ZESZYTY NAUKOWE POLITECHNIKI ŚLĄSKIEJ 2004

Seria: MATEMATYKA-FIZYKA z. 91 Nr kol. 1652

Beata BAJORSKA

ON LOCALLY GRADED GROUPS

Summary. Locally graded groups were introduced in 1970. A group G is called locally graded if every nontrivial finitely generated subgroup in G has a proper normal subgroup of finite index. We prove some properties of locally graded groups. We also recall six known problems (three of which are equivalent), concerning positive laws, that have an affirmative answer in the class of locally graded groups. For four of the problems the general solution is still not known.

O GRUPACH LOKALNIE STOPNIOWYCH

Streszczenie. Grupy lokalnie stopniowe* zostały zdefiniowane w 1970 roku. Grupę nazywa się lokalnie stopniową, jeśli każda jej nietrywialna skończenie generowana podgrupa posiada właściwą podgrupę normalną skończonego indeksu. Udowodnimy kilka własności grup lokalnie stopniowych. Przedstawimy również sześć znanych problemów dotyczących tożsamości pół grupowych (z czego trzy równoważne), które mają pozytywne rozwiązanie w klasie grup lokalnie stopniowych. Cztery z tych problemów nadal pozostają otwarte.

*Tłumaczenie pochodzi od autorki

1. Introduction

A group G is *locally graded* if every nontrivial finitely generated subgroup in G has a proper normal subgroup of finite index (or equivalently – if every nontrivial finitely generated subgroup in G has a nontrivial finite image). For example, locally finite groups are locally graded. Indeed, let H be an arbitrary nontrivial finitely generated subgroup in a locally finite group G. Then H is finite and the trivial group $\{1\}$ is a proper normal subgroup of finite index in H, providing G to be locally graded.

The class of locally graded groups was introduced by Černikov in 1970 and considered in [9, 14, 22] among the others. This is a very extensive class of groups – it contains for instance all residually finite groups, all locally soluble groups and all known groups of intermediate growth, in fact it contains all groups, usually considered in textbooks. Thus it seems to be a natural restriction while solving problems, since (as we will show in Theorem 2. below) locally graded groups are these and exactly these, that do not have finitely generated infinite simple factors. This lets avoid the groups which need methods of Adian and Ol'shanskii.

There are known only a few types of groups which are not locally graded, like infinite Burnside groups and groups obtained by Ol'shanskii's geometric methods (e.g. the Tarski-Ol'shanskii monster and Ol'shanskii-Storozhev groups). The monster is a 2-generator infinite group, every proper subgroup of whose is a finite cyclic group of order p for some (very large) prime number p. If we suppose that the monster is locally graded, then, as a finitely generated group, it should contain a proper normal subgroup of finite index. But the group itself is infinite while all its subgroups are finite, and hence none can have finite index, a contradiction. So the Tarski-Ol'shanskii monster is not locally graded. The remaining two types of groups are not locally graded by Corollary 1. in section 3.

Knowing some examples, we have a lot of possibilities of creating new examples of groups that are not locally graded. For instance, since the class of locally graded groups is closed for taking subgroups (see (i) in Theorem 1. below), then also all groups containing subgroups isomorphic to non-locally graded groups are not locally graded theirselves – for instance direct products of groups, at least one of which is not locally graded.

Infinite Burside and Ol'shanskii-Storozhev groups are negative answers to two known problems (which we recall in section 3 as Problem 1. and 2., respectively) that have positive solution in the class of locally graded groups. Another example of a problem with a negative solution in general and a positive one for locally graded groups can be found in [6]. In section 3 we also recall other four problems concerning positive laws in groups (three of which are equivalent) that have an affirmative answer in the class of locally graded groups, but whose general solutions are not known yet.

2. Properties of locally graded groups

Theorem 1. *The class of locally graded groups is closed for*
(*i*) *subgroups*
(*ii*) *extensions*
(*iii*) *cartesian products*
(*iv*) *properties "locally" and "residually", that is if a group is locally- or residually-(locally graded), then it is a locally graded group.*

Proof. (*i*) Let G be a locally graded group, H – its subgroup and let C be an arbitrary finitely generated subgroup in H. Since C is also a subgroup in G, then by definition it contains a proper normal subgroup of finite index in G, which – by Lagrange Theorem – is also of finite index in H. Thus any subgroup of a locally graded group is locally graded itself.

