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CONNECTIONS BETWEEN THE PRIMITIVE 5-TH ROOTS OF UNITY AND FIBONACCI NUMBERS – PART I

Summary. In this paper a new method of investigating the identities for Fibonacci and Lucas numbers is presented. This method is based on some fundamental identities for powers of the number called the golden ratio and the conjugate number to it. Those identities give also possibility to connect in the interesting way Fibonacci and Lucas numbers with many important sequences of numbers, like Bernoulli numbers, binomial coefficients or δ-Fibonacci numbers.

ZWIĄZKI POMIĘDZY PIERWIASTKAMI PIERWOTNYMI STOPNIA PIĄTEGO Z JEDYNKI A LICZBAMI FIBONACCIEGO – CZĘŚĆ I

Streszczenie. W artykule przedstawiono nową metodę dowodzenia tożsamości dla liczb Fibonacciego i Lucasa. Bazuje ona na pewnych fundamentalnych tożsamościach dla potęg liczby zwanej złotą proporcją oraz liczby do niej sprzężonej. Tożsamości te pozwalają także ciekawie powiązać liczby Fibonacciego i Lucasa z wieloma ważnymi ciągami liczbowymi, takimi jak: liczby Bernoullego, współczynniki dwumianowe czy liczby δ-Fibonacciego.

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1. Introduction

The author's fascination with Fibonacci, Lucas and complex numbers has been reflected in the following two nice identities (discovered independently by Rabinowitz [13] and Grzymkowski and Wituła [10]):

$$(1+\xi+\xi^4)^n = F_{n+1}+F_n(\xi+\xi^4) \quad \text{and} \quad (1+\xi^2+\xi^3)^n = F_{n+1}+F_n(\xi^2+\xi^3), \tag{1}$$

where $\xi^5 = 1$, $\xi \in \mathbb{C}$ and $\xi \neq 1$, and F_n denote the nth Fibonacci number.

The scope of the present paper is to convince the readers, that these identities constitute a new profound method of proving and generating another identities, properties as well as other facts concerning Fibonacci and Lucas numbers. Moreover, these identities seem to enable providing shorter, clearer and more creative proofs. They offer the alternative kinds of proofs based on Binet's expressions [11, 16], matrix representations [6, 7, 12, 15], the generating functions methods [11, 18] and finally purely combinatoric considerations [2].

Let us also notice, that the identities (1) have been already used by other researchers. For example S. Alikhani and Y. Peng [1] on the ground of the (1) have proven, that α^n, where $\alpha := \frac{1+\sqrt{5}}{2}$ and $n \in \mathbb{N}$ (α is called the golden ratio [3]) can not be a root of any chromatic polynomials. R. Wituła [21] has used the identities (1) for generalization of the definition of Fibonacci and Lucas numbers for the real indices, which, afterwards, has he applied to extend the concept of the power of matrix for the real exponents. Meanwhile R. Wituła and D. Słota [19] have discussed the one parameter generalization of the identities (1):

$$\big(1+\delta(\xi+\xi^4)\big)^n = a_n(\delta)+b_n(\delta)(\xi+\xi^4) \tag{2}$$

and

$$\big(1+\delta(\xi^2+\xi^3)\big)^n = a_n(\delta)+b_n(\delta)(\xi^2+\xi^3), \tag{3}$$

where the specified above polynomials $a_n(\delta)$ and $b_n(\delta)$, $n = 0, 1, \ldots$, called δ-Fibonacci numbers, satisfy the following reccurent relations:

$$a_0(\delta) = 1, \qquad b_0(\delta) = 0,$$
$$\begin{bmatrix} a_{n+1}(\delta) \\ b_{n+1}(\delta) \end{bmatrix} = \begin{bmatrix} 1 & \delta \\ \delta & 1-\delta \end{bmatrix} \begin{bmatrix} a_n(\delta) \\ b_n(\delta) \end{bmatrix}. \tag{4}$$

Finally D. Gerdemann in paper [8] has used the first of identities (1) for analyzing so called Golden Ratio Division Algorithm. In consequence he has received there the semi-combinatorial proof of the following beautiful theorem.

Theorem 1. *For nonconsecutive integers $a_1, \ldots, a_k$, the following two statements are equivalent ($\forall m \in \mathbb{N}$):*

$$m\,F_n = F_{n+a_1} + F_{n+a_2} + \ldots + F_{n+a_k},$$
$$m = \alpha^{a_1} + \alpha^{a_2} + \ldots + \alpha^{a_k}.$$

For example, it can be verified that

$$6 = \alpha^3 + \alpha + \alpha^{-4}, \qquad 7 = \alpha^4 + \alpha^{-4},$$

which, by the theorem implies, the identities

$$6\,F_n = F_{n+3} + F_{n+1} + F_{n-4}$$

and

$$7\,F_n = F_{n+4} + F_{n-4}.$$

In the context of this theorem we propose to the Reader to take a look at the Remark 15., to get acquainted with the important Zeckendorf's theorem.

2. Basic identities

Let us set

$$\alpha := 2\cos\frac{\pi}{5} = \frac{1+\sqrt{5}}{2} \qquad \text{and} \qquad \beta := -2\cos\left(\frac{2}{5}\pi\right) = \frac{1-\sqrt{5}}{2}.$$

Then, we have

$$\alpha + \beta = 1, \qquad \alpha\beta = -1 \tag{5}$$

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \qquad n \in \mathbb{Z}, \tag{6}$$

$$L_n = \alpha^n + \beta^n, \qquad n = 0, 1, 2, \ldots, \tag{7}$$

where L_n denote the nth Lucas number. Now let $\xi \in \mathbb{C}$, $\xi^5 = 1$, $\xi \neq 1$. The following two lemmas form the basic technical collection of formulas and identities, which are explored in the next sections of this paper.

