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ON SOME REAL AND COMPLEX LIMITS

Summary. In this paper a problem of existence of some complex limit, in the context of existence of the proper real limits, is discussed. On that occasion a completely elementary method of proving the following implication is presented: the existence of limit $\lim\limits_{x\to 0} \frac{\sin x}{x} = 1$ implies (for every $n \in \mathbb{N}$):

$$\lim_{x\to 0} x^{-n-1} \left(\sin x - \sum_{k=1}^{n} (-1)^{k-1} \frac{x^k}{(2\,k-1)} \right) = \frac{(-1)^n}{(2\,n+1)!}.$$

O PEWNYCH GRANICACH RZECZYWISTYCH I ZESPOLONYCH

Streszczenie. W artykule omówiono zagadnienie istnienia pewnej granicy zespolonej w kontekście istnienia odpowiednich granic rzeczywistych. Przy okazji zaprezentowano całkowicie elementarną metodę dowodzenia implikacji: z istnienia granicy $\lim\limits_{x\to 0} \frac{\sin x}{x} = 1$ wynika istnienie granic (dla każdego $n \in \mathbb{N}$):

$$\lim_{x\to 0} x^{-n-1} \left(\sin x - \sum_{k=1}^{n} (-1)^{k-1} \frac{x^k}{(2\,k-1)} \right) = \frac{(-1)^n}{(2\,n+1)!}.$$

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The problem contributing to the considered task arised in the course of discussing of certain real limits in view of their generalization into a complex case (surely, in the possibly elementary way, i.e. by means of poor technical tools). The problem is stated as follows: is it true that from the fact

$$\lim_{\mathbb{R}\ni x\to 0} \frac{e^x - 1}{x} = 1 \tag{1}$$

one can also deduce, that

$$\lim_{\mathbb{C}\ni z\to 0} \frac{e^z - 1}{z} = 1 \qquad ? \tag{2}$$

And, if the implication does not work, what possible additional assumptions are needed for (2). It seems, that the relation (1) is definitely insufficient to derive (2), which may be proved in the following way: Let us assume, that $x \in \mathbb{R}$. Then we derive

$$\frac{e^{ix} - 1}{ix} = \frac{\sin x}{x} + i\frac{1 - \cos x}{x},$$

from which we get

$$\lim_{\mathbb{R}\ni x\to 0} \frac{e^{ix} - 1}{ix} = \lim_{\mathbb{R}\ni x\to 0} \frac{\sin x}{x} + i\left(\lim_{\mathbb{R}\ni x\to 0} \frac{1 - \cos x}{x^2} \cdot x\right) = 1, \tag{3}$$

where limit (1) does not occur at all, but, instead, another limit is used:

$$\lim_{\mathbb{R}\ni x\to 0} \frac{\sin x}{x} = 1. \tag{4}$$

Remark 1. From (4) it can be easily concluded, that

$$\lim_{\mathbb{R}\ni x\to 0} \frac{1 - \cos x}{x^2} = \frac{1}{2}, \tag{5}$$

because

$$\lim_{x\to 0} \frac{1 - \cos x}{x^2} = \lim_{x\to 0} \left(\frac{\sin x}{x}\right)^2 \cdot \frac{1}{1 + \cos x} = \frac{1}{2}.$$

The answer to our principal problem concerning the limit (2),can be formulated in form of the following theorem

$$\lim_{\mathbb{C}\ni z\to 0} \frac{e^z - 1}{z} = 1 \iff \lim_{\mathbb{R}\ni x\to 0} \frac{e^x - 1}{x} = 1 \quad \text{and} \quad \lim_{\mathbb{R}\ni x\to 0} \frac{\sin x}{x} = 1. \tag{6}$$

Surely, a full proof of this theorem (strictly speaking, proof of the right to left implication) requires more work, thus we invite the readers to participate in making this proof. Let us start with the following task:

Let us prove, that equality (2) holds, by using only the general theorems for limits, limit (4) and the following two facts:

(i) the function $(-1,0)\cup(0,1)\ni x\mapsto \frac{e^x-1-x}{x^2}$ is bounded

(ii) the function $(-1,0)\cup(0,1)\ni x\mapsto \frac{\sin x-x}{x^3}$ is bounded.

In the next few remarks we want to prove, in the elementary manner, that limit (4) implies the existence of limit $\lim_{x\to 0}(\sin x-x)x^{-3}$ and, in consequence, the condition (ii) follows, whereas, limit (1) implies the existence of limit $\lim_{x\to 0}(e^x-1-x)x^{-2}$ which means, that (i) is true. Accordingly, from (1) and (4) we receive the relation (2), which is our main aim.

