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CONNECTIONS BETWEEN THE PRIMITIVE 5-TH ROOTS OF UNITY AND FIBONACCI NUMBERS – PART II

Summary. This paper constitute a continuation of previous paper of the authors. Also in here, many of the classical and relatively recently discovered identities for Fibonacci and Lucas numbers are proven. It seems, that the proposed approach to the problem of proving identities for those numbers is absolutely original.

ZWIĄZKI POMIĘDZY PIERWIASTKAMI PIERWOTNYMI STOPNIA PIĄTEGO Z JEDYNKI A LICZBAMI FIBONACCIEGO – CZĘŚĆ II

Streszczenie. Praca jest kontynuacją wcześniejszego artykułu autorów. Dotyczy nowej, oryginalnej metody dowodzenia różnych tożsamości i zależności dla liczb Fibonacciego i Lucasa.

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1. Sums of certain series

The author's interest in problems concerning the Fibonacci, Lucas and complex numbers express in the following two nice identities (discovered independently by Rabinowitz [5] and Grzymkowski and Wituła [1]):

$$(1+\xi+\xi^4)^n = F_{n+1}+F_n(\xi+\xi^4) \quad \text{and} \quad (1+\xi^2+\xi^3)^n = F_{n+1}+F_n(\xi^2+\xi^3), \tag{1}$$

where $\xi^5 = 1$, $\xi \in \mathbb{C}$ and $\xi \neq 1$, and F_n denotes the nth Fibonacci number.

Immediately from (1) (see also (14), (15), (9) and (10) in [7]):

$$\sum_{n=0}^{\infty} \frac{1}{F_{n+1}+2F_n\cos\frac{2}{5}\pi} = \sum_{n=0}^{\infty} \frac{\sqrt{5}}{L_{n+1}+2L_n\cos\frac{2}{5}\pi} = \frac{1+2\cos\frac{2}{5}\pi}{2\cos\frac{2}{5}\pi} = \frac{\sqrt{5}+3}{2};$$

$$\sum_{n=0}^{\infty} \frac{1}{F_{kn+1}+2F_{kn}\cos\frac{2}{5}\pi} = \sum_{n=0}^{\infty} \frac{\sqrt{5}}{L_{kn+1}+2L_{kn}\cos\frac{2}{5}\pi} = \frac{F_{k+1}+2F_k\cos\frac{2}{5}\pi}{F_{k+1}+2F_k\cos\frac{2}{5}\pi-1}. \tag{2}$$

Hence, we obtain, for example

$$\sum_{n=0}^{\infty} \frac{1}{F_{3n+1}+2F_{3n}\cos\frac{2}{5}\pi} = \sum_{n=0}^{\infty} \frac{\sqrt{5}}{L_{3n+1}+2L_{3n}\cos\frac{2}{5}\pi} = \frac{\sqrt{5}+3}{4}.$$

In this paper the symbol L_n denotes the nth Lucas number. We also have

$$\sum_{n=0}^{\infty} \left(F_{n+1}+2F_n\cos\frac{4}{5}\pi\right) = \sum_{n=0}^{\infty} -\frac{\sqrt{5}}{5}\left(L_{n+1}+2L_n\cos\frac{4}{5}\pi\right) = \frac{1}{-2\cos\frac{4}{5}\pi} = \frac{\sqrt{5}-1}{2};$$

$$\sum_{n=0}^{\infty} \left(F_{kn+1}+2F_{kn}\cos\frac{4}{5}\pi\right) = \sum_{n=0}^{\infty} -\frac{\sqrt{5}}{5}\left(L_{kn+1}+2L_{kn}\cos\frac{4}{5}\pi\right) = \frac{1}{1-F_{k+1}-2F_k\cos\frac{4}{5}\pi}. \tag{3}$$

Hence, we derive, for example

$$\sum_{n=0}^{\infty} \left(F_{3n+1}+2F_{3n}\cos\frac{4}{5}\pi\right) = \sum_{n=0}^{\infty} -\frac{\sqrt{5}}{5}\left(L_{3n+1}+2L_{3n}\cos\frac{4}{5}\pi\right) = \frac{\sqrt{5}+1}{4}.$$