(*ii*) Let G be a group and N – its locally graded normal subgroup such that $T := G/N$ is locally graded. Let H be an arbitrary nontrivial finitely generated subgroup in G. There are two possibilities:

(1°) *H belongs to N*. Since N is locally graded and H is its finitely generated nontrivial subgroup, then H has a proper normal subgroup of finite index.

(2°) *H does not belong to N*. Then the factor group $K := HN/N \cong H/H \cap N$ is a nontrivial finitely generated subgroup in the locally graded group T. Hence there is a proper normal subgroup M of finite index in K. By the theorem on homomorphisms [13, 4.2.1] we conclude that the full inverse image of M under the map $H \to K$ is a proper normal subgroup of finite index in H.

Thus, in both cases, H has a proper normal subgroup of finite index. Hence every group which is an extension of a locally graded group by a locally graded group is locally graded itself.

(iii) Let G be a cartesian product of groups $G_i, i \in I$, each of which is locally graded and let H be a nontrivial subgroup in G, finitely generated by $h_1, \dots, h_n$. Each element h_k, $k = 1, \dots, n$ can be treated as a sequence: $h_k := (h_{k1}, h_{k2}, \dots, h_{ki}, \dots)$, $h_{ki} \in G_i, i \in I$. We consider groups generated by h_{ij}, namely $H_i := gp(h_{1i}, \dots, h_{ni}) \subseteq G_i$ for every $i \in I$. Then H is a subcartesian product of groups H_i. Since H is nontrivial, then at least one of groups H_i is nontrivial, H_j let say. Next, since G_j is locally graded, then H_j is locally graded by (i). Hence, as a finitely generated locally graded group, H_j contains a normal subgroup M_j of finite index. Since there exists a nontrivial epimorphism $H \to H_j$, then it follows from [13, 4.2.1], that H has a proper normal subgroup of finite index. Thus we conclude, that every group which is a cartesian product of locally graded groups is locally graded itself.

$(iv)(a)$ Let G be locally-(locally graded), that is let every finitely generated subgroup in G be locally graded. If H is an arbitrary finitely generated subgroup in G, then it is locally graded, so in particular H contains a proper normal subgroup of finite index, and hence G is locally graded. Thus every locally-(locally graded) group is locally graded itself.

$(iv)(b)$ Let G be residually-(locally graded) and let H be its nontrivial subgroup, finitely generated by $h_1, \dots, h_n$. By assumption, there exists a family of normal subgroups $\mathcal{N} := \{N_i, i \in I\}$ in G with the trivial intersection, such that every factor group G/N_i is locally graded. Since $\bigcap N_i = \{1\}$, then there exists a group $N_j \in \mathcal{N}$ that does not contain the elements $h_1, \dots, h_n$. Hence we obtain, that $T_j := HN_j/N_j$ is a finitely generated subgroup in the locally graded group G/N_j. Then by definition, T_j has a normal subgroup of finite index. Since there exists a nontrivial epimorphism $H \to T_j$, then it follows from [13, 4.2.1] that H has a proper normal subgroup of finite index in G, so G is locally graded. Thus every residually-(locally graded) group is locally graded itself. □

Lemma 1. *The class of locally graded groups is not closed for taking homomorphic images.*

Proof. Finite groups are obviously locally graded and by (*iv*) in Theorem 1., residually finite groups also. Hence absolutely free groups, as residually finite ones [19, 6.1.9], are locally graded. Since every group is a homomorphic image of some absolutely free group, and there exist groups which are not locally graded, then the statement follows. Hence if group is locally graded, not every of its homomorphic images have to be locally graded itself. □

Let G be a group, H – its subgroup and let N be normal in H, that is $G > H \rhd N$. Then the factor group H/N is called *a factor of G*. The theorem below gives another definition of locally graded groups.

Theorem 2. *A group does not contain a finitely generated infinite simple factor if and only if it is locally graded.*

Proof. Suppose that G is a locally graded group and H/N is its (nontrivial) infinite simple factor, finitely generated by cosets $h_1N, h_2N, \ldots, h_nN$. We will show that it leads to contradiction. Let us consider the subgroup M of H, generated by $h_1, \ldots, h_n$. So under the canonical mapping $H \to H/N$, the subgroup M has the same image as H, that is $H/N = MN/N \cong M/M \cap N$. It follows then, that there exists a finitely generated infinite simple factor which is the image of a finitely generated subgroup in G. Hence we can assume that H/N itself is exactly like this. Thus by definition H, as a finitely genereted subgroup of a locally graded group, should contain a proper normal subgroup K of finite index. That implies that H/N should contain the proper normal subgroup KN/N of finite index, that is H/KN should be finite. Since H/N is simple, then $KN/N=\{1\}$ which means that $K \subseteq N$. Then H/N should be finite, which gives the required contradiction. Hence if a group contains a finitely generated infinite simple factor, it is not locally graded.