Lemma 2. *The following identities hold*

$$1 + \xi + \xi^2 + \xi^3 + \xi^4 = 0, \tag{8}$$

$$\xi + \xi^4 = \xi + \xi^{-1} = \xi + \overline{\xi} = 2\cos(\arg\xi), \tag{9}$$

$$\xi^2 + \xi^3 = \xi^2 + \xi^{-2} = \xi^2 + \overline{\xi^2} = 2\cos(2\arg\xi), \tag{10}$$

$$1 + \xi + \xi^4 = \alpha \tag{11}$$

and

$$1 + \xi^2 + \xi^3 = -\xi - \xi^4 = 1 - \alpha = \beta. \tag{12}$$

Both above identities hold for $\xi = \exp(i2\pi/5)$.

Lemma 3. *The following identities are satisfied*

a)

$$(1 + \xi + \xi^4)(1 + \xi^2 + \xi^3) = -1, \tag{13}$$

hence

$$(1 + \xi^2 + \xi^3)^{-1} = \xi^2 + \xi^3 \tag{14}$$

and

$$(1 + \xi + \xi^4)^{-1} = \xi + \xi^4, \tag{15}$$

$$(1 - \xi - \xi^4)(1 - \xi^2 - \xi^3) = 1, \tag{16}$$

$$(1 + \xi + \xi^4)(1 - \xi^2 - \xi^3) = 3 + 2(\xi + \xi^4), \tag{17}$$

and in the more general form

$$\big(a + b(\xi + \xi^4)\big)\big(c + d(\xi^2 + \xi^3)\big) = ac - ad - bf + (bc - ad)(\xi + \xi^4); \tag{18}$$

b)

$$(1 + \xi + \xi^4)^2 - 1 = 1 + \xi + \xi^4 \tag{19}$$

$$\frac{1 + \xi + \xi^4}{1 + \xi^2 + \xi^3} = -(1 + \xi + \xi^4)^2 = -(2 + \xi + \xi^4); \tag{20}$$

c)

$$(1 + \xi + \xi^4)^k(1 + \xi^2 + \xi^3)^m + (1 + \xi^2 + \xi^3)^k(1 + \xi + \xi^4)^m = \\ = F_{k+1}L_m - F_kL_{m+1} = (-1)^kL_{m-k}. \tag{21}$$

d)

$$2(1 + \xi + \xi^4)^k(1 + \xi^2 + \xi^3)^m = (-1)^kL_{m-k} + \sqrt{5}(-1)^{k+1}F_{m-k}. \tag{22}$$

e)

$$\frac{a+b(\xi+\xi^4)}{c+d(\xi+\xi^4)}=\frac{bd+ad-ac}{d^2-c^2+cd}+\frac{ad-bc}{d^2-c^2+cd}(\xi+\xi^4). \tag{23}$$

Proof.

ad a) We have

$$(1+\xi+\xi^4)(1+\xi^2+\xi^3)=1+2(\xi+\xi^2+\xi^3+\xi^4)=-1+2(1+\xi+\xi^2+\xi^3+\xi^4)=-1;$$

ad c) By (1) we obtain

$$\begin{aligned}
&(1+\xi+\xi^4)^k(1+\xi^2+\xi^3)^m+(1+\xi^2+\xi^3)^k(1+\xi+\xi^4)^m\\
&\quad=\left[(1+\xi+\xi^4)(1+\xi^2+\xi^3)\right]^k\left[(1+\xi^2+\xi^3)^{m-k}+(1+\xi+\xi^4)^{m-k}\right]\\
&\quad=(-1)^k\left[F_{m-k+1}+F_{m-k}(\xi^2+\xi^3)+F_{m-k+1}+F_{m-k}(\xi+\xi^4)\right]\\
&\quad=(-1)^k\left[2F_{m-k+1}-F_{m-k}\right]=(-1)^kL_{m-k}.
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
&(1+\xi+\xi^4)^k(1+\xi^2+\xi^3)^m+(1+\xi+\xi^4)^m(1+\xi^2+\xi^3)^k\\
&\quad=\left(F_{k+1}+F_k(\xi+\xi^4)\right)\left(F_{m+1}+F_m(\xi^2+\xi^3)\right)\\
&\quad+\left(F_{k+1}+F_k(\xi^2+\xi^3)\right)\left(F_{m+1}+F_m(\xi+\xi^4)\right)\\
&\quad=2F_{k+1}F_{m+1}+F_{k+1}F_m(\xi+\xi^2+\xi^3+\xi^4)+F_kF_{m+1}(\xi+\xi^2+\xi^3+\xi^4)\\
&\quad+2F_kF_m(\xi+\xi^4)(\xi^2+\xi^3)\\
&\quad=2F_{k+1}F_{m+1}-F_{k+1}F_m-F_kF_{m+1}-2F_kF_m\\
&\quad=F_{k+1}(2F_{m+1}-F_m)-F_k(F_{m+1}+2F_m)=F_{k+1}L_m-F_kL_{m+1}.
\end{aligned}$$

It should be noticed that (22) also implies (21):

$$(-1)^kL_{m-k}+(-1)^mL_{k-m}=(-1)^kL_{m-k}+(-1)^m(-1)^{k-m}L_{m-k}=2(-1)^kL_{m-k}$$

and

$$(-1)^{k+1}F_{m-k}+(-1)^{m+1}F_{k-m}=(-1)^{k+1}F_{m-k}+(-1)^{m+1}(-1)^{k-m+1}F_{m-k}=0.$$

ad d) Similarly as above, by applying identities (1), we obtain

$$\begin{aligned}
&(1+\xi+\xi^4)^k(1+\xi^2+\xi^3)^m=\left(F_{k+1}+F_k(\xi+\xi^4)\right)\left(F_{m+1}+F_m(\xi^2+\xi^3)\right)\\
&\quad=F_{k+1}F_{m+1}+F_kF_{m+1}(\xi+\xi^4)+F_{k+1}F_m(\xi^2+\xi^3)+F_kF_m(\xi+\xi^4)(\xi^2+\xi^3)\\
&\quad=F_{k+1}F_{m+1}-F_kF_m-F_{k+1}F_m+(F_kF_{m+1}-F_{k+1}F_m)(\xi+\xi^4)
\end{aligned}$$

$$= F_{k+1}F_{m-1} - F_kF_m + (-1)^{k+1}F_{m-k}(\xi+\xi^4)$$
$$= (-1)^k F_{m-k-1} + (-1)^{k+1}F_{m-k}(\xi+\xi^4)$$
$$= \frac{1}{2}(-1)^k(2F_{m-k-1}+F_{m-k}) + \frac{\sqrt{5}}{2}(-1)^{k+1}F_{m-k}$$
$$= \frac{1}{2}(-1)^k L_{m-k} + \frac{\sqrt{5}}{2}(-1)^{k+1}F_{m-k}.$$