2. Inner products of the vectors composed of Fibonacci and Lucas numbers

The next object of our consideration is to prove the following identities

$$\sum_{k=0}^{n} F_{2k+1} = F_{2n+2}, \quad \sum_{k=0}^{n} F_{2k} = F_{2n+1} - 1, \tag{4}$$

$$\sum_{k=0}^{n} L_{2k} = L_{2n+1} + 1, \tag{5}$$

$$\sum_{k=0}^{n} L_k F_k = F_{2n+1} - 1, \tag{6}$$

$$\sum_{k=0}^{n} F_k^2 = \frac{1}{4}(L_{2n+1} + F_{2n+1} - 2F_{n+1}^2) \tag{7}$$

and

$$\sum_{k=0}^{n} L_k^2 = \frac{1}{4}(3L_{2n+1} - 5F_{2n+1} + 10F_{n+1}^2) + 2. \tag{8}$$

Proof. First we note that (by (19) and (15) in [7]):

$$\sum_{k=0}^{n}(1+\xi+\xi^4)^{2k} = \frac{(1+\xi+\xi^4)^{2n+2}-1}{(1+\xi+\xi^4)^2-1}$$

$$= \frac{(1+\xi+\xi^4)^{2n+2}-1}{1+\xi+\xi^4} = (1+\xi+\xi^4)^{2n+1} - (1+\xi+\xi^4)^{-1}$$

$$= F_{2n+2} + F_{2n+1}(\xi+\xi^4) - (\xi+\xi^4) = F_{2n+2} + (F_{2n+1}-1)(\xi+\xi^4) \tag{9}$$

$$= F_{2n+2} + \frac{\sqrt{5}-1}{2}(F_{2n+1}-1) = \frac{1}{2}(L_{2n+1}+1) + \frac{\sqrt{5}}{2}(F_{2n+1}-1). \tag{10}$$

On the other hand, we have

$$\sum_{k=0}^{n}(1+\xi+\xi^4)^{2k} = \sum_{k=0}^{n}\left(F_{2k+1} + F_{2k}(\xi+\xi^4)\right)$$

$$= \sum_{k=0}^{n} F_{2k+1} + (\xi+\xi^4)\sum_{k=0}^{n} F_{2k} \tag{11}$$

$$= \sum_{k=0}^{n} \frac{1}{2}(2F_{2k+1} - F_{2k}) + \frac{\sqrt{5}}{2}\sum_{k=0}^{n} F_{2k}$$

$$= \frac{1}{2}\sum_{k=0}^{n} L_{2k} + \frac{\sqrt{5}}{2}\sum_{k=0}^{n} F_{2k}. \tag{12}$$

Comparing (9) with (11) and the rational part of identity (10) with the rational part of (12), the identities (4) and (5) can be derived.

Now, we obtain

$$\begin{aligned}\sum_{k=0}^{n}(1+\xi+\xi^4)^{2k} &= \sum_{k=0}^{n}(F_{k+1}+(\xi+\xi^4)F_k)^2 \\ &= \sum_{k=0}^{n}\left(F_{k+1}+\frac{\sqrt{5}-1}{2}F_k\right)^2 = \sum_{k=0}^{n}\left(\frac{1}{2}L_k+\frac{\sqrt{5}}{2}F_k\right)^2 \\ &= \frac{1}{4}\left(\sum_{k=0}^{n}L_k^2+5\sum_{k=0}^{n}F_k^2\right)+\frac{\sqrt{5}}{2}\sum_{k=0}^{n}L_kF_k. \end{aligned} \tag{13}$$

Hence, by (10), we derive

$$\sum_{k=0}^{n}L_k^2+5\sum_{k=0}^{n}F_k^2+2\sqrt{5}\sum_{k=0}^{n}L_kF_k = 2(L_{2n+1}+1)+2\sqrt{5}(F_{2n+1}-1),$$

which, by the irrationality of $\sqrt{5}$, yields (6) and the following identity

$$\sum_{k=0}^{n}L_k^2+5\sum_{k=0}^{n}F_k^2 = 2(L_{2n+1}+1). \tag{14}$$

Now, let's observe that

$$\sum_{k=0}^{n}(L_k+F_k)^2 = \sum_{k=0}^{n}(F_{k+1}+F_{k-1}+F_k)^2 = 4\sum_{k=0}^{n}F_{k+1}^2.$$

On the other hand, we have

$$\sum_{k=0}^{n}(L_k+F_k)^2 = \sum_{k=0}^{n}L_k^2+2\sum_{k=0}^{n}L_kF_k+\sum_{k=0}^{n}F_k^2.$$

