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ON SOME INEQUALITY OF THE TRYGONOMETRIC TYPE

Summary. In the paper the following inequality

$$\cos(\alpha(x) + \gamma \sin x) > \sin(\beta(x) + \delta \cos x)$$

is discussed. The conditions, which the functions and parameters $(\alpha(x), \beta(x), \gamma, \delta)$ are required to satisfy, so that the inequality holds, are given. Also the value $\min\{\cos(\sin x) - \sin(\cos x) : x \in \mathbb{R}\}$ is derived. Some applications for other inequalities are given in this paper.

O PEWNEJ NIERÓWNOŚCI TRYGONOMETRYCZNEJ

Streszczenie. W artykule omawiana jest następująca nierówność:

$$\cos(\alpha(x) + \gamma \sin x) > \sin(\beta(x) + \delta \cos x).$$

Podano warunki dla funkcji i parametrów $(\alpha(x), \beta(x), \gamma, \delta)$, które gwarantują zachodzenie tej nierówności. Wyznaczono wartość:

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$\min\{\cos(\sin x) - \sin(\cos x) : x \in \mathbb{R}\}$. W artykule zaproponowano także pewne zastosowania otrzymanych wyników do analizy innych nierówności trygonometrycznych.

1. One simple proof

In support of the following, so called, summation formula (for complex numbers from the unit circle), we have

$$e^{i\varphi} + e^{i\psi} = 2\cos\Big(\frac{\varphi-\psi}{2}\Big) e^{i\frac{\varphi+\psi}{2}}.$$

From this formula we easy obtain two special identities

$$1 + e^{i\varphi} = 2\cos\frac{\varphi}{2}\, e^{i\frac{\varphi}{2}},$$
$$1 - e^{i\varphi} = 1 + e^{i(\pi+\varphi)} = 2\cos\frac{\pi+\varphi}{2}\, e^{i\frac{\pi+\varphi}{2}} = -i\,2\sin\frac{\varphi}{2}\, e^{i\frac{\varphi}{2}}$$

and the following one

$$\begin{aligned}\cos\varphi - \sin\psi &= \mathrm{Re}\big(e^{i\varphi} + i\,e^{i\psi}\big) = \\ &= \mathrm{Re}\big(e^{i\varphi} + e^{i(\frac{\pi}{2}+\psi)}\big) = \mathrm{Re}\Big(2\cos\Big(\frac{1}{2}(\varphi-\psi-\frac{\pi}{2})\Big)\, e^{i\frac{1}{2}(\varphi+\psi+\frac{\pi}{2})}\Big) = \\ &= 2\cos\Big(\frac{1}{2}(\varphi-\psi-\frac{\pi}{2})\Big)\cdot\cos\Big(\frac{1}{2}(\varphi+\psi+\frac{\pi}{2})\Big). \quad (1)\end{aligned}$$

Hence we get the next identity

$$\begin{aligned}\cos\big(\alpha(x) + \gamma\sin x\big) - \sin\big(\beta(x) + \delta\cos x\big) &= \\ &= 2\cos\Big(\frac{1}{2}\big(\alpha(x) - \beta(x) + \gamma\sin x - \delta\cos x - \frac{\pi}{2}\big)\Big)\times\end{aligned}$$

$$\begin{aligned}&\times\cos\Big(\frac{1}{2}\big(\alpha(x) + \beta(x) + \gamma\sin x + \delta\cos x + \frac{\pi}{2}\big)\Big) = \\ &= 2\cos\Big(\frac{1}{2}\big(\alpha(x) - \beta(x) + \sqrt{\gamma^2+\delta^2}\sin(x - x_0) - \frac{\pi}{2}\big)\Big)\times \\ &\times\cos\Big(\frac{1}{2}\big(\alpha(x) + \beta(x) + \sqrt{\gamma^2+\delta^2}\sin(x + x_0) - \frac{\pi}{2}\big)\Big), \quad (2)\end{aligned}$$

where $\cos x_0 = \frac{\gamma}{\sqrt{\gamma^2+\delta^2}}$, $\sin x_0 = \frac{\delta}{\sqrt{\gamma^2+\delta^2}}$ and $x_0 \in [0, 2\pi)$.

