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FINITE VERSION OF CARLEMAN'S AND KNOPP'S INEQUALITIES

Summary. In this note the estimations of the finite versions of the Carleman's, the Knopp's and the Amghibech's inequalities are improved.

SKOŃCZONE WERSJE NIERÓWNOŚCI CARLEMANA I KNOPPA

Streszczenie. W artykule poprawiono oszacowania w skończonych wersjach nierówności Carlemana, Knoppa i Amghibecha.

The main object of this note is to improve the estimation of the finite version of the Carleman's inequality (1). Usually on the right side of this inequality there appears the expression: $(1+\frac{1}{n})^n$ (replacing the number e from the standard infinite case). It turns out that the above expression can be replaced by $(1+\frac{1}{n})^{n-1}$.

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Theorem 1. ***(Finite version of the Carleman's inequality).*** *Let* $x_k > 0$, $1 \leqslant k \leqslant n$. *Then we have*

$$\sum_{k=1}^{n} \left(x_1\, x_2\, \ldots\, x_k\right)^{1/k} \leqslant \left(1 + \frac{1}{n}\right)^{n-1} \sum_{k=1}^{n} x_k. \tag{1}$$

Proof. Our proof is a simple adaptation of the proof of Theorem 1 from the Ping and Guozheng paper [3]. Let $c_k > 0$, $1 \leqslant k \leqslant n$. Subsequently, by using the A-G inequality, we obtain:

$$\sum_{k=1}^{n} \left(x_1\, x_2\, \ldots\, x_k\right)^{1/k} = \sum_{k=1}^{n} \left(\frac{c_1\, x_1 \cdot c_2\, x_2 \cdot \ldots \cdot c_k\, x_k}{c_1\, c_2\, \ldots\, c_k}\right)^{1/k} \leqslant$$

$$= \sum_{k=1}^{n} \left(c_1\, c_2\, \ldots\, c_k\right)^{-1/k} \frac{1}{k} \sum_{l=1}^{k} c_l\, x_l = \sum_{l=1}^{n} c_l\, x_l \sum_{k=l}^{n} \frac{1}{k} \left(c_1\, c_2\, \ldots\, c_k\right)^{-1/k} =$$

(We will now define numbers c_k in such a way, that $c_1\, c_2\, \ldots\, c_k = (k+1)^k$. Accordingly we have $c_k = k\,(1 + \frac{1}{k})^k$, for every $k = 1, 2, \ldots, n$.)

$$= \sum_{l=1}^{n} c_l\, x_l \sum_{k=l}^{n} \frac{1}{k\,(k+1)} = \sum_{l=1}^{n} \left(\frac{1}{l} - \frac{1}{n+1}\right) c_l\, x_l =$$

$$= \sum_{l=1}^{n} \left(1 - \frac{l}{n+1}\right) \left(1 + \frac{1}{l}\right)^{l} x_l < \left(1 - \frac{1}{n+1}\right) \left(1 + \frac{1}{n}\right)^{n} \sum_{l=1}^{n} x_l =$$

$$= \left(1 + \frac{1}{n}\right)^{n-1} \sum_{l=1}^{n} x_l,$$

since the sequence $\{(1 + \frac{1}{n})^n\}_{n=1}^{\infty}$ is increasing. □

Readers interested in the review of the state of knowledge about the Carleman's inequality and its generalizations should refer to [4].

Corollary 2. ***(The strengthenning of the Amghibech's inequality [1]).*** *Let* A *be a real symmetric positive definite matrix of degree* n. *Let* A_k *be the* $k \times k$ *matrix, placed in the left upper corner of* A. *Then we have*

$$\sum_{k=1}^{n} \left(\det A_k\right)^{1/k} \leqslant \left(1 + \frac{1}{n}\right)^{n-1} \mathrm{Tr}(A), \tag{2}$$

where $\mathrm{Tr}(A)$ *denotes the trace of the matrix* A.

Proof. The proof can be derived from the Hadamard's inequality and from the inequality (1) (see [1]). □

Remark 3. From the A-G inequality, the following inequality can be deduced:

$$\sum_{k=1}^{n} \left(x_1 x_2 \ldots x_k\right)^{1/k} \leqslant \sum_{k=1}^{n} \left(H_n - H_{k-1}\right) x_k, \tag{3}$$

where $H_0 := 0$, $H_k := \sum\limits_{l=1}^{k} \frac{1}{l}$ for $k = 1, 2, \ldots$, the numbers H_k are called harmonic numbers [2]. The above inequality occurs the equality only if $x_1 = x_2 = \ldots = x_n$.

Remark 4. Let us set

$$f(x_1, x_2, \ldots, x_n) := \frac{\sum_{k=1}^{n} \left(x_1 x_2 \ldots x_k\right)^{1/k}}{\sum_{k=1}^{n} x_k}$$

and

$$M_n := \max_{x_1, x_2, \ldots, x_n > 0} \{f(x_1, x_2, \ldots, x_n)\}.$$

Then the sequence $\{M_n\}_{n=1}^{\infty}$ is increasing. Moreover, we have:

$$M_2 = f\left(1, 3 - 2\sqrt{2}\right) = \frac{1 + \sqrt{2}}{2},$$

$$M_3 = f\left(1, \tfrac{1}{4}, \tfrac{1}{16}\right) = \frac{4}{3},$$

and by numerical calculations:

$$M_4 \approx 1.420844385409614, \quad M_5 \approx 1.486353228963051, \quad \textit{etc.}$$

Remark 5. Similarly, the finite version of the Knopp's inequality discussed in [5] (the case when $p = 1$) can be rendered in a more subtle way. We can deduce, that for $x_1, x_2, \ldots, x_n > 0$:

$$\sum_{k=1}^{n} \frac{k}{\sum_{j=1}^{k} 1/x_j} \leqslant 2\left(1 - \frac{1}{(n+1)^2}\right) \sum_{j=1}^{n} x_j,$$

which follows immediately from the proof given in [5].

Let us set

$$g(x_1, x_2, \ldots, x_n) := \left(\sum_{k=1}^{n} \frac{k}{\sum_{j=1}^{k} 1/x_j}\right) \left(\sum_{j=1}^{n} x_j\right)^{-1}.$$

and

$$M_n^* := \max_{x_1, x_2, \ldots, x_n > 0} \{g(x_1, x_2, \ldots, x_n)\}.$$

Then we have:

$$M_2^* = g(1, \tfrac{1}{3}) = \frac{9}{8},$$

and by numerical calculations

$$M_3^* \approx 1.204692944799793, \quad M_4^* \approx 1.261100464767132, \quad \textit{etc.}$$

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Streszczenie

W artykule poprawiono oszacowania w skończonych wersjach nierówności Carlemana, Knoppa i Amghibecha. Po stronie majoranty w nieskończonej, klasycznej wersji nierówności Carlemana występuje liczba e. W skończonej n składnikowej wersji tej nierówności zwykle pojawia się mnożnik $(1+\frac{1}{n})^n$. W pracy udowodniono, że można go zastąpić przez $(1+\frac{1}{n})^{n-1}$. W konsekwencji otrzymujemy poprawę oszacowania w pewnej nierówności Amghibecha dla macierzy rzeczywistych symetrycznych, dodatnio określonych. Podobnie otrzymujemy wzmocnienie pewnej nierówności Knoppa.