Remark 2. It is known, that there exists the limit

$$g=\lim_{x\to 0}\frac{\sin x-x}{x^3}. \tag{7}$$

(see Remark 3.; and also, papers [1,2]). The value of g will be determined on the grounds of (5) and the properties of limits, written below

$$\begin{aligned} g&=\lim_{x\to 0}\frac{\sin(2x)-2x}{(2x)^3}=\frac{1}{4}\lim_{x\to 0}\frac{\sin x\cos x-x}{x^3}=\\ &=\frac{1}{4}\lim_{x\to 0}\left(\cos x\frac{\sin x-x}{x^3}+\frac{\cos x-1}{x^2}\right)=\frac{1}{4}\left(g-\frac{1}{2}\right), \end{aligned}$$

which implies $g=-\frac{1}{6}$.

Remark 3. Now let us present an elementary method of proving the existence of limit $\lim_{x\to 0}\frac{\sin x-x}{x^3}$, basing only on the assumption (4) (without using the concept of derivative).

We shall utilize the following transformation

$$\begin{aligned} \sin x-x&=\sin x-\sin x\cos x+\sin x\cos x-x\\ &=(1-\cos x)\sin x+\frac{1}{2}(\sin 2x-2x), \end{aligned}$$

from which the following identity may be derived

$$\frac{\sin 2x-2x}{(2x)^3}-\frac{1}{4}\cdot\frac{\sin x-x}{x^3}=\frac{(\cos x-1)}{x^2}\cdot\frac{\sin x}{x},\qquad x\in\mathbb{R}\setminus\{0\}, \tag{8}$$

and, subsequently, by iteration, the following general formula can be obtained

$$\begin{aligned} &\frac{\sin x-x}{x^3}-\frac{1}{x^2}\left(\frac{\sin(2^{-N}x)}{2^{-N}x}-1\right)=\\ &\qquad=\sum_{k=0}^{N-1}\left(\frac{1}{4^k}\cdot\frac{\sin(2^{-k}x)-2^{-k}x}{(2^{-k}x)^3}-\frac{1}{4^{k+1}}\cdot\frac{\sin(2^{-k-1}x)-(2^{-k-1}x)}{(2^{-k-1}x)^3}\right)\\ &\qquad=\sum_{k=0}^{N-1}\frac{1}{4^{k+1}}\cdot\frac{\cos(2^{-k-1}x)-1}{(2^{-k-1}x)^2}\cdot\frac{\sin(2^{-k-1}x)}{2^{-k-1}x}. \end{aligned}\tag{9}$$

From (4) and (5) we have, that for any $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$, such that

$$-\frac{1}{2} - \varepsilon \leqslant \frac{\cos(y) - 1}{y^2} \cdot \frac{\sin y}{y} \leqslant -\frac{1}{2} + \varepsilon$$

whenever $|y| \leqslant \delta$. So, if $0 < |x| \leqslant \delta$, then from (9) we may derive the following estimations:

$$\Big(-\frac{1}{2} - \varepsilon\Big)\frac{1}{4} \cdot \frac{1 - 4^{-N}}{1 - 4^{-1}} + \frac{1}{x^2}\Big(\frac{\sin(2^{-N}x)}{2^{-N}x} - 1\Big) \leqslant$$
$$\leqslant \frac{\sin x - x}{x^3} \leqslant \frac{1}{x^2}\Big(\frac{\sin(2^{-N}x)}{2^{-N}x} - 1\Big) + \Big(-\frac{1}{2} + \varepsilon\Big)\frac{1}{4} \cdot \frac{1 - 4^{-N}}{1 - 4^{-1}},$$

i.e.,

$$\frac{1}{3}\Big(-\frac{1}{2} - \varepsilon\Big)(1 - 4^{-N}) + \frac{1}{x^2}\Big(\frac{\sin(2^{-N}x)}{2^{-N}x} - 1\Big) \leqslant$$
$$\leqslant \frac{\sin x - x}{x^3} \leqslant \frac{1}{x^2}\Big(\frac{\sin(2^{-N}x)}{2^{-N}x} - 1\Big) + \frac{1}{3}\Big(-\frac{1}{2} + \varepsilon\Big)(1 - 4^{-N}).$$

Hence, with N converging to the limit, it can be finally derived:

$$\frac{1}{3}\Big(-\frac{1}{2} - \varepsilon\Big) \leqslant \frac{\sin x - x}{x^3} \leqslant \frac{1}{3}\Big(-\frac{1}{2} + \varepsilon\Big),$$

which proves the existence of the limit: $\lim\limits_{x\to 0} \frac{\sin x - x}{x^3} = -\frac{1}{6}$.