□

Remark 4. The following identities hold

$$(1+\xi+\xi^2)^n = \xi^{n-1}(F_n + F_{n+1}\xi + F_n\xi^2), \tag{24}$$

$$(1+\xi+\xi^3)^n = \xi^{3n}(F_{n+1} + F_n\xi^2 + F_n\xi^3) \tag{25}$$

and

$$(1+\xi^2+\xi^4)^n = \xi^{2n}(F_{n+1} + F_n\xi^2 + F_n\xi^3). \tag{26}$$

Remark 5. (Only the case of (26) is discussed below)
We have (for some $a_n, b_n, c_n \in \mathbb{Z}$):

$$(1+\xi^2+\xi^4)^n = a_n + b_n\xi^2 + c_n\xi^4 \quad \Leftrightarrow \quad n \equiv 1 (\mathrm{mod}\ 5),$$
$$(1+\xi^2+\xi^4)^{5n+1} = F_{5n+1} + F_{5n+2}\xi^2 + F_{5n+1}\xi^4,$$
$$(1+\xi^2+\xi^4)^n = a_n + b_n\xi^3 + c_n\xi^4 \quad \Leftrightarrow \quad n \equiv 2 (\mathrm{mod}\ 5),$$
$$(1+\xi^2+\xi^4)^{5n+2} = F_{5n+2} - F_{5n+2}\xi^3 + F_{5n+1}\xi^4,$$
$$(1+\xi^2+\xi^4)^n = a_n + b_n\xi + c_n\xi^2 \quad \Leftrightarrow \quad n \equiv 3 (\mathrm{mod}\ 5),$$
$$(1+\xi^2+\xi^4)^{5n+3} = -F_{5n+3} + F_{5n+2}\xi - F_{5n+3}\xi^2,$$
$$(1+\xi^2+\xi^4)^n = a_n + b_n\xi + c_n\xi^3 \quad \Leftrightarrow \quad n \equiv 4 (\mathrm{mod}\ 5),$$
$$(1+\xi^2+\xi^4)^{5n+4} = F_{5n+4} + F_{5n+4}\xi + F_{5n+5}\xi^3,$$
$$(1+\xi^2+\xi^4)^n = a_n + b_n\xi^2 + c_n\xi^3 \quad \Leftrightarrow \quad n \equiv 0 (\mathrm{mod}\ 5),$$
$$(1+\xi^2+\xi^4)^{5n} = F_{5n+1} + F_{5n}\xi^2 + F_{5n}\xi^3.$$

Remark 6. We have

$$(1+\xi+\xi^4)^{n+1} + (1+\xi+\xi^4)^{n-1} = L_{n+1} + L_n(\xi+\xi^4) \tag{27}$$

and

$$(1+\xi+\xi^4)^{n+1} - (1+\xi+\xi^4)^{n-1} = F_{n+1} + F_n(\xi+\xi^4). \tag{28}$$

Remark 7. We have (by (27) and (15)):

$$L_{n+1} + L_n(\xi + \xi^4) = \left[(1+\xi+\xi^4) + (1+\xi+\xi^4)^{-1}\right](1+\xi+\xi^4)^n = \\ = \left[2(\xi+\xi^4)+1\right](1+\xi+\xi^4)^n.$$

Hence for $\xi = \exp(2\pi i/5)$, we get

$$L_{n+1} + L_n(\xi + \xi^4) = \sqrt{5}(1+\xi+\xi^4)^n. \tag{29}$$

Another formula can also be deduced

$$L_{n+1} + L_n(\xi^2 + \xi^3) = -\sqrt{5}(1+\xi^2+\xi^3)^n. \tag{30}$$

Remark 8. Both, from the identities (1) (adding or subtracting them properly by sides) like from the identities (29) and (30) (again, adding or subtracting them properly by sides) we can easily generate the Binet formulas (6) and (7). More precisely, the following relations can be easily deduced

identities (1) $\Longleftrightarrow$ (6) and (7) $\Longleftrightarrow$ (29) and (30).

Immediately from identities (1), (29) and (30) the next result follows.

Theorem 9. *Let* $\{k_n\}_{n=1}^{\infty}$ *be a sequence of positive integers. Then the following identities hold true*

$$\prod_{n=1}^{N}\left(F_{k_n+1} + \frac{\sqrt{5}-1}{2}F_{k_n}\right) = \left(\frac{1+\sqrt{5}}{2}\right)^{\sum_{n=1}^{N} k_n},$$

$$\prod_{n=1}^{N}\left(F_{k_n+1} - \frac{\sqrt{5}+1}{2}F_{k_n}\right) = \left(\frac{1-\sqrt{5}}{2}\right)^{\sum_{n=1}^{N} k_n},$$

$$\prod_{n=1}^{N}\left(F_{k_n+1} \pm \sqrt{5}\,F_{k_n} + F_{k_n-1}\right) = 2^N\left(\frac{1\pm\sqrt{5}}{2}\right)^{\sum_{n=1}^{N} k_n},$$

$$\prod_{n=1}^{N}\left(L_{k_n+1} + \frac{\sqrt{5}-1}{2}L_{k_n}\right) = (\sqrt{5})^N\left(\frac{1+\sqrt{5}}{2}\right)^{\sum_{n=1}^{N} k_n},$$

$$\prod_{n=1}^{N}\left(L_{k_n+1} - \frac{\sqrt{5}+1}{2}L_{k_n}\right) = (-\sqrt{5})^N\left(\frac{1-\sqrt{5}}{2}\right)^{\sum_{n=1}^{N} k_n},$$

$$\prod_{n=1}^{N}\left(L_{k_n+1} \pm \sqrt{5}\,L_{k_n} + L_{k_n-1}\right) = (\pm 2\sqrt{5})^N\left(\frac{1\pm\sqrt{5}}{2}\right)^{\sum_{n=1}^{N} k_n}.$$

We note, that these six identities act as link between Fibonacci and Lucas sequences and many other special sequences of positive integers.

