xi(\tau)\|\,d\tau + \\ + C\int_0^t \exp[-\gamma(t-\tau)]\,\|f(\tau,\xi(\tau))\|\,d\tau + \\ + C\int_0^t \exp[-\gamma(t-\tau)]\left[\int_D \|h(\tau,x)\|\,\|u(\tau,x)\|\,dx\right]d\tau. \quad (13)$$

Using (10) we have:

$$\|\xi(t)\| \leqslant C\exp(-\gamma t)\,\|\xi(0)\| + C\int_0^t \exp[-\gamma(t-\tau)]\,\|B(\tau)\|\,\|\xi(\tau)\|\,d\tau + \\ + C\int_0^t \exp[-\gamma(t-\tau)]\,\|f(\tau,\xi(\tau))\|\,d\tau + \\ + C\,\|\eta(x)\|\,\|\xi(0)\|\int_0^t \exp[-\gamma(t-\tau)]\left[\int_D \|h(\tau,x)\|dx\right]d\tau. \quad (14)$$

Using the assumption f) we have

$$\|\xi(t)\| \leqslant C\exp(-\gamma t)\,\|\xi(0)\|\cdot\left[1 + \int_0^t \exp(\gamma\tau)\,[\|B(\tau)\| + \overline{c}]\,d\tau + \right. \\ \left. + \|\eta(x)\|\int_0^t \exp(\gamma\tau)\left[\int_D \|h(t,x)\|\,dx\right]d\tau\right] \quad (15)$$

and so

$$\|\xi(t)\| \leqslant C\exp(-\gamma t)\|\xi(0)\|\cdot\left[1+\int_0^t \exp(\gamma\tau)\left[\|B(\tau)\|+\overline{c}+ + \|\eta(x)\|\int_D \|h(t,x)\|\,dx\right]d\tau\right]. \quad (16)$$

From the assumption that $\|B(\tau)\|$, $\|\eta(x)\|$, $\|h(t,x)\|$ are bounded, there exists a constant M_1 such that

$$\|\beta(\tau)\|+\overline{c}+\|\eta(x)\|\int_D \|h(t,x)\|\,dx \leqslant M_1 \quad (17)$$

and so

$$\|\xi(t)\| \leqslant C\|\xi(0)\|\exp(-\gamma t)\left[1+M_1\int_0^t \exp(\gamma\tau)d\tau\right]. \quad (18)$$

After some simple transformations and putting

$$C_1 = C\left(1+\frac{M_1}{\gamma}\right), \quad (19)$$

we have

$$\|\xi(t)\| \leqslant C_1\|\xi(0)\|. \quad (20)$$

Using the Gronwall-Bellman inequality and assumption (14) we have

$$\lim_{t\to\infty}\|\xi(t)\| = 0. \quad (21)$$

Let 2^0 be fulfilled. Then from (13) and (11) we have:

$$\|\xi(t)\| \leqslant C\exp(-\gamma t)\|\xi(0)\| + C\int_0^t \exp[-\gamma(t-\tau)]\|B(\tau)\|\,\|\xi(\tau)\|\,d\tau+ + C\int_0^t \exp[-\gamma(t-\tau)]\|\xi(\tau,\xi)\|\,d\tau+ + C\|\eta(x)\|\int_0^t \exp[-\gamma(t-\tau)]\left[\int_D \|h(\tau,x)\|dx\right]\|\xi(\tau)\|\,d\tau. \quad (22)$$

Under the assumption f) after some simple transformations we have

$$\|\xi(t)\| \leqslant C\exp(-\gamma t)\|\xi(0)\| + \\ + C\int_0^t \left[\|B(\tau)\| + \overline{c} + \|\eta(x)\| \int_D \|h(\tau,x)\| dx\right] \|\xi(\tau)\| \, d\tau. \quad (23)$$

Using the Gronwell-Bellman inequality and (17) from (23) we have

$$\|\xi(t)\| \leqslant C\|\xi(0)\| \exp[-t(\gamma - M_1 C)], \quad (24)$$

for $\gamma \geqslant M_1 C$:

$$\|\xi(t)\| \leqslant C\|\xi(0)\|, \quad (25)$$

for $\gamma > M_1 C$ from (24) we have

$$\lim_{t\to\infty} \|\xi(t)\| = 0. \quad (26)$$

Joining the results for 1^0 and 2^0 from (10), (11), (20), (25) we have the inequality

$$\|u(t,x)\| < m\|\xi(0)\| \|\eta(x)\|, \quad (27)$$

where $m = \max\{1, C, C_1\}$. Let

$$\|(u|\xi)\| = \|u(t,x)\| + \|\xi(t)\|. \quad (28)$$

Using the above estimations we have

$$\|(u|\xi)\| = m\|n(x)\| \|\xi(0)\| + m\|\xi(0)\| = m[\|\eta(x)\| + 1] \|\xi(0)\|, \quad (29)$$

and putting

$$\overline{m} = m[\|\eta(x)\| + 1], \quad (30)$$

we have

$$\|(u|\xi)\| \leqslant \overline{m}\|\xi(0)\|, \quad (31)$$

so that solution of the system is stable in the Mowczan sense [8,9]. □

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Streszczenie

Rozpatrzono mieszany dyskretno-ciągły układ w postaci:

$$\frac{\partial u(t,x)}{\partial t} = \sum_{i,j=1}^{n} a_{ij}(t,x)\frac{\partial^2 u(t,x)}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(t,x)\frac{\partial u(t,x)}{\partial x_i} + \\ + c(t,x)\,u(t,x) + g(t,x,u(t,x))$$

dla $(t,x) \in \Omega$,

$$u(0,x) = \varphi_0(x) = \eta(x)\cdot|\xi(0)| \qquad \text{dla } x \in D,$$

$$u(t,x) = \eta(x)\cdot|\xi(t)| \qquad \text{dla } (t,x) \in \Gamma,$$

$$\frac{d\xi(t)}{dt} = (A + B(t))\,\xi(t) + f(t,\xi(t)) + \int_D h(t,x)\,u(t,x)dx,$$

gdzie: $c(t,x)$, $g(t,x,u(t,x))$, $u(t,x)$, $\eta(t,x)$ – funkcje skalarne; $\xi(t)$, $f(t,\xi(t))$, $h(t,x)$ – wymiarowe funkcje wektorowe; $A, B(t)$ – $n \times n$ wymiarowe macierze. Rozpatrzyliśmy wyłącznie klasyczne rozwiązania powyższego układu. Układ został określony w $n+1$ wymiarowym walcu $\Omega = \langle 0,T\rangle \times D$, $D \in R^n$, $x = [x_1, x_2, \ldots, x_n]$. Wykorzystując uogólnioną zasadę maksimum z pracy [6], ustalone zostały warunki stabilności rozwiązań w sensie Lapunova-Mowczana.