Remark 4. Similarly like in Remark 2., from (1) we can easily find the value of

$$q := \lim_{x\to 0} \frac{e^x - 1 - x}{x^2}. \tag{10}$$

Definitely, we have

$$\frac{e^{2x} - 1 - 2x}{(2x)^2} = \frac{1}{4x^2}\big((e^x - 1)(e^x + 1) - 2(e^x - 1) + 2(e^x - 1) - 2x\big)$$
$$= \frac{1}{4}\Big(\frac{e^x - 1}{x}\Big)^2 + \frac{1}{2} \cdot \frac{e^x - 1 - x}{x^2}, \tag{11}$$

which implies $q = \frac{1}{4} + \frac{1}{2}q$, i.e. $q = \frac{1}{2}$ is derived. Moreover, from (11) we obtain the following relation

$$\frac{e^{2x} - 1 - 2x}{(2x)^2} - \frac{1}{2} \cdot \frac{e^x - 1 - x}{x^2} = \frac{1}{4}\Big(\frac{e^x - 1}{x}\Big)^2,$$

from which the existence of limit (10) results only if formula (1) is assumed (like in Remark 3., from formula (8) the existence of the limit $\lim\limits_{x\to 0} \frac{\sin x - x}{x^3}$ may be deduced).

At the end of this note we propose the reader to find the following limits but only by using limit (2) and the general facts for limits:

b) $\lim\limits_{z=x+iy\to 0} \frac{1}{x+iy}\left((e^x-1)\cos\frac{y}{2}+i(e^x+1)\sin\frac{y}{2}\right)$;

c) $\lim\limits_{z=x+iy\to 0} \frac{e^x+e^{iy}-2}{x+iy}$;

d) $\lim\limits_{z=x+iy\to 0} \frac{e^x-e^{iy}+2iy}{x+iy}$;

e) $\lim\limits_{z=x+iy\to 0} \frac{(e^x+e^y-2)^2}{x+iy}$ and $\lim\limits_{z=x+iy\to 0} \frac{(e^{ix}+e^{iy}-2)^2}{x+iy}$.

Taking advantage of the fact, that $\lim\limits_{n\to\infty}\left(1+\frac{z}{n}\right)^n = e^z$ for any $z\in\mathbb{C}$, it is possible to calculate the following two limits:

f) $\lim\limits_{n\to\infty} \frac{\left(1+\frac{r+i\varphi}{n}\right)^n+\left(1+\frac{r+i\psi}{n}\right)^n}{\left(1+\frac{2r+i(\varphi+\psi)}{2n}\right)^n}$, $\qquad r,\varphi,\psi\in\mathbb{R}$;

g) $\lim\limits_{n\to\infty} n\left[\frac{\left(1+\frac{\varphi+i\varphi}{n}\right)^n+\left(1+\frac{\varphi-i\varphi}{n}\right)^n}{2e^{\varphi}\cos\varphi}-1\right]$, $\qquad \varphi\in\mathbb{R}$.

Solutions. *a)* Let $z=x+iy$, $x,y\in\mathbb{R}$. Accordingly, we have

$$e^z = e^x(\cos y + i\sin y),$$
$$e^z-1-z = e^x(\cos y-1)+e^x-1-x+i\big(e^x(\sin y-y)+y(e^x-1)\big),$$
$$\frac{e^z-1-z}{z} = e^x\cdot\frac{\cos y-1}{y^2}\cdot\frac{y^2}{|z|^2}\cdot\overline{z}+\frac{e^x-1-x}{x^2}\cdot\frac{x^2}{|z|^2}\cdot\overline{z}+$$
$$+i\left[e^x\cdot\frac{\sin y-y}{y^3}\cdot\frac{y^3}{|z|^2}\cdot\overline{z}+\frac{e^x-1}{x}\cdot\frac{xy}{|z|^2}\cdot\overline{z}\right],$$

which, in view of relation (1) (resulting from (i)), the boundedness of the functions:

$$\frac{x^2}{|z|^2},\quad \frac{y^2}{|z|^2},\quad \frac{xy}{|z|^2}$$

(all three of them are defined in the set $\mathbb{C}\setminus\{0\}$)

$$\frac{y^3}{|z|^2}\qquad \text{(determined in the annulus } 0<|z|<1\text{)},$$

and the remaining assumptions of point a), implies relation (2).