Corollary 10. ***(A bridge between Fibonacci, Lucas and Bernoulli numbers)***

We have

$$\prod_{n=1}^{N-1}\left(F_{n^k+1}+\frac{\sqrt{5}-1}{2}F_{n^k}\right)=\left(\frac{1+\sqrt{5}}{2}\right)^{\int_0^N B_k(x)\,dx},$$

$$\prod_{n=1}^{N-1}\left(F_{n^k+1}-\frac{\sqrt{5}+1}{2}F_{n^k}\right)=\left(\frac{1-\sqrt{5}}{2}\right)^{\int_0^N B_k(x)\,dx},$$

$$\prod_{n=1}^{N-1}\left(F_{n^k+1}\pm\sqrt{5}\,F_{n^k}+F_{n^k-1}\right)=2^{N-1}\left(\frac{1\pm\sqrt{5}}{2}\right)^{\int_0^N B_k(x)\,dx},$$

$$\prod_{n=1}^{N-1}\left(L_{n^k+1}+\frac{\sqrt{5}-1}{2}L_{n^k}\right)=(\sqrt{5})^{N}\left(\frac{1+\sqrt{5}}{2}\right)^{\int_0^N B_k(x)\,dx},$$

$$\prod_{n=1}^{N-1}\left(L_{n^k+1}-\frac{\sqrt{5}+1}{2}L_{n^k}\right)=(-\sqrt{5})^{N-1}\left(\frac{1-\sqrt{5}}{2}\right)^{\int_0^N B_k(x)\,dx},$$

$$\prod_{n=1}^{N-1}\left(L_{n^k+1}\pm\sqrt{5}\,L_{n^k}+L_{n^k-1}\right)=(\pm 2\sqrt{5})^{N-1}\left(\frac{1\pm\sqrt{5}}{2}\right)^{\int_0^N B_k(x)\,dx},$$

since

$$\sum_{n=1}^{N-1} n^k=\int_0^N B_k(x)\,dx=\sum_{r=0}^{k}\binom{k}{r}B_r\,\frac{N^{k-r+1}}{k-r+1},$$

where B_r are Bernoulli numbers defined by the following recursion formula [4, 14]:

$$B_0=1,\quad \binom{n}{n-1}B_{n-1}+\binom{n}{n-2}B_{n-2}+\ldots+\binom{n}{0}B_0=0,\qquad n=2,3,\ldots$$

(we note that $B_{2k+1}=0$, $k=1,2,\ldots$) and $B_k(x)$ denote the k-th Bernoulli polynomial defined by

$$B_k(x)=\sum_{l=0}^{k}\binom{k}{l}B_l\,x^{k-l}.$$

Corollary 11. ***(A bridge between Fibonacci numbers, Lucas numbers and binomial coefficients)***

We have

$$\prod_{k=1}^{\lfloor (n+1)/2 \rfloor} \left(F_{\binom{n-k}{k-1}+1} + \frac{\sqrt{5}-1}{2} F_{\binom{n-k}{k-1}} \right) = \left(\frac{1+\sqrt{5}}{2} \right)^{F_n},$$

$$\prod_{k=1}^{\lfloor (n+1)/2 \rfloor} \left(F_{\binom{n-k}{k-1}+1} - \frac{\sqrt{5}+1}{2} F_{\binom{n-k}{k-1}} \right) = \left(\frac{1-\sqrt{5}}{2} \right)^{F_n},$$

$$\prod_{k=1}^{\lfloor (n+1)/2 \rfloor} \left(F_{\binom{n-k}{k-1}+1} \pm \sqrt{5}\, F_{\binom{n-k}{k-1}} + F_{\binom{n-k}{k-1}-1} \right) = 2^{\lfloor (n+1)/2 \rfloor} \left(\frac{1 \pm \sqrt{5}}{2} \right)^{F_n},$$

$$\prod_{k=1}^{\lfloor (n+1)/2 \rfloor} \left(L_{\binom{n-k}{k-1}+1} + \frac{\sqrt{5}-1}{2} L_{\binom{n-k}{k-1}} \right) = (\sqrt{5})^{\lfloor (n+1)/2 \rfloor} \left(\frac{1+\sqrt{5}}{2} \right)^{F_n},$$

$$\prod_{k=1}^{\lfloor (n+1)/2 \rfloor} \left(L_{\binom{n-k}{k-1}+1} - \frac{\sqrt{5}+1}{2} L_{\binom{n-k}{k-1}} \right) = (-\sqrt{5})^{\lfloor (n+1)/2 \rfloor} \left(\frac{1-\sqrt{5}}{2} \right)^{F_n},$$

$$\prod_{k=1}^{\lfloor (n+1)/2 \rfloor} \left(L_{\binom{n-k}{k-1}+1} \pm \sqrt{5}\, L_{\binom{n-k}{k-1}} + L_{\binom{n-k}{k-1}-1} \right) = (\pm 2\sqrt{5})^{\lfloor (n+1)/2 \rfloor} \left(\frac{1 \pm \sqrt{5}}{2} \right)^{F_n},$$

since the following identity hold (see [11]):

$$F_n = \sum_{k=1}^{\lfloor (n+1)/2 \rfloor} \binom{n-k}{k-1}.$$

Corollary 12. *We have*

$$\left(F_{F_{n+1}+1} + \frac{\sqrt{5}-1}{2} F_{F_{n+1}} \right) \left(F_{F_{n-1}+1} + \frac{\sqrt{5}-1}{2} F_{F_{n-1}} \right) = \left(\frac{1+\sqrt{5}}{2} \right)^{L_n},$$