Conversely, suppose that G is not locally graded. Then by definition, there exists a finitely generated subgroup H in G, which is infinite and has no proper normal subgroup of finite index. If N is a maximal normal subgroup in H (which exists in each finitely generated group by Zorn's Lemma), then H/N is simple. Next, by assumption N has infinite index, which means that H/N is infinite. Moreover H/N is finitely generated as the image of the finitely generated group H. Hence, by the law of contraposition, it follows that if a group G does not contain a finitely generated infinite simple factor, it is locally graded, which finishes the proof. □

We consider the following properties of a group G:

P1 Each proper subgroup in G is nilpotent-by-finite.

P2 G is torsion.

P3 G is not simple.

Theorem 3. *(i) Let G be any group. If G has a subgroup of finite index, then the Property* ***P1*** *implies that G is nilpotent-by-finite.*

(ii) Let G be a locally graded group. The Properties ***P1*** *and* ***P2*** *imply that G is locally finite.*

(iii) Let G be a locally graded group. The Properties ***P1***, ***P2*** *and* ***P3*** *imply that G is nilpotent-by-finite.*

Proof. (*i*) If G has a subgroup of finite index, then G has a normal subgroup N of finite index by [13, 12.2.2], and by **P1**, N is nilpotent-by-finite. Then G is (nilpotent-by-finite)-by-finite, hence is nilpotent-by-(finite-by-finite) [17, 21.51] and thus nilpotent-by-finite [13, 23.1.1].

(*ii*) If H is a finitely generated subgroup in a locally graded group G, then H has a subgroup of finite index, and by (*i*), is nilpotent-by-finite. A torsion finitely generated nilpotent group is finite [13, 16.1.6], and hence a torsion nilpotent-by-finite group is also finite [13, 23.1.1]. Thus any finitely generated subgroup in G is finite providing G to be locally finite.

(*iii*) By **P3**, G has a nontrivial proper normal subgroup H, which by (*ii*) is finite. Then the centralizer of H has a finite index in G by [13, 3.1.4], and by (*i*), G is nilpotent-by-finite. □

3. A few problems concerning positive laws

Let us remind some notions. An n-ary relation

$$u(x_1, \ldots, x_n) = v(x_1, \ldots, x_n)$$

is called *a positive (or semigroup) relation*, if it does not involve inverses of any x_i. For instance, $x_2^3 x_1 = x_5^{11} x_1^2$ is positive, while $x_2^{-1} x_4 = x_3^8$ is not. We say that an n-tuple $g_1, \ldots, g_n$ of elements in a group G *satisfies a relation* $u(x_1, \ldots, x_n) = v(x_1, \ldots, x_n)$ if the equality $u(g_1, \ldots, g_n) = v(g_1, \ldots, g_n)$ holds in G. If every n-tuple in G satisfies the same n-ary positive relation $u = v$ then we say, that G *satisfies the n-ary positive (or semigroup) law* $u = v$. For instance, if every pair of elements in G commute, that is if satisfies $x_1 x_2 = x_2 x_1$, then this relation becomes a positive law in G, and in this case G is called abelian. Since every n-ary positive law implies 2-ary law (if substitute $x_i \to xy^i$ for instance), then it is enough to consider 2-ary (called also binary) laws only.

We recall here six known problems concerning positive laws in groups. The reader is referred to [3] for proofs and futher details.

A group G is of *a finite exponent* if there exists $n \in \mathbb{N}$ such that every element in G satisfies the relation $x^n = 1$. The following problem was posed in 1902 [8] by Burnside:

Problem 1. *Is it true that every finitely generated group of a finite exponent is finite?*

The negative answer was given in 1968 by Adian and Novikov [1], however the complete solution is still not known (e.g. the case of 5-generated groups). The comprehensive description of all Burnside problems and the history of their solutions can be found in [2] (see also [12] for some new results).