Now if we suppose that for every $x \in \mathbb{R}$

$$0 \leqslant \beta(x) \leqslant \frac{\pi}{2} - \sqrt{\gamma^2+\delta^2} \quad \text{and} \quad |\alpha(x)| \leqslant \frac{\pi}{2} - \sqrt{\gamma^2+\delta^2} - \beta(x), \tag{3}$$

then we obtain

$$\beta(x) + \sqrt{\gamma^2+\delta^2} - \frac{\pi}{2} \leqslant \alpha(x) \leqslant \frac{\pi}{2} - \sqrt{\gamma^2+\delta^2} - \beta(x),$$

which implies two following estimations

$$\begin{aligned} -\pi = \sqrt{\gamma^2+\delta^2} - \frac{\pi}{2} - \sqrt{\gamma^2+\delta^2} - \frac{\pi}{2} \leqslant \\ \leqslant \alpha(x) - \beta(x) + \sqrt{\gamma^2+\delta^2}\sin(x - x_0) - \frac{\pi}{2} \leqslant \\ \leqslant \alpha(x) + \beta(x) + \sqrt{\gamma^2+\delta^2} - \frac{\pi}{2} \leqslant \frac{\pi}{2} - \sqrt{\gamma^2+\delta^2} + \sqrt{\gamma^2+\delta^2} - \frac{\pi}{2} = 0 \end{aligned}$$

and

$$\begin{aligned} 0 = \sqrt{\gamma^2+\delta^2} - \frac{\pi}{2} - \sqrt{\gamma^2+\delta^2} + \frac{\pi}{2} \leqslant \alpha(x) - \beta(x) - \sqrt{\gamma^2+\delta^2} + \frac{\pi}{2} \leqslant \\ \leqslant \alpha(x) + \beta(x) + \sqrt{\gamma^2+\delta^2}\sin(x + x_0) + \frac{\pi}{2} \leqslant \\ \leqslant \frac{\pi}{2} - \sqrt{\gamma^2+\delta^2} + \sqrt{\gamma^2+\delta^2} + \frac{\pi}{2} = \pi. \end{aligned}$$

The above considerations lead to the theorem.

Theorem 1. *If the conditions (3) are satisfied, then the inequality holds*

$$\cos\big(\alpha(x) + \gamma \sin x\big) \geqslant \sin\big(\beta(x) + \delta \cos x\big), \qquad x \in \mathbb{R}. \tag{4}$$

Additionally, if $\alpha(x) - \beta(x) > \sqrt{\gamma^2+\delta^2} - \frac{\pi}{2}$ *for* $x = -\frac{\pi}{2} + x_0 + 2k\pi$, $k \in \mathbb{Z}$ *and* $\alpha(x) + \beta(x) < \frac{\pi}{2} - \sqrt{\gamma^2+\delta^2}$ *for* $x = \frac{\pi}{2} - x_0 + 2k\pi$, $k \in \mathbb{Z}$, *then the inequality (4) is sharp for every* $x \in \mathbb{R}$.

Moreover, if $\sqrt{\gamma^2+\delta^2} \in [\pi, 2\pi]$, $\beta(x) \equiv -\frac{1}{2}\sqrt{\gamma^2+\delta^2}$ *and*

$$\frac{1}{2}\Big(\sqrt{\gamma^2+\delta^2} - \pi\Big) \leqslant \alpha(x) \leqslant \frac{1}{2}\Big(3\pi - \sqrt{\gamma^2+\delta^2}\Big)$$

then it can be deduced, that

$$\cos\big(\alpha(x) + \gamma \sin x\big) \geqslant \sin\big(-\frac{1}{2}\sqrt{\gamma^2+\delta^2} + \delta \cos x\big) \tag{5}$$

for every $x \in [-x_0, \frac{\pi}{2} - x_0]$, *such that* $-\frac{\pi}{2} \in [-2x_0, \frac{\pi}{2} - 2x_0]$.