$$\left(F_{F_{n+1}+1} - \frac{\sqrt{5}+1}{2} F_{F_{n+1}} \right) \left(F_{F_{n-1}+1} - \frac{\sqrt{5}+1}{2} F_{F_{n-1}} \right) = \left(\frac{1-\sqrt{5}}{2} \right)^{L_n},$$

$$\left(F_{F_{n+1}+1} \pm \sqrt{5}\, F_{F_{n+1}} + F_{F_{n+1}-1} \right) \cdot$$

$$\cdot \left(F_{F_{n-1}+1} \pm \sqrt{5}\, F_{F_{n-1}} + F_{F_{n-1}-1} \right) = 4 \left(\frac{1 \pm \sqrt{5}}{2} \right)^{L_n},$$

$$\left(L_{F_{n+1}+1} + \frac{\sqrt{5}-1}{2} L_{F_{n+1}} \right) \left(L_{F_{n-1}+1} + \frac{\sqrt{5}-1}{2} L_{F_{n-1}} \right) = 5 \left(\frac{1+\sqrt{5}}{2} \right)^{L_n},$$

$$\left(L_{F_{n+1}+1} - \frac{\sqrt{5}+1}{2} L_{F_{n+1}} \right) \left(L_{F_{n-1}+1} - \frac{\sqrt{5}+1}{2} L_{F_{n-1}} \right) = 5 \left(\frac{1-\sqrt{5}}{2} \right)^{L_n},$$

$$\left(L_{F_{n+1}+1} \pm \sqrt{5}\, L_{F_{n+1}} + L_{F_{n+1}-1}\right) \cdot$$
$$\cdot \left(L_{F_{n-1}+1} \pm \sqrt{5}\, L_{F_{n-1}} + L_{F_{n-1}-1}\right) = 20 \left(\frac{1 \pm \sqrt{5}}{2}\right)^{L_n},$$

since $F_{n+1} + F_{n-1} = L_n$, $n \in \mathbb{N}$.

Corollary 13. *We have*

$$\left(F_{L_{n+1}+1} + \frac{\sqrt{5}-1}{2} F_{L_{n+1}}\right)\left(F_{L_{n-1}+1} + \frac{\sqrt{5}-1}{2} F_{L_{n-1}}\right) = \left(\frac{1+\sqrt{5}}{2}\right)^{5F_n},$$
$$\left(F_{L_{n+1}+1} - \frac{\sqrt{5}+1}{2} F_{L_{n+1}}\right)\left(F_{L_{n-1}+1} - \frac{\sqrt{5}+1}{2} F_{L_{n-1}}\right) = \left(\frac{1-\sqrt{5}}{2}\right)^{5F_n},$$
$$\left(F_{L_{n+1}+1} \pm \sqrt{5}\, F_{L_{n+1}} + F_{L_{n+1}-1}\right) \cdot$$
$$\cdot \left(F_{L_{n-1}+1} \pm \sqrt{5}\, F_{L_{n-1}} + F_{L_{n-1}-1}\right) = 4 \left(\frac{1 \pm \sqrt{5}}{2}\right)^{5F_n},$$
$$\left(L_{L_{n+1}+1} + \frac{\sqrt{5}-1}{2} L_{L_{n+1}}\right)\left(L_{L_{n-1}+1} + \frac{\sqrt{5}-1}{2} L_{L_{n-1}}\right) = 5\left(\frac{1+\sqrt{5}}{2}\right)^{5F_n},$$
$$\left(L_{L_{n+1}+1} - \frac{\sqrt{5}+1}{2} L_{L_{n+1}}\right)\left(L_{L_{n-1}+1} - \frac{\sqrt{5}+1}{2} L_{L_{n-1}}\right) = 5\left(\frac{1-\sqrt{5}}{2}\right)^{5F_n},$$
$$\left(L_{L_{n+1}+1} \pm \sqrt{5}\, L_{L_{n+1}} + L_{L_{n+1}-1}\right) \cdot$$
$$\cdot \left(L_{L_{n-1}+1} \pm \sqrt{5}\, L_{L_{n-1}} + L_{L_{n-1}-1}\right) = 20 \left(\frac{1 \pm \sqrt{5}}{2}\right)^{5F_n},$$

since $L_{n+1} + L_{n-1} = 5\,F_n$, $n \in \mathbb{N}$.

Corollary 14. ***(A bridge between Fibonacci, Lucas and δ-Fibonacci numbers)***

For positive integers δ we get

$$\prod_{k=0}^{n}\left(F_{1+\binom{n}{k}F_{k-1}\delta^k} + \frac{\sqrt{5}-1}{2} F_{\binom{n}{k}F_{k-1}\delta^k}\right) = \left(\frac{1+\sqrt{5}}{2}\right)^{a_n(-\delta)},$$
$$\prod_{k=1}^{n}\left(F_{1+\binom{n}{k}F_k\delta^k} + \frac{\sqrt{5}-1}{2} F_{\binom{n}{k}F_k\delta^k}\right) = \left(\frac{1+\sqrt{5}}{2}\right)^{-b_n(-\delta)},$$
$$\prod_{k=0}^{n}\left(F_{1+\binom{n}{k}F_{k-1}\delta^k} - \frac{\sqrt{5}+1}{2} F_{\binom{n}{k}F_{k-1}\delta^k}\right) = \left(\frac{1-\sqrt{5}}{2}\right)^{a_n(-\delta)},$$
$$\prod_{k=1}^{n}\left(F_{1+\binom{n}{k}F_k\delta^k} - \frac{\sqrt{5}+1}{2} F_{\binom{n}{k}F_k\delta^k}\right) = \left(\frac{1-\sqrt{5}}{2}\right)^{-b_n(-\delta)},$$
$$\prod_{k=0}^{n}\left(F_{1+\binom{n}{k}F_{k-1}\delta^k} \pm \sqrt{5}\, F_{\binom{n}{k}F_{k-1}\delta^k} + F_{-1+\binom{n}{k}F_{k-1}\delta^k}\right) = 2^{n+1}\left(\frac{1 \pm \sqrt{5}}{2}\right)^{a_n(-\delta)},$$