In 1953 [15] Mal'cev proved that every finitely generated nilpotent group satisfies a positive law (see also [16]). Hence it follows that every group G which is an extension of a nilpotent group N by a group K of a finite exponent also satisfies a positive law (if N satisfies $u(x,y) = v(x,y)$, then G satisfies $u(x^k, y^k) = v(x^k, y^k)$ for instance, where k is the exponent of K). Thus the following problem arises:

Problem 2. *If a group satisfies a positive law, must it be nilpotent-by-finite exponent?*

A hard work was done in finding groups that confirm this conjecture (see e.g. [7,10,20,21,23,24]). All these groups are locally graded, so Theorem 5. below gives the most general positive answer to this problem. After more than forty years, only in 1996, this problem was solved negatively [18].

A group G is called *n-Engel* if it satisfies the law $[\dots[[x,y],y],\dots,y]=1$ where y is repeated n times and $[x,y] := x^{-1}y^{-1}xy$ (or $[x,y] := xyx^{-1}y^{-1}$). The following question was posed by Shirshov in 1963 [25]:

Problem 3. *Does every n-Engel group satisfy a positive law?*

The general answer is known only for 2- and 3-Engel groups (see [23]) and for 4-Engel groups [24].

The three problems listed below are open.

A group G satisfies a *finite disjunction of positive relations* if there exists a finite set of positive relations $S := \{u_i(x,y) = v_i(x,y),\ i = 1,2,\dots,m\}$ such that every pair of elements in G satisfies some relation in S. In [4] Boffa posed the question:

Problem 4. *If a group satisfies a finite disjunction of positive relations, does it satisfy a positive law?*

A group G is called *n-collapsing* if for every n-element subset $S \subseteq G$, the inequality $|S^n| < n^n$ holds. A group is called *collapsing* if it is n-collapsing for some n. In the paper [21] by Shalev the following question appears:

Problem 5. *Does every collapsing group satisfy a positive law?*

Let G^ω denote the cartesian product of countably many copies of G indexed by natural numbers $\mathbb{N}$ (called *the cartesian power* of G). Let G^* denote *the ultrapower* of G modulo a fixed nonprincipal ultrafilter over natural numbers (i.e. the image of G^ω with respect to the congruence defined by a nonprincipal ultrafilter). Boffa in [5] considered the following problem:

Problem 6. *Suppose that G^* contains no free non-abelian subsemigroup. Does G^ω also contain no free non-abelian subsemigroups?*

Theorem 4. (cf. Theorem 2, [3]) *Problems 4., 5. and 6. are equivalent.*

Theorem 5. (cf. Theorems 1, 3 and 4, [3]) *All six problems formulated above have positive solutions in the class of locally graded groups.*

Corollary 1. *Infinite Burnside groups and Ol'shanskii-Storozhev groups are not locally graded.*

Proof. As it was mentioned, those two types of groups give a negative solution to Problem 1. and 2. respectively, hence by Theorem 5. they cannot be locally graded. □

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Beata Bajorska
Instytut Matematyki
Politechnika Śląska
Kaszubska 23
44-100 Gliwice

Streszczenie

Grupy lokalnie stopniowe zostały zdefiniowane w 1970 roku. Grupę nazywa się lokalnie stopniową, jeśli każda jej nietrywialna skończenie generowana podgrupa posiada właściwą podgrupę normalną skończonego indeksu. Klasa grup lokalne stopniowych jest zamknięta ze względu na podgrupy, rozszerzenia i iloczyny kartezjańskie ((*i*) – (*iii*) w Twierdzeniu 1.), ale nie jest zamknięta ze względu na obrazy homomorficzne (Lemat 1.). Pokazano również, że grupa lokalnie-(lokalnie stopniowa) jest lokalnie stopniowa oraz że grupa rezydualnie-(lokalnie stopniowa) jest lokalnie stopniowa ((*iv*)

w Twierdzeniu 1.). Twierdzenie 2. podaje inną definicję grup lokalnie stopniowych – mianowicie, grupy lokalnie stopniowe to takie, które nie posiadają skończenie generowanych nieskończonych sekcji prostych.

Przedstawionych zostało również sześć znanych problemów dotyczących tożsamości pół grupowych (z czego trzy równoważne), które mają pozytywne rozwiązanie w klasie grup lokalnie stopniowych. Cztery z tych problemów nadal pozostają otwarte. Problem 1. (Ograniczony Problem Burnside'a) i Problem 2. (Problem Malcewa) mają negatywne rozwiązanie w ogólności, co dostarcza przykładów grup, które nie są lokalnie stopniowe – nieskończone grupy Burnside'a oraz grupy Olszańskiego-Storożewa (Wniosek 1.).