Corollary 2. *The following inequalities hold*

1^o $\cos(\gamma \sin x) > \sin(\delta \cos x), \qquad \sqrt{\gamma^2+\delta^2} \leqslant \frac{\pi}{2}, \quad x \in \mathbb{R};$

2^o $\cos\Big(\big(\frac{\pi}{2}-\sqrt{2}\big)\sin^k x + \sin x\Big) \geqslant \sin\Big(\big(\frac{\pi}{2}-\sqrt{2}\big)\cos^2 x + \cos x\Big), \ k \geqslant 2, \quad x \in \mathbb{R}$
and

$$\cos\Big(\big(\frac{\pi}{2}-\sqrt{2}\big)\cos^k x + \sin x\Big) \geqslant \sin\Big(\big(\frac{\pi}{2}-\sqrt{2}\big)\sin^2 x + \cos x\Big), \ k \geqslant 2, \quad x \in \mathbb{R};$$

3^o $\cos\Big(\big(\frac{\pi}{2}-\sqrt{2}\big)\chi_A(x) + \sin x\Big) \geqslant \sin\Big(\big(\frac{\pi}{2}-\sqrt{2}\big)\chi_{\mathbb{R}\setminus A}(x) + \cos x\Big),$
where $\chi_B(x)$ means the characteristic function of the set B, for every $B \subset \mathbb{R}$.

In the next two sections our aim will be to find the following value

$$\min\big\{x \in \mathbb{R} : \ \cos(\sin x) > \sin(\cos x)\big\}.$$

2. Some auxiliary functions

Let us denote

$$\begin{aligned} f(x) &= \frac{\sin x}{x}, && x \neq 0,\\ g(x) &= \frac{\cos x}{x}, && x \neq 0,\\ h(x) &= f(\sin x) - g(\cos x), && x \neq k\,\frac{\pi}{2}, \ k \in \mathbb{Z}, \end{aligned}$$

and

$$\eta(x) = \cos(\sin x) - \sin(\cos x).$$

Lemma 3. *We have*

$$f'(x) = \frac{1}{x^2}\big(x - \operatorname{tg} x\big)\cos x, \qquad x \neq (2k+1)\frac{\pi}{2}, \ k \in \mathbb{Z},$$

and

$$g'(x) = -\frac{1}{x^2}\big(x + \operatorname{ctg} x\big)\sin x, \qquad x \neq k\,\pi, \ k \in \mathbb{Z}.$$

Corollary 4. *Let* $\operatorname{tg} x_0 = x_0$, $x_0 \in \left(\pi, \frac{3}{2}\pi\right)$. *Then function* $f(x)$ *is decreasing in the interval* $(0, x_0]$ *from the value* 1 *to* $\frac{\sin x_0}{x_0}$.

Corollary 5. *Let* $\operatorname{ctg} x_1 = -x_1$, $x_1 \in \left(\frac{\pi}{2}, \pi\right)$. *Then function* $g(x)$ *is decreasing in the interval* $(0, x_1)$ *from* $+\infty$ *to the value* $\frac{\cos x_1}{x_1}$.

Directly from Corollary 4. and 5. we obtain the following two lemmas.

Lemma 6. *Function* $f(\sin x)$ *(as composition of increasing function with decreasing function) is decreasing in the interval* $(0, \frac{\pi}{2}]$ *from the value* 1 *to* $\sin(1)$, *however in the interval* $[\frac{\pi}{2}, \pi)$ *(as composition of two decreasing functions) is increasing from the value* $\sin(1)$ *to* 1.

Lemma 7. *Function* $g(\cos x)$ *in the interval* $[0, \frac{\pi}{2})$ *is increasing from the value* $\cos(1)$ *to* $+\infty$, *however in the interval* $(\frac{\pi}{2}, \pi)$ *this function (equal to* $-g(|\cos x|)$*), as composition of one increasing function and two decreasing functions) is increasing from* $-\infty$ *to the value* $-\cos(1)$.