$$\prod_{k=1}^{n}\left(F_{1+\binom{n}{k}F_k\delta^k} \pm \sqrt{5}\,F_{\binom{n}{k}F_k\delta^k} + F_{-1+\binom{n}{k}F_k\delta^k}\right) = 2^n\left(\frac{1\pm\sqrt{5}}{2}\right)^{-b_n(-\delta)},$$

$$\prod_{k=0}^{n}\left(L_{1+\binom{n}{k}F_{k-1}\delta^k} + \frac{\sqrt{5}-1}{2}\,L_{\binom{n}{k}F_{k-1}\delta^k}\right) = 5^{(n+1)/2}\left(\frac{1+\sqrt{5}}{2}\right)^{a_n(-\delta)},$$

$$\prod_{k=1}^{n}\left(L_{1+\binom{n}{k}F_k\delta^k} + \frac{\sqrt{5}-1}{2}\,L_{\binom{n}{k}F_k\delta^k}\right) = 5^{n/2}\left(\frac{1+\sqrt{5}}{2}\right)^{-b_n(-\delta)},$$

etc.

since we have (see [19]):

$$a_n(\delta) = \sum_{k=0}^{n}\binom{n}{k}F_{k-1}\,(-\delta)^k, \tag{31}$$

and

$$b_n(\delta) = \sum_{k=1}^{n}\binom{n}{k}(-1)^{k-1}\,F_k\,\delta^k, \tag{32}$$

(moreover, we define $F_{n+1} = F_n + F_{n-1}$, $n \in \mathbb{Z}$).

Remark 15. We note that Theorem 9. is connecting with the following very important Zeckendorf's theorem [9]:

For every number $n \in \mathbb{N}$ there exists exactly one increasing sequence $2 \leqslant k_1 < \ldots < k_r$, where $r = r(n) \in \mathbb{N}$, such that $k_{i+1} - k_i \geqslant 2$ for $i = 1, 2, \ldots, r-1$, and

$$n = F_{k_1} + F_{k_2} + \ldots + F_{k_r}.$$

For example we have

$$1000 = 987 + 13 = F_{16} + F_7,$$

that is

$$\left(F_{988} \pm \sqrt{5}\,F_{987} + F_{986}\right)\left(F_{14} \pm \sqrt{5}\,F_{13} + F_{12}\right) = 4\left(\frac{1\pm\sqrt{5}}{2}\right)^{1000},$$

and

$$\left(L_{988} \pm \sqrt{5}\,L_{987} + L_{986}\right)\left(L_{14} \pm \sqrt{5}\,L_{13} + L_{12}\right) = 20\left(\frac{1\pm\sqrt{5}}{2}\right)^{1000}.$$

At the end of this section, the following Lemma about linear independence will be presented.

Lemma 16. *a) Any two numbers among the following one's ($\xi = \exp(i2\pi/5)$)*

$$1, \quad \xi + \xi^4 = 2\cos\frac{2\pi}{5}, \quad \xi^2 + \xi^3 = -2\cos\frac{\pi}{5}$$

are linearly independent over $\mathbb{Q}$.

b) Let $f_k : \mathbb{C} \to \mathbb{C}$ and $g_k : \mathbb{C} \to \mathbb{C}$, $k = 1, 2$ be continuous functions. Then for any $a, b \in \mathbb{R}$, which are linearly independent over $\mathbb{Q}$, the following statement holds:

if

$$f_1(\delta)a + g_1(\delta)b \equiv f_2(\delta)a + g_2(\delta)b, \quad \textit{for} \quad \delta \in \mathbb{Q},$$

then

$$f_1(\delta) = f_2(\delta) \quad \textit{and} \quad g_1(\delta) = g_2(\delta), \quad \textit{for} \quad \delta \in \mathbb{C}.$$

3. Applications of identities (1) to the investigation on Fibonacci and Lucas numbers

3.1. Generalization of de Vries' identity

In this section the following generalizations of de Vries' equality will be proven (see [17], where only the case of $a = l = 1$ is discussed):

$$\sum_{k=0}^{n} \binom{n}{k} a^{n-k} F_{kl+i} = \frac{\alpha^i(a+\alpha^l)^n - \beta^i(a+\beta^l)^n}{\alpha - \beta} \tag{33}$$

$$= \frac{\sqrt{5}}{5}\left[\alpha^i(a+\alpha^l)^n - \beta^i(a+\beta^l)^n\right] \tag{34}$$

and

$$\sum_{k=0}^{n} \binom{n}{k} a^{n-k} L_{kl+i} = \alpha^i(a+\alpha^l)^n + \beta^i(a+\beta^l)^n. \tag{35}$$

The identity (35) can be explained by applying the Binet formula (7). Now the following expression will be considered

$$S = (1+\xi+\xi^4)^i(a + (1+\xi+\xi^4)^l)^n.$$

By (11) we have $S = \alpha^i(a+\alpha^l)^n$.

On the other hand, by binomial formula and by (1), it can be deduced that

$$
\begin{aligned}
S &= \sum_{k=0}^{n} \binom{n}{k} a^{n-k} (1+\xi+\xi^4)^{kl+i} \\
&= \sum_{k=0}^{n} \binom{n}{k} a^{n-k} (F_{kl+i+1} + F_{kl+i}(\xi+\xi^4)) \\
&\text{(by (9), for } \xi = \exp\left(i\tfrac{2}{5}\pi\right)) \\
&= \sum_{k=0}^{n} \binom{n}{k} a^{n-k} F_{kl+i+1} + \frac{\sqrt{5}-1}{2} \sum_{k=0}^{n} \binom{n}{k} a^{n-k} F_{kl+i} \\
&= \frac{1}{2} \sum_{k=0}^{n} \binom{n}{k} a^{n-k} (2F_{kl+i+1} - F_{kl+i}) + \frac{\sqrt{5}}{2} \sum_{k=0}^{n} \binom{n}{k} a^{n-k} F_{kl+i} \\
&= \frac{1}{2} \sum_{k=0}^{n} \binom{n}{k} a^{n-k} L_{kl+i} + \frac{\sqrt{5}}{2} \sum_{k=0}^{n} \binom{n}{k} a^{n-k} F_{kl+i}.
\end{aligned}
$$

Hence we obtain the identity

$$\sqrt{5} \sum_{k=0}^{n} \binom{n}{k} a^{n-k} F_{kl+i} = 2S - \sum_{k=0}^{n} \binom{n}{k} a^{n-k} L_{kl+i}, \tag{36}$$

which by (35) implies the identity (34).