By the last two equalities and by (6) we conclude

$$\sum_{k=0}^{n}L_k^2-3\sum_{k=0}^{n}F_k^2 = 4F_{n+1}^2-2F_{2n+1}+2. \tag{15}$$

Identities (14) and (15) lead to identities (7) and (8). □

3. Reduction formulas

The following reduction formulas will be proven in this section

$$F_{k+l} = F_{k+1}F_l + F_kF_{l-1}, \tag{16}$$

$$L_{k+l} = F_{l+1}L_k + F_l L_{k-1}, \tag{17}$$

$$F_{k+1}^2 - F_k\, F_{k+2} = (-1)^k, \tag{18}$$

$$F_{k-l} = (-1)^l\,(F_k F_{l+1} - F_{k+1} F_l), \tag{19}$$

$$F_u F_v - F_{u-r} F_{v+r} = (-1)^{u-r} F_{v-u+r} F_r, \tag{20}$$

$$F_{r+s+t+1} = F_{r+1}\, F_{s+1}\, F_{t+1} + F_{r+1}\, F_s\, F_t + F_r\, F_s\, F_{t+1} + \\ + F_r\, F_{s+1}\, F_t - F_r\, F_s\, F_t, \tag{21}$$

and

$$F_{r+s+t} = F_{r+1}\, F_{s-1}\, F_t + F_{r-1}\, F_s\, F_{t+1} + F_r\, F_{s+1}\, F_{t-1} + 2\, F_r\, F_s\, F_t \tag{22}$$

etc.

Proof. We have

$$(1+\xi+\xi^4)^{k+l} = F_{k+l+1} + (\xi+\xi^4)F_{k+l} = F_{k+l+1} + \frac{\sqrt{5}-1}{2}F_{k+l} = \\ = \frac{1}{2}(2F_{k+l+1} - F_{k+l}) + \frac{\sqrt{5}}{2}F_{k+l} = \frac{1}{2}L_{k+l} + \frac{\sqrt{5}}{2}F_{k+l}. \tag{23}$$

On the other hand, we obtain

$$\begin{aligned}(1+\xi+\xi^4)^{k+l} &= (1+\xi+\xi^4)^k(1+\xi+\xi^4)^l \\ &= (F_{k+1} + (\xi+\xi^4)F_k)(F_{l+1} + (\xi+\xi^4)F_l) \\ &= F_{k+1}F_{l+1} + (\xi+\xi^4)(F_kF_{l+1} + F_{k+1}F_l) + (\xi+\xi^4)^2 F_kF_l \\ &= F_{k+1}F_{l+1} + \frac{\sqrt{5}-1}{2}(F_kF_{l+1} + F_{k+1}F_l) + \frac{3-\sqrt{5}}{2}F_kF_l \\ &= \frac{1}{2}(2F_{k+1}F_{l+1} - F_kF_{l+1} - F_{k+1}F_l + 3F_kF_l) \\ &+ \frac{\sqrt{5}}{2}(F_kF_{l+1} + F_{k+1}F_l - F_kF_l) \\ &= \frac{1}{2}(F_{l+1}L_k + F_lL_{k-1}) + \frac{\sqrt{5}}{2}(F_kF_{l+1} + F_{k-1}F_l).\end{aligned} \tag{24}$$

Hence, by comparing (23) and (24) and by the irrationality of $\sqrt{5}$, identities (16) and (17) follow. Now, by (13) in [7], we evaluate

$$(-1)^k = (1+\xi+\xi^4)^k(1+\xi^2+\xi^3)^k = \\ = \big(F_{k+1} + F_k(\xi+\xi^4)\big)\big(F_{k+1} + F_k(\xi^2+\xi^3)\big) = \\ = F_{k+1}^2 - F_k^2 - F_kF_{k+1} = F_{k+1}^2 - F_k\, F_{k+2}, \tag{25}$$

which gives the identity (18).