At the end, from Lemma 6. and 7., we get the basic result.

Theorem 8. *The function* $h(x)$ *(as a sum of functions* $f(\sin x)$ *and* $-g(\cos x)$ *i.e. two decreasing functions) is decreasing in the interval* $(0, \frac{\pi}{2})$ *and this function has exactly one real root* $z_1 \in (0, \frac{\pi}{4})$.

Proof. Because we have

$$\lim_{x \to 0_+} h(x) = 1 - \cos(1) > 0,$$

$$\lim_{x \to \frac{\pi}{2}_-} h(x) = \sin(1) - (+\infty) = -\infty,$$

therefore there exists exactly one root $z_1 \in (0, \frac{\pi}{2})$ of the function $h(x)$. Because $\frac{\sqrt{2}}{2} < \frac{3}{4} < \frac{\pi}{4}$, thus $h(\frac{\pi}{4}) = \sqrt{2}(\sin(\frac{\sqrt{2}}{2}) - \cos(\frac{\sqrt{2}}{2})) < 0$ and in consequence $z_1 < \frac{\pi}{4}$. After direct calculation we obtain

$$z_1 \simeq 0.69272857018683388834 7365\,\text{rad} \simeq 39.69042341983760570558 80^\circ$$

and

$$f(\sin z_1) = g(\cos z_1) \simeq 0.93339618940889841184 6964.$$

□

3. The minimum of the function $\eta(x)$

Theorem 9. *We evaluate*

$$\begin{aligned}\min\{\eta(x):\ x\in\mathbb{R}\} &= \eta(z_1) = \\ &= \frac{\cos(\cos z_1)+\sin(\sin z_1)}{\cos(\sin z_1)+\sin(\cos z_1)}\cdot\frac{\cos(\cos z_1)}{\cos z_1}\cdot\big(\cos z_1-\sin z_1\big) = \\ &= \frac{\cos z_1+\sin z_1}{\cos(\sin z_1)+\sin(\cos z_1)}\cdot\frac{\cos^2(\cos z_1)}{\cos^2 z_1}\cdot\big(\cos z_1-\sin z_1\big)\simeq \\ &\simeq 0.107126944872952996112029.\end{aligned}$$

Proof. We have

$$\eta'(x) = -\frac{1}{2}\,h(x)\,\sin 2x,$$

from where we obtain

$$\begin{aligned}\eta'(x) &< 0 \qquad \text{for } x\in(0,z_1),\\ \eta'(x) &> 0 \qquad \text{for } x\in(z_1,\tfrac{\pi}{2})\end{aligned}$$

and $\eta'(z_1)=0$. Let us remind, that $f(\sin z_1)=g(\cos z_1)$, which implies the following equalities

$$\begin{aligned}\eta(z_1) &= \frac{\cos^2(\cos z_1)-\sin^2(\sin z_1)}{\cos(\sin z_1)+\sin(\cos z_1)} = \\ &= \frac{g^2(\cos z_1)\cos^2(z_1)-f^2(\sin z_1)\sin^2(z_1)}{\cos(\sin z_1)+\sin(\cos z_1)} = \\ &= \frac{\cos^2(z_1)-\sin^2(z_1)}{\cos(\sin z_1)+\sin(\cos z_1)}\cdot g^2(\cos z_1)\end{aligned}$$

or

$$\begin{aligned}\eta(z_1) &= \frac{\cos(\cos z_1)+\sin(\sin z_1)}{\cos(\sin z_1)+\sin(\cos z_1)}\big(g(\cos z_1)\,\cos z_1 - f(\sin z_1)\,\sin z_1\big) = \\ &= \frac{\cos(\cos z_1)+\sin(\sin z_1)}{\cos(\sin z_1)+\sin(\cos z_1)}g(\cos z_1)\big(\cos z_1-\sin z_1\big).\end{aligned}$$

□

4. Another problem

In this section we will find the maximum value $p > 0$, such that

$$(\cos(\sin x))^p \geqslant \sin(\cos x), \qquad x \in \left(0, \frac{\pi}{2}\right).$$