Corollary 17. *We have:*

$$\sum_{k=0}^{n} \binom{n}{k} F_k = F_{2n}, \tag{37}$$

$$\sum_{k=0}^{n} \binom{n}{k} F_{k+1} = F_{n+1}^2 + F_n^2 = F_{2n-1}. \tag{38}$$

3.2. Some binomial – Lucas identities

Now, the following three equalities will be proven

$$1 = \binom{2n}{n}(-1)^n + \sum_{k=0}^{n-1} \binom{2n}{k} (-1)^k L_{2(n-k)}, \tag{39}$$

$$1 = \sum_{k=0}^{n-1} \binom{2n-1}{k} (-1)^k L_{2(n-k)-1} \tag{40}$$

and

$$2=\sum_{k=0}^{n}\binom{n}{k}(-1)^k L_{n-2k}. \tag{41}$$

Proof. First, by (8), (13) and by (21), we obtain

$$\begin{aligned}1&=2+\xi+\xi^2+\xi^3+\xi^4=(2+\xi+\xi^2+\xi^3+\xi^4)^{2n}=\\&=\left((1+\xi+\xi^4)+(1+\xi^2+\xi^3)\right)^{2n}=\\&=\sum_{k=0}^{n-1}\binom{2n}{k}\left[(1+\xi+\xi^4)^k(1+\xi^2+\xi^3)^{2n-k}+(1+\xi+\xi^4)^{2n-k}(1+\xi^2+\xi^3)^k\right]+\\&\quad+\binom{2n}{n}\left((1+\xi+\xi^4)(1+\xi^2+\xi^3)\right)^n=\\&=\sum_{k=0}^{n-1}\binom{2n}{k}(-1)^k L_{2(n-k)}+\binom{2n}{n}(-1)^n.\end{aligned}$$

The proof of formula (40) runs in the similar way

$$\begin{aligned}1&=\left((1+\xi+\xi^4)+(1+\xi^2+\xi^3)\right)^{2n-1}=\\&=\sum_{k=0}^{n-1}\binom{2n-1}{k}\left[(1+\xi+\xi^4)^k(1+\xi^2+\xi^3)^{2n-1-k}+\right.\\&\quad\left.+(1+\xi+\xi^4)^{2n-1-k}(1+\xi^2+\xi^3)^k\right]=\sum_{k=0}^{n-1}\binom{2n-1}{k}(-1)^k L_{2(n-k)-1}.\end{aligned}$$

Finally, the identity (41) will be proven

$$\begin{aligned}1&=2+\xi+\xi^2+\xi^3+\xi^4=(2+\xi+\xi^2+\xi^3+\xi^4)^n=\\&=\left((1+\xi+\xi^4)+(1+\xi^2+\xi^3)\right)^n=\\&=\sum_{k=0}^{n}\binom{n}{k}(1+\xi+\xi^4)^k(1+\xi^2+\xi^3)^{n-k}\overset{(22)}{=}\\&=\frac{1}{2}\sum_{k=0}^{n}\binom{n}{k}(-1)^k L_{n-2k}+\frac{\sqrt{5}}{2}\sum_{k=0}^{n}\binom{n}{k}(-1)^{k+1}F_{n-2k},\end{aligned}$$

which, because of the irrationality of $\sqrt{5}$, implies (41) and the following equality

$$\sum_{k=0}^{n}\binom{n}{k}(-1)^{k+1}F_{n-2k}=0.$$

□

Remark 18. We note that the following generalizations of (39) and (40), respectively, are true (see [5]):

$$L_m^{2n} = (-1)^{mn}\binom{2n}{n} + \sum_{k=0}^{n-1}(-1)^{mk}\binom{2n}{k} L_{(2n-2k)m},$$

and

$$L_m^{2n+1} = \sum_{k=0}^{n}(-1)^{mk}\binom{2n+1}{k} L_{(2n+1-2k)m}.$$

Remark 19. By using δ-Fibonacci numbers $a_n(\delta)$ and $b_n(\delta)$ we obtain the relations

$$\sum_{r=0}^{n}\binom{n}{r} F_{rk+(n-r)l+1} = \left(F_{k+1}+F_{l+1}\right)^n a_n\Big(\frac{F_k+F_l}{F_{k+1}+F_{l+1}}\Big), \tag{42}$$

$$\sum_{r=0}^{n}\binom{n}{r} F_{rk+(n-r)l} = \left(F_{k+1}+F_{l+1}\right)^n b_n\Big(\frac{F_k+F_l}{F_{k+1}+F_{l+1}}\Big), \tag{43}$$

$$5^{(n-2)/2}\sum_{r=0}^{n}\binom{n}{r} L_{rk+(n-r)l+1} = \left(L_{k+1}+L_{l+1}\right)^n a_n\Big(\frac{L_k+L_l}{L_{k+1}+L_{l+1}}\Big), \tag{44}$$

$$5^{(n-2)/2}\sum_{r=0}^{n}\binom{n}{r} L_{rk+(n-r)l} = \left(L_{k+1}+L_{l+1}\right)^n b_n\Big(\frac{L_k+L_l}{L_{k+1}+L_{l+1}}\Big). \tag{45}$$