By (23) in [7] and by (18) we receive

$$\frac{F_{k+1}+F_k(\xi+\xi^4)}{F_{l+1}+F_l(\xi+\xi^4)} =$$
$$= \frac{F_kF_l+F_{k+1}F_l-F_{k+1}F_{l+1}}{F_l^2-F_{l+1}^2+F_lF_{l+1}} + \frac{F_{k+1}F_l-F_kF_{l+1}}{F_l^2-F_{l+1}^2+F_lF_{l+1}}(\xi+\xi^4) =$$
$$= (-1)^{l+1}(F_{k+2}F_l-F_{k+1}F_{l+1}) + (-1)^l(F_kF_{l+1}-F_{k+1}F_l)(\xi+\xi^4). \quad (26)$$

On the other hand, we have

$$(1+\xi+\xi^4)^{k-l} = F_{k-l+1}+F_{k-l}(\xi+\xi^4) =$$
$$= \frac{(1+\xi+\xi^4)^k}{(1+\xi+\xi^4)^l} = \frac{F_{k+1}+F_k(\xi+\xi^4)}{F_{l+1}+F_l(\xi+\xi^4)}, \quad (27)$$

which by (26) implies (19).

To prove the identity (20), first we note, that

$$(1+\xi+\xi^4)^{u+v} = (1+\xi+\xi^4)^u(1+\xi+\xi^4)^v =$$
$$= \big(F_{u+1}+F_u(\xi+\xi^4)\big)\big(F_{v+1}+F_v(\xi+\xi^4)\big) =$$
$$= F_{u+1}F_{v+1}+F_uF_v+(F_uF_{v+1}+F_{u+1}F_v-F_uF_v)(\xi+\xi^4). \quad (28)$$

Hence, replacing u by $u-r$ and v by $v+r$, we obtain

$$(1+\xi+\xi^4)^{u+v} = F_{u-r+1}F_{v+r+1}+F_{u-r}F_{v+r}+$$
$$+(F_{u-r}F_{v+r+1}+F_{u-r+1}F_{v+r}-F_{u-r}F_{v+r})(\xi+\xi^4). \quad (29)$$

Finally, by comparing parts without the element $(\xi+\xi^4)$ in (28) and in (29), we get

$$F_{u+1}F_{v+1}-F_{u-r+1}F_{v+r+1} = F_{u-r}F_{v+r}-F_uF_v$$
$$\text{(after the next } (u-r) \text{ iterations)}$$
$$= (-1)^{u-r+1}(F_rF_{v-u+r}-F_0F_{v-u+2r})$$
$$= (-1)^{u-r+1}F_rF_{v-u+r}.$$

Proof the identities (21) and (22) (like also of the proper reduction identities for Lucas numbers and identities for $F_{\sum_{j=1}^n s_j}$ and $L_{\sum_{j=1}^n s_j}$) proceeds in the following way

$$\begin{aligned}
F_{r+s+t+1} + F_{r+s+t}\left(\xi+\xi^4\right) &= \left(1+\xi+\xi^4\right)^{r+s+t} = \\
&= \left(1+\xi+\xi^4\right)^r \left(1+\xi+\xi^4\right)^s \left(1+\xi+\xi^4\right)^t = \\
&= \left(F_{r+1} + F_r\left(\xi+\xi^4\right)\right)\left(F_{s+1} + F_s\left(\xi+\xi^4\right)\right)\left(F_{t+1} + F_t\left(\xi+\xi^4\right)\right) = \\
&= F_{r+1}\,F_{s+1}\,F_{t+1} + \left(F_{r+1}\,F_{s+1}\,F_t + F_{r+1}\,F_s\,F_{t+1} + F_r\,F_{s+1}\,F_{t+1}\right)\left(\xi+\xi^4\right) + \\
&+ \left(F_{r+1}\,F_s\,F_t + F_r\,F_s\,F_{t+1} + F_r\,F_{s+1}\,F_t\right)\left(1-\xi-\xi^4\right) + F_r\,F_s\,F_t\left(2\left(\xi+\xi^4\right)-1\right) = \\
&= F_{r+1}\,F_{s+1}\,F_{t+1} + F_{r+1}\,F_s\,F_t + F_r\,F_s\,F_{t+1} + F_r\,F_{s+1}\,F_t - F_r\,F_s\,F_t + \\
&+ \left(\xi+\xi^4\right)\left(F_{r+1}\,F_{s-1}\,F_t + F_{r-1}\,F_s\,F_{t+1} + F_r\,F_{s+1}\,F_{t-1} + 2\,F_r\,F_s\,F_t\right).
\end{aligned}$$