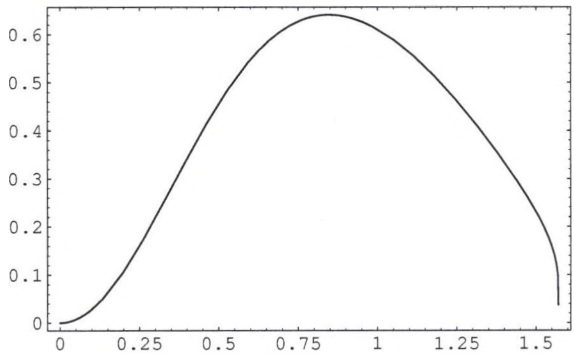

Fig. 1. Plot of the function $r(x)$
Rys. 1. Wykres funkcji $r(x)$

We will give only the numerical solution of the considered problem. The plot of the function

$$r(x) = \frac{\ln(\cos(\sin x))}{\ln(\sin(\cos x))}$$

is presented in the Figure 1. The function $r(x)$ has the maximum for the argument

$$x_0 \approx 0.84702526408125332281975$$

and the value of this maximum is equal to

$$r_{max} = r(x_0) \approx 0.6410192376328008246798234.$$

Hence the required maximum value of p is equal to

$$p_{max} = \frac{1}{r_{max}} \approx 1.560015583452483583231204.$$

5. Some applications of the function $\eta(x)$

We have the identities

$$\sin\varphi + \sin\psi + \sin(\varphi+\psi) = \operatorname{Im}\big(e^{i\varphi} + e^{i\psi} + e^{i(\varphi+\psi)}\big) = \\ = \operatorname{Im}\Big(\big(1+e^{i\varphi}\big)\big(1+e^{i\psi}\big)\Big) = 4\cos\frac{\varphi}{2}\cos\frac{\psi}{2}\operatorname{Im}\big(e^{i\frac{\varphi+\psi}{2}}\big) = \\ = 4\cos\frac{\varphi}{2}\cos\frac{\psi}{2}\sin\frac{\varphi+\psi}{2}, \tag{6}$$

$$1 + \cos\varphi + \cos\psi + \cos(\varphi+\psi) = \operatorname{Re}\big(1 + e^{i\varphi} + e^{i\psi} + e^{i(\varphi+\psi)}\big) = \\ = \operatorname{Re}\Big(\big(1+e^{i\varphi}\big)\big(1+e^{i\psi}\big)\Big) = \operatorname{Re}\Big(4\cos\frac{\varphi}{2}\cos\frac{\psi}{2}e^{i\frac{\varphi+\psi}{2}}\big) = \\ = 4\cos\frac{\varphi}{2}\cos\frac{\psi}{2}\cos\frac{\varphi+\psi}{2}, \tag{7}$$

$$1 + \cos\varphi - \cos\psi - \cos(\varphi+\psi) = \operatorname{Re}\big(1 + e^{i\varphi} - e^{i\psi} - e^{i(\varphi+\psi)}\big) = \\ = \operatorname{Re}\Big(\big(1+e^{i\varphi}\big)\big(1-e^{i\psi}\big)\Big) = \operatorname{Re}\Big(-4\,i\cos\frac{\varphi}{2}\sin\frac{\psi}{2}e^{i\frac{\varphi+\psi}{2}}\big) = \\ = 4\cos\frac{\varphi}{2}\sin\frac{\psi}{2}\sin\frac{\varphi+\psi}{2}. \tag{8}$$