Sketch of the proof. First we get

$$\left(\left(1+\xi+\xi^4\right)^k + \left(1+\xi^2+\xi^3\right)^l\right)^n = \sum_{r=0}^{n}\binom{n}{r}\left(1+\xi+\xi^4\right)^{kr}\left(1+\xi+\xi^4\right)^{(n-r)l} =$$

$$= \sum_{r=0}^{n}\binom{n}{r}\left(F_{kr+1}+\left(\xi+\xi^4\right)F_{kr}\right)\left(F_{(n-r)l+1}+\left(\xi+\xi^4\right)F_{(n-r)l}\right) =$$

(be reduction formula (12) in [20])

$$= \sum_{r=0}^{n}\binom{n}{r}F_{rk+(n-r)l+1} + \left(\xi+\xi^4\right)\sum_{r=0}^{n}\binom{n}{r}F_{rk+(n-r)l}.$$

On the other hand, we obtain

$$\left(\left(1+\xi+\xi^4\right)^k + \left(1+\xi^2+\xi^3\right)^l\right)^n =$$

$$= \left(F_{k+1}+F_{l+1}+\left(\xi+\xi^4\right)\left(F_k+F_l\right)\right)^n =$$

$$= \left(F_{k+1}+F_{l+1}\right)^n\left(a_n\Big(\frac{F_k+F_l}{F_{k+1}+F_{l+1}}\Big) + \left(\xi+\xi^4\right)b_n\Big(\frac{F_k+F_l}{F_{k+1}+F_{l+1}}\Big)\right).$$

We note, that by (31), (32) and by Lemma 16. the formulas (42) and (43) follow.

□

3.3. Generalization of some Church and Bicknell's identities

We have

$$F_{n-1}^k = \sum_{l=0}^{k} (-1)^l \binom{k}{l} F_n^l F_{n(k-l)+l+1}, \tag{46}$$

$$0 = \sum_{l=0}^{k} (-1)^l \binom{k}{l} F_n^l F_{n(k-l)+l} \tag{47}$$

and

$$F_{kn+s} = \sum_{l=0}^{k} \binom{k}{l} F_{n-1}^{k-l} F_n^l F_{l+s} \tag{48}$$

(the last identity is a generalization of some Church and Bicknell's identities – see identities (27) and (28) on [11] page 238).

Proof. Immediately from (1) the following identities can be deduced

$$(1+\xi+\xi^4)^n = F_{n+1} + F_n(\xi+\xi^4) = F_{n-1} + F_n(1+\xi+\xi^4),$$

i.e.

$$F_{n-1} = (1+\xi+\xi^4)^n - F_n(1+\xi+\xi^4).$$

Hence, we obtain

$$\begin{aligned} F_{n-1}^k &= \sum_{l=0}^{k} \binom{k}{l} (-1)^l F_n^l (1+\xi+\xi^4)^{n(k-l)+l} \\ &= \sum_{l=0}^{k} \binom{k}{l} (-1)^l F_n^l \left(F_{n(k-l)+l+1} + F_{n(k-l)+l}(\xi+\xi^4)\right) \end{aligned}$$

which implies (46) and (47). Now, we note that

$$(1+\xi+\xi^4)^{kn+s} = F_{kn+s+1} + F_{kn+s}(\xi+\xi^4). \tag{49}$$

On the other hand, we can transform the left side of (49) in the following way

$$\begin{aligned} (1+\xi+\xi^4)^{kn+s} &= \left((1+\xi+\xi^4)^n\right)^k (1+\xi+\xi^4)^s = \\ &= \left(F_{n+1} + F_n(\xi+\xi^4)\right)^k (1+\xi+\xi^4)^s = \left(F_{n-1} + F_n(1+\xi+\xi^4)\right)^k (1+\xi+\xi^4)^s \\ &= \sum_{l=0}^{k} \binom{k}{l} F_{n-1}^{k-l} F_n^l (1+\xi+\xi^4)^{l+s} = \sum_{l=0}^{k} \binom{k}{l} F_{n-1}^{k-l} F_n^l \left(F_{l+s+1} + F_{l+s}(\xi+\xi^4)\right) \\ &= \sum_{l=0}^{k} \binom{k}{l} F_{n-1}^{k-l} F_n^l F_{l+s+1} + (\xi+\xi^4) \sum_{l=0}^{k} \binom{k}{l} F_{n-1}^{k-l} F_n^l F_{l+s}, \end{aligned}$$

which, by (49) and Lemma 16, yields (48). □

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Streszczenie

W pracy przedstawiono nową metodę dowodzenia tożsamości dla liczb Fibonacciego i Lucasa. Prezentowana metoda jest alternatywna do dowodów opartych na wzorach Bineta, reprezentacji macierzowej dla liczb Fibonacciego i Lucasa, wykorzystujących funkcje tworzące dla tych liczb czy wreszcie opartych na rozważaniach natury kombinatorycznej (metody zliczeniowe). Metoda polega na wykorzystaniu pewnych fundamentalnych tożsamości dla potęg liczby zwanej złotą proporcją oraz liczby do niej sprzężonej, odkrytych niezależnie przez wielu autorów (m.in. S. Rabinowitza i R. Witułę). Tożsamości te znane są w literaturze, lecz wykorzystywano je zazwyczaj selektywnie do analizy wybranych problemów. Wyjątek stanowi tu praca R. Wituły i D. Słoty [19], w której autorzy badali liczby δ-Fibonacciego, czyli wielomianowe uogólnienia klasycznych liczb Fibonacciego oraz Lucasa, za pomocą ogólniejszych, niż powyższe tożsamości dla potęg złotego stosunku. Okazuje się, że tożsamości otrzymywane wówczas dla liczb δ-Fibonacciego znacząco różnią się od tożsamości dla klasycznych liczb Fibonacciego i Lucasa. Stąd pomysł na niniejszą pracę, która bazując na prostszej tożsamości wyjściowej, pozwala generować, uogólniać, a nawet prowadzić do nowych zależności dla liczb Fibonacciego i Lucasa w sposób łatwy, szybki i naturalny, wykorzystując jedynie proste rozważania algebraiczne.