□

Remark 1. We note, that from equalities (16) and (17) the next two equalities can be derived:

$$F_{2n} = F_n L_n \tag{30}$$

and

$$L_{2n} = F_{n+1} L_n + F_n L_{n-1}. \tag{31}$$

The identity (31) can also be proved in the following way

$$\begin{aligned}
L_{2n+1} + L_{2n}\left(\xi+\xi^4\right) &= \left(1+\xi+\xi^4\right)^{2n} = \left(1+\xi+\xi^4\right)^n \left(1+\xi+\xi^4\right)^n = \\
&= \left(F_{n+1} + F_n\left(\xi+\xi^4\right)\right)\left(L_{n+1} + L_n\left(\xi+\xi^4\right)\right) = \\
&= L_{n+1}\,F_{n+1} + L_n\,F_n + \left(F_{n+1}\,L_n + L_{n+1}\,F_n - L_n\,F_n\right)\left(\xi+\xi^4\right) = \\
&= L_{n+1}\,F_{n+1} + L_n\,F_n + \left(F_{n+1}\,L_n + L_{n-1}\,F_n\right)\left(\xi+\xi^4\right),
\end{aligned}$$

which leads to (31) and

$$L_{2n+1} = L_{n+1} F_{n+1} + L_n F_n.$$

Remark 2. Proofs of presented in this section reduction formulas for Fibonacci and Lucas numbers are given in [2], to the context of graph colorings.

4. Identities with the powers of Fibonacci and Lucas numbers

By (29) in [7] we have (in all equalities in this section we put $\xi = \exp(2\pi i/5)$):

$$\begin{aligned} 5F_{2n+1} + 5F_{2n}(\xi+\xi^4) &= \left(\sqrt{5}(1+\xi+\xi^4)^n\right)^2 = \\ &= \left(L_{n+1} + L_n(\xi+\xi^4)\right)^2 \\ &= L_{n+1}^2 + 2L_{n+1}L_n(\xi+\xi^4) + L_n^2(1-\xi-\xi^4) \\ &= L_{n+1}^2 + L_n^2 + (2L_{n+1}L_n - L_n^2)(\xi+\xi^4), \end{aligned} \tag{32}$$

which implies

$$L_{n+1}^2 + L_n^2 = L_{2n+2} + L_{2n} = 5F_{2n+1} \tag{33}$$

and

$$2L_{n+1}L_n - L_n^2 = 5F_{2n}, \tag{34}$$

i.e.

$$F_{2n} = L_n F_n.$$

Remark 3. Identity (34) is the correct version of the identity (44), page 97 of [3].

Next, we consider the following expression

$$\begin{aligned} 5L_{3n+1} + 5L_{3n}(\xi+\xi^4) &= \left(\sqrt{5}(1+\xi+\xi^4)^n\right)^3 = \left(L_{n+1} + L_n(\xi+\xi^4)\right)^3 = \\ &= L_{n+1}^3 + 3L_{n+1}^2L_n(\xi+\xi^4) + 3L_{n+1}L_n^2(1-\xi-\xi^4) + L_n^3\left(2(\xi+\xi^4)-1\right) = \\ &= L_{n+1}^3 + 3L_{n+1}L_n^2 - L_n^3 + (3L_{n+1}^2L_n - 3L_{n+1}L_n^2 + 2L_n^3)(\xi+\xi^4) = \\ &= L_{n+1}^3 + 2L_{n+1}L_n^2 + L_n^2L_{n-1} + (3L_{n+1}L_nL_{n-1} + 2L_n^3)(\xi+\xi^4) = \\ &= L_{n+1}^3 + L_{n+1}L_n^2 + 5F_nL_n^2 + (3L_{n+1}L_nL_{n-1} + 2L_n^3)(\xi+\xi^4), \end{aligned} \tag{35}$$

which implies

$$\begin{aligned} 5L_{3n} &= 3L_{n+1}L_nL_{n-1} + 2L_n^3 = L_n(3L_{n+1}L_{n-1} + 2L_n^2) = \\ &= L_n\left[3(L_{2n} + 3(-1)^{n-1}) + 2(L_{2n} + 2(-1)^n)\right] = 5L_n(L_{2n} - (-1)^n), \end{aligned} \tag{36}$$

i.e.

$$L_{3n} = L_n(L_{2n} - (-1)^n), \tag{37}$$

and

$$5L_{3n+1} = L_{n+1}^3 + L_{n+1}L_n^2 + 5F_nL_n^2 = (L_{n+1}^2 + L_n^2)L_{n+1} + 5F_nL_n^2 = \\ = (L_{2n} + L_{2n+2})L_{n+1} + 5F_nL_n^2 = 5F_{2n+1}L_{n+1} + 5F_nL_n^2, \quad (38)$$

i.e.