Immediately from equation (8) we obtain

$$\cos\varphi - \sin\psi = \cos\varphi - \cos\big(\frac{\pi}{2} - \psi\big) = \\ = \cos\big(\frac{\pi}{2} + \varphi - \psi\big) - 1 + 4\cos\frac{\varphi}{2}\sin\frac{\frac{\pi}{2}-\psi}{2}\sin\Big(\frac{\frac{\pi}{2}+\varphi-\psi}{2}\Big) = \\ = -\sin(\varphi-\psi) - 1 + 4\cos\frac{\varphi}{2}\sin\frac{\frac{\pi}{2}-\psi}{2}\sin\Big(\frac{\frac{\pi}{2}+\varphi-\psi}{2}\Big) \tag{9}$$

and hence (for $\varphi = \sin x$ and $\psi = \cos x$) we get the estimation

$$4\cos\frac{\sin x}{2}\sin\frac{\frac{\pi}{2}-\cos x}{2}\sin\Big(\frac{\frac{\pi}{2}+\sin x-\cos x}{2}\Big) = \\ = 1 + \sin(\sin x - \cos x) + \eta(x) \geqslant 1 + \sin\Big(\sqrt{2}\sin\big(x - \frac{\pi}{4}\big)\Big) + \eta(z_1) \geqslant \\ \geqslant 1 - \sin(\sqrt{2}) + \eta(z_1) \approx 0.119360998880217469042895. \tag{10}$$

This estimation is very good, because the function $r_1(x) = 1 + \sin(\sin x - \cos x) + \eta(x)$ has the minimum value equal to $r_{1min} \approx 0.120100736258068063753489$.

Similarly, from equation (6) we obtain

$$\begin{aligned}\cos\varphi - \sin\psi &= \sin(\frac{\pi}{2}+\varphi) + \sin(-\psi) = \\ &= -\sin(\frac{\pi}{2}+\varphi-\psi) + 4\cos\Big(\frac{\frac{\pi}{2}+\varphi}{2}\Big)\cos\frac{-\psi}{2}\sin\Big(\frac{\frac{\pi}{2}+\varphi-\psi}{2}\Big) = \\ &= -\cos(\varphi-\psi) + 4\cos\Big(\frac{\frac{\pi}{2}+\varphi}{2}\Big)\cos\frac{\psi}{2}\sin\Big(\frac{\frac{\pi}{2}+\varphi-\psi}{2}\Big) \end{aligned} \tag{11}$$

and hence (for $\varphi = \sin x$ and $\psi = \cos x$) we receive

$$\begin{aligned}4\cos\Big(\frac{\frac{\pi}{2}+\sin x}{2}\Big)\cos\frac{\cos x}{2}\sin\Big(\frac{\frac{\pi}{2}+\sin x-\cos x}{2}\Big) &= \\ = \cos(\sin x - \cos x) + \eta(x) \geqslant \cos\Big(\sqrt{2}\sin(x-\frac{\pi}{4})\Big) + \eta(z_1) &\geqslant \\ \geqslant \cos(\sqrt{2}) + \eta(z_1) \approx 0.263070639638327469566677. \end{aligned} \tag{12}$$

This estimation is also very good, because the function: $r_2(x) = \cos(\sin x - \cos x) + \eta(x)$ has the minimum value equal to $r_{2min} \approx 0.265230413973439632397303$.

Remark 10. All numerical calculations were made in *Mathematica.*[1]

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[1] *Mathematica* is registered trademark of Wolfram Research Inc.

Streszczenie

W artykule omawiana jest następująca nierówność:

$$\cos\bigl(\alpha(x) + \gamma \sin x\bigr) > \sin\bigl(\beta(x) + \delta \cos x\bigr).$$

Podano warunki dla funkcji i parametrów $(\alpha(x), \beta(x), \gamma, \delta)$, które gwarantują zachodzenie tej nierówności. Badano także wartość: $\min\{\cos(\sin x) - \sin(\cos x) : x \in \mathbb{R}\}$. Zagadnienie to jest uogólnieniem problemu zaproponowanego przez autorów w American Mathematical Monthly [4]. Problem ten dotyczył udowodnienia omówionej we Wniosku 2 nierówności 1^o. W artykule wyznaczono także numerycznie maksymalną wartość $p > 0$, takiego że $(\cos(\sin x))^p \geqslant \sin(\cos x)$ dla $x \in (0, \pi/2)$. W ostatnim rozdziale zaproponowano także zastosowania otrzymanych wyników do analizy innych nierówności trygonometrycznych.