b) From Euler formulae we get

$$\begin{aligned}(e^x-1)\cos\frac{y}{2}+i(e^x+1)\sin\frac{y}{2} &= \\ &= \frac{1}{2}\big[(e^x-1)(e^{iy/2}+e^{-iy/2})+(e^x+1)(e^{iy/2}-e^{-iy/2})\big] = \\ &= \frac{1}{2}e^{-iy/2}\big[(e^x-1)(e^{iy}+1)+(e^x+1)(e^{iy}-1)\big] = e^{-iy/2}[e^{x+iy}-1].\end{aligned}$$

Hence, limit *b)* is equal to

$$\lim_{z=x+iy\to 0} e^{-iy/2}\cdot\frac{e^z-1}{z}=1.$$

c) We have

$$\begin{aligned}\lim_{x,y\to 0}\frac{e^x+e^{iy}-2}{x+iy} &= \lim_{x,y\to 0}\frac{e^{x+iy}-1-(e^x-1)(e^{iy}-1)}{x+iy} \\ &= \lim_{z=x+iy\to 0}\frac{e^z-1}{z}+\lim_{x,y\to 0}-i\frac{xy}{x^2+y^2}\cdot\frac{e^x-1}{x}\cdot\frac{e^{iy}-1}{iy}\cdot(x-iy),\end{aligned}$$

from which, in view of the boundedness of function $0\neq z=x+iy\mapsto\frac{-ixy}{x^2+y^2}$ and by (3) (bearing in mind, that the latter follows from (4)), we obtain

$$\lim_{x,y\to 0}\frac{e^x+e^{iy}-2}{x+iy}=1.$$

d) We easily find

$$e^x-e^{iy}=e^x-e^{-iy}+(e^{-iy}-e^{iy})=e^{-iy}(e^{x+iy}-1)-2i\sin y,$$

from which we obtain

$$\begin{aligned}\lim_{x,y\to 0}\frac{e^x-e^{iy}+2iy}{x+iy} &= \\ &= \lim_{z=x+iy\to 0}e^{-iy}\frac{e^z-1}{z}+2i\lim_{x,y\to 0}\frac{y-\sin y}{y^3}\cdot\frac{y^2}{x^2+y^2}\cdot y(x-iy).\end{aligned}$$

f) The sought limit is equal to the expression

$$\frac{e^{r+i\varphi}+e^{r+i\psi}}{e^{r+i(\varphi+\psi)/2}}=\frac{e^{i\varphi}+e^{i\psi}}{e^{i(\varphi+\psi)/2}}=2\cos\left(\frac{\varphi-\psi}{2}\right).$$

g) We have

$$\left(1+\frac{z}{n}\right)^n=\exp\left(n\ln\left(1+\frac{z}{n}\right)\right)=e^z\left(1-\frac{z^2}{2n}+\frac{z^3(8z+3)}{24n^2}+\ldots\right),$$

from which we obtain

$$n\left[\left(1+\frac{\varphi+i\varphi}{n}\right)^n+\left(1+\frac{\varphi-i\varphi}{n}\right)^n-2e^{\varphi}\cos\varphi\right]=$$

$$=-\frac{1}{2}\left((\varphi+i\varphi)^2e^{\varphi+i\varphi}+(\varphi-i\varphi)^2e^{\varphi-i\varphi}\right)+O\left(\frac{1}{n}\right)$$

$$=-ie^{\varphi}\varphi^2\left(e^{i\varphi}-e^{-i\varphi}\right)+O\left(\frac{1}{n}\right)=2e^{\varphi}\varphi^2\sin\varphi+O\left(\frac{1}{n}\right),$$

i.e.,

$$\lim_{n\to\infty} n\left[\frac{\left(1+\frac{\varphi+i\varphi}{n}\right)^n+\left(1+\frac{\varphi-i\varphi}{n}\right)^n}{2e^{\varphi}\cos\varphi}-1\right]=\varphi^2\operatorname{tg}\varphi.$$

References

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Streszczenie

W pracy zostało omówione zagadnienie istnienia pewnej granicy zespolonej w kontekście istnienia odpowiednich granic rzeczywistych. Przy okazji zaprezentowano całkowicie elementarną metodę dowodzenia implikacji: z istnienia granicy:

$$\lim_{x\to 0}\frac{\sin x}{x}=1$$

wynika istnienie granic (dla każdego $n\in\mathbb{N}$):

$$\lim_{x\to 0} x^{-n-1}\left(\sin x-\sum_{k=1}^{n}(-1)^{k-1}\frac{x^k}{(2k-1)}\right)=\frac{(-1)^n}{(2n+1)!}.$$