$$L_{3n+1} = F_{2n+1}L_{n+1} + F_nL_n^2. \quad (39)$$

We note that from the first two equalities of (35), by (18), the below identity can be concluded

$$\left((1+\xi+\xi^4)^{n+1} + (1+\xi+\xi^4)^{n-1}\right)^3 = 5\left[(1+\xi+\xi^4)^{3n+1} + (1+\xi+\xi^4)^{3n-1}\right]. \quad (40)$$

Now, the following expression will be investigated

$$25F_{4n+1} + 25F_{4n}(\xi+\xi^4) = 25(1+\xi+\xi^4)^{4n} = \\ = \left(\sqrt{5}(1+\xi+\xi^4)^n\right)^4 = \left(L_{n+1} + L_n(\xi+\xi^4)\right)^4 = \\ = L_{n+1}^4 + 4L_{n+1}^3L_n(\xi+\xi^4) + 6L_{n+1}^2L_n^2(1-\xi-\xi^4) + \\ + 4L_{n+1}L_n^3(2(\xi+\xi^4)-1) + L_n^4(2-3(\xi+\xi^4)) = \\ = L_{n+1}^4 + 6L_{n+1}^2L_n^2 - 4L_{n+1}L_n^3 + 2L_n^4 + \\ + (4L_{n+1}^3L_n - 6L_{n+1}^2L_n^2 + 8L_{n+1}L_n^3 - 3L_n^4)(\xi+\xi^4), \quad (41)$$

which implies two new identities

$$25F_{4n+1} = L_{n+1}^4 + 6L_{n+1}^2L_n^2 - 4L_{n+1}L_n^3 + 2L_n^4 \\ = L_{n+1}^4 + 6L_{n+1}L_n^2(L_{n+1} - L_n) + 2L_n^3(L_{n+1} + L_n) \\ = L_{n+1}^4 + 6L_{n+1}L_n^2L_{n-1} + 2L_n^3L_{n+2} \quad (42)$$

and

$$25F_{4n} = 4L_{n+1}^3L_n - 6L_{n+1}^2L_n^2 + 8L_{n+1}L_n^3 - 3L_n^4 \\ = 4L_{n+1}^2L_n(L_{n+1} - L_n) - 2L_{n+1}L_n^2(L_{n+1} - L_n) + 3L_{n+1}L_n^3 + \\ + 3L_n^3(L_{n+1} - L_n) \\ = 2L_{n+1}^2L_n(L_{n+1} - L_n) + 2L_{n+1}L_n(L_{n+1} - L_n)^2 + 3L_n^3(2L_{n+1} - L_n) \\ = 2L_{n+1}^2L_nL_{n-1} + 2L_{n+1}L_nL_{n-1}^2 + 3L_n^3(L_{n+1} + L_{n-1}) \\ = 2L_{n+1}L_nL_{n-1}(L_{n+1} + L_{n-1}) + 3L_n^3(L_{n+1} + L_{n-1}) \\ = (2L_{n+1}L_nL_{n-1} + 3L_n^3)(L_{n+1} + L_{n-1}) \\ = 5F_n(2L_{n+1}L_nL_{n-1} + 3L_n^3) = 5F_{2n}(2L_{n+1}L_{n-1} + 3L_n^2), \quad (43)$$

i.e.

$$5L_{2n} = 2L_{n+1}L_{n-1} + 3L_n^2. \quad (44)$$

5. Generalizations

The following identities form the generalizations of the identity (40):

$$\begin{aligned}
&\big(L_{n+1} + L_n(\xi + \xi^4)\big)^{2k+1} = \\
&= \big((1+\xi+\xi^4)^{n+1} + (1+\xi+\xi^4)^{n-1}\big)^{2k+1} \\
&= 5^k\big((1+\xi+\xi^4)^{(2k+1)n+1} + (1+\xi+\xi^4)^{(2k+1)n-1}\big) \\
&= 5^k L_{(2k+1)n+1} + 5^k L_{(2k+1)n}(\xi+\xi^4),
\end{aligned} \tag{45}$$

$$\begin{aligned}
&\big(L_{n+1} + L_n(\xi + \xi^4)\big)^{2k} = \\
&= \big((1+\xi+\xi^4)^{n+1} + (1+\xi+\xi^4)^{n-1}\big)^{2k} \\
&= 5^k(1+\xi+\xi^4)^{2kn} = 5^k F_{2kn+1} + 5^k F_{2kn}(\xi+\xi^4),
\end{aligned} \tag{46}$$

and

$$\big(F_{n+1} + F_n(\xi+\xi^4)\big)^k = (1+\xi+\xi^4)^{kn} = F_{kn+1} + F_{kn}(\xi+\xi^4). \tag{47}$$

From this, the two below identities can be generated

$$\begin{aligned}
L_{5n+1} &= L_{n+1}^5 + 10L_{n+1}^3 L_n^2 - 10L_{n+1}^2 L_n^3 + 10L_{n+1}L_n^4 - 3L_n^5 \\
&= L_{n+1}^5 + 10L_{n+1}^2 L_n^2 L_{n-1} + 10L_{n-1}L_n^4 + 7L_n^5 \\
&= L_{n+1}^5 + 10L_n^2 L_{n-1}(L_{n+1}^2 + L_n^2) + 7L_n^5 \\
&\text{(by (33))} \\
&= L_{n+1}^5 + 50L_n^2 L_{n-1}F_{2n+1} + 7L_n^5
\end{aligned} \tag{48}$$

and

$$\begin{aligned}
25L_{5n} &= 5L_{n+1}^4 L_n - 10L_{n+1}^3 L_n^2 + 20L_{n+1}^2 L_n^3 - 15L_{n+1}L_n^4 + 5L_n^5 \\
&= 5L_n(L_{n+1}^4 - 2L_{n+1}^3 L_n + 4L_{n+1}^2 L_n^2 - 3L_{n+1}L_n^3 + L_n^4) \\
&= 5L_n(L_{n+1}^3 L_{n-1} - L_{n+1}^2 L_n L_{n-1} + 3L_{n+1}L_n^2 L_{n-1} + L_n^4) \\
&= 5L_n\big((L_{2n} - 3(-1)^n)^2 + (L_{2n} + 2(-1)^n)(4L_{2n} - 7(-1)^n)\big) \\
&= 25L_n(L_{2n}^2 - (-1)^n L_{2n} - 1),
\end{aligned}$$

i.e.

$$L_{5n} = L_n(L_{2n}^2 - (-1)^n L_{2n} - 1) \tag{49}$$

(Carlitz, 1970);

$$F_{2n+1} = F_{n+1}^2 + F_n^2, \tag{50}$$
$$F_{2n} = 2F_nF_{n+1} - F_n^2, \tag{51}$$
$$F_{3n+1} = F_{n+1}^3 + 3F_{n+1}F_n^2 - F_n^3, \tag{52}$$
$$\begin{aligned} F_{3n} &= 3F_{n+1}^2F_n - 3F_{n+1}F_n^2 + 2F_n^3 = \\ &= 3F_{n+1}F_nF_{n-1} + 2F_n^3, \end{aligned} \tag{53}$$
$$\begin{aligned} &= F_n\left(\frac{3}{5}(L_{2n} + 3(-1)^n\right) + \frac{2}{5}(L_{2n} - 2(-1)^n)\Big) \\ &= F_n\big(L_{2n} + (-1)^n\big) \end{aligned} \tag{54}$$

(Koshy, 1999);

$$\begin{aligned} F_{4n+1} &= F_{n+1}^4 + 6F_{n+1}^2F_n^2 - 4F_{n+1}F_n^3 + 2F_n^4 = \\ &= F_{n+1}^4 + 2F_{n+1}^2F_n^2 + 4F_{n+1}F_n^2F_{n-1} + 2F_n^4. \end{aligned} \tag{55}$$

Hence

$$\begin{aligned} 25F_{4n+1} &= L_{2n+2}^2 + 2L_{2n}L_{2n+2} + 6L_{2n}^2 - 20 \\ &= (L_{2n+2} + L_{2n})^2 + 5L_{2n}^2 - 20, \end{aligned} \tag{56}$$

which provides to

$$5F_{4n+1} = 5F_{2n+1}^2 + L_{2n}^2 - 4, \tag{57}$$
$$F_{4n} = 4F_{n+1}^3F_n - 6F_{n+1}^2F_n^2 + 8F_{n+1}F_n^3 - 3F_n^4. \tag{58}$$

From identities (55) and (58) we obtain

$$\begin{aligned} F_{4n+2} &= F_{n+1}^4 + 4F_{n+1}^3F_n + 4F_{n+1}F_n^3 - F_n^4 \\ &= F_{n+1}^4 - F_n^4 + 4F_{n+1}F_n(F_{n+1}^2 + F_n^2) \\ &= (F_{n+1}^2 + 4F_{n+1}F_n - F_n^2)(F_{n+1}^2 + F_n^2), \end{aligned} \tag{59}$$

i.e., by (49):

$$\begin{aligned} L_{2n+1} &= F_{n+1}^2 + 4F_{n+1}F_n - F_n^2 \\ &= (F_{n+1} + 2F_n)^2 - 5F_n^2 = L_{n+1}^2 - 5F_n^2; \end{aligned} \tag{60}$$
$$\begin{aligned} F_{5n+1} &= F_{n+1}^5 + 10F_{n+1}^3F_n^2 - 10F_{n+1}^2F_n^3 + 10F_{n+1}F_n^4 - 3F_n^5 \\ &= F_{n+1}^5 + 10F_{n+1}F_n^2(F_{n+1}^2 - F_{n+1}F_n + F_n^2) - 3F_n^5 \\ &= F_{n+1}^5 + 2F_{n+1}F_n^2(2L_{2n} + (-1)^n) - 3F_n^5 \\ &= F_{n+1}^5 + \frac{2}{5}F_{n+1}\big(2L_{2n}^2 + 5(-1)^nL_{2n} + 2\big) - 3F_n^5; \end{aligned} \tag{61}$$

$$\begin{aligned}
F_{5n} &= 5F_n(F_{n+1}^4 - 2F_{n+1}^3 F_n + 4F_n^2 F_{n+1}^2 - 3F_n^3 F_{n+1} + F_n^4)\\
&= 5F_n\left[(F_{n+1}^2 + F_n^2)(F_{n+1}^2 - F_{n+1}F_n + F_n^2)\right.\\
&\left.- F_{n+1}F_n(F_n^2 + F_{n-1}^2)\right]\\
&= \frac{1}{5}F_n(4L_{2n}^2 + L_{2n+1}L_{2n-1} + 5(-1)^n L_{2n})\\
&= F_n(L_{4n} + (-1)^n L_{2n} + 1)\\
&= F_n\left(L_{2n}^2 + (-1)^n L_{2n} - 1\right);
\end{aligned} \tag{62}$$

$$\begin{aligned}
F_{7n-1} &= F_{n+1}^7 - 7F_{n+1}^6 F_n + 42F_{n+1}^5 F_n^2 - 105F_{n+1}^4 F_n^3 +\\
&+ 175F_{n+1}^3 F_n^4 - 168F_{n+1}^2 F_n^5 + 91F_{n+1}F_n^6 - 21F_n^7\\
&= F_{n+1}^7 - 7F_n\left[F_{n-1}^6 + F_n^3(-F_n^2 L_n - F_{n+2}F_n F_{n-1} - 5F_{n+1}F_{n-1}^2)\right].
\end{aligned} \tag{63}$$

Remark 4. Many general identities for sum of powers of Fibonacci and Lucas numbers have been proven relatively recently by H. Prodinger in the paper [4], for example

$$F_{(2k+1)(2n+1)} = \sum_{l=0}^{k} 5^l (-1)^{k-l} \frac{2k+1}{2l+1}\binom{k+l}{k-l} F_{2n+1}^{2l+1},$$

$$F_{(2n+1)(2m-2j)} = F_{2n+1} \sum_{l=0}^{m-j-1} \binom{m-j+l}{m-j-l-1} L_{2n+1}^{2l+1},$$

etc.

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Streszczenie

Praca jest kontynuacją wcześniejszego artykułu autorów [7]. Wykorzystuje się w niej nową metodę dowodzenia różnych tożsamości i zależności dla liczb Fibonacciego i Lucasa. Metoda oparta jest na wykorzystaniu fundamentalnego wzoru dla potęg liczby zwanej złotą proporcją. W pracy otrzymano oryginalne wzory dla sum pewnych szeregów, tożsamości dla iloczynów skalarnych skończonych wektorów o składowych złożonych z liczb Fibonacciego i Lucasa oraz wzory redukcyjne związane z potęgami tych liczb.