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ORE CONGRUENCES AND THEIR PROPERTIES

Summary. Let F and $\mathcal{F}$ be a free group and a free semigroup, respectively, with common set of free generators. Then $\mathcal{F} \subset F$ and every natural homomorphism $F \to F/N$ defines the congruence ρ_N on $\mathcal{F}$. Different normal subgroups N can define the same congruence. We show that this is impossible if the congruence satisfies one of Ore conditions.

KONGRUENCJE ORE I ICH WŁASNOŚCI

Streszczenie. Niech F i $\mathcal{F}$ będą, odpowiednio, grupą wolną i półgrupą wolną o wspólnym zbiorze wolnych generatorów. Wtedy $\mathcal{F} \subset F$ oraz każdy homomorfizm naturalny $F \to F/N$ definiuje kongruencję ρ_N w $\mathcal{F}$. Różne podgrupy normalne mogą definiować tę samą kongruencję. Pokazemy, że jest to niemożliwe, gdy kongruencja spełnia jeden z warunków Ore.

1. Preliminaries

Let X be a set of free generators of the group F and of the semigroup $\mathcal{F}$ containing the identity e. We denote by S and by$G$ any semigroup and any group, respectively. We denote by $ngp\, A$ a normal subgroup of a group G generated by the set A. The following definitions are known:

A congruence ρ on $\mathcal{F}$ is *cancellative* if for all $s_1, s_2, t_1, t_2 \in \mathcal{F}$, the implication holds: if $(t_1 s_1 t_2, t_1 s_2 t_2) \in \rho$ then $(s_1, s_2) \in \rho$.

A group G is a *group of fractions* of a semigroup S if $S \subseteq G$ and $G = SS^{-1} = S^{-1}S$.

A congruence ρ on $\mathcal{F}$ satisfies *right (left) Ore condition*, if for all $s_1, s_2 \in S$ there exist $t_1, t_2 \in S$, such that $(t_1 s_1, t_2 s_2) \in \rho$ ($(s_1 t_1, s_2 t_2) \in \rho$). We say that ρ is the right (left) Ore congruence if ρ satisfies right (left) Ore conditions. If ρ satisfies both Ore conditions then we call ρ Ore congruence.

A semigroup S satisfies right (left) Ore condition if for all $s_1, s_2 \in S$ there exist $t_1, t_2 \in S$ such that $t_1 s_1 = t_2 s_2$ ($s_1 t_1 = s_2 t_2$).

It is clear that

Corollary 1. *A congruence ρ on $\mathcal{F}$ satisfies right (left) Ore condition if and only if $\mathcal{F}/\rho$ satisfies right (left) Ore condition.*

Definition 1. *We say that the normal subgroup N in F is the normal Ore subgroup if the following equalities holds:*

$$F = \mathcal{F}\mathcal{F}^{-1}N = \mathcal{F}^{-1}\mathcal{F}N.$$

2. Embedding of a semigroup into a group

Every group is cancellative and every congruence in a group is cancellative. So, the cancellation is the necessary condition for embedding semigroup into a group. However, it is not sufficient [2, § 1.10].

The Ore conditions and cancellation are sufficient conditions for embedding semigroup S into G (Ore, 1931). In fact, it is also necessary condition [4] for embedding such that $G = SS^{-1} = S^{-1}S$.

The first necessary and sufficient conditions for embedding a semigroup into group were given by Malcev [6] in 1931. Later there appeared another publications concerning this topic [5, 8].

Let ρ be a congruence on $\mathcal{F}$ and we denote $\mathcal{A}_\rho := \{ab^{-1}, (a, b) \in \rho\}$.

Lemma 1. *If ρ is a cancellative congruence on $\mathcal{F}$ satisfying Ore conditions, then the semigroup $S \cong \mathcal{F}/\rho$ is embeddable into a group G such that $G \cong F/N_\rho$, where*

$$N_\rho = ngp\ \mathcal{A}_\rho, \quad N_\rho \cap \mathcal{F}\mathcal{F}^{-1} = \mathcal{A}_\rho.$$

Proof. The construction of the embedding is given in [2, § 12.3]. The idea is the following: since $\mathcal{F} \subset F$, the congruence ρ defines a congruence $\rho^{\#}$ on F (a congruence closure of ρ in F). Hence $\rho^{\#}$ is cancellative and $(g, h) \in \rho^{\#}$ if and only if $(gh^{-1}, e) \in \rho^{\#}$, that is $gh^{-1} \in [e]_\rho$. If $(g, e) \in \rho^{\#}$, $(h, e) \in \rho^{\#}$ then $(e, g^{-1}) \in \rho^{\#}$, $(g^{-1}, e) \in \rho^{\#}$ and $(g^{-1}h, e) \in \rho^{\#}$. Thus for every s, $(sgs^{-1}, e) \in \rho^{\#}$. Hence ρ defines $[e]_\rho$ as the normal subgroup in F generated by the set $\mathcal{A}_\rho$, so $N_\rho := [e]_\rho = ngp\,\mathcal{A}_\rho$. The second equality is shown in [2, § 12.8]. □

From [3] (also [2, § 1.5]), every normal subgroup $N \lhd F$ defines a congruence ρ on $\mathcal{F}$ such that: $(g, h) \in \rho$ if and only if $gh^{-1} \in N$, $g, h \in \mathcal{F}$.

Due to the facts above, the following diagram is commutative:

$$\begin{array}{ccc} \mathcal{F} & \xrightarrow{\ i\ } & F \\ {\scriptstyle\varphi}\big\downarrow & & \big\downarrow{\scriptstyle\varphi^{\#}} \\ \mathcal{F}/\rho & \xrightarrow{\ i_1\ } & F/N_\rho \end{array}$$

where i, i_1 are embeddings and φ, $\varphi^{\#}$ are cannonical epimorphisms corresponding to the congruences ρ, $\rho^{\#}$.

Theorem 1. *Let a group G be generated by a subsemigroup S. Then the following conditions are equivalent:*

(i) S satisfies both Ore conditions.

(ii) G is a group of fractions of S.

(iii) $G \cong F/N$, *where* N *is a normal Ore subgroup.*

(iv) $S \cong \mathcal{F}/\rho$, *where* ρ *is a cancellative Ore congruence.*

Proof. $(i) \Rightarrow (ii)$ If S satisfies right Ore condition then for all $s_i, s_j \in S$ there exist $t_i, t_j \in S$ such that $t_i s_i = t_j s_j$, so

$$s_i s_j^{-1} = t_i^{-1} t_j. \tag{1}$$

Let $s_1, s_2, s_3 \in S$. By (1) we get $s_1^{-1} s_2 s_3^{-1} = s_1^{-1} t_2^{-1} t_3 \in S^{-1}S$ and

$$S^{-1}SS^{-1} \subseteq S^{-1}S. \tag{2}$$

Multiplying this inclusion from the right by S we get

$$(S^{-1}S)^2 \subseteq (S^{-1}SS^{-1})S \overset{(2)}{\subseteq} (S^{-1}S)S = S^{-1}S. \tag{3}$$

Assume that

$$(S^{-1}S)^{k-1} = S^{-1}S. \tag{4}$$

Then

$$(S^{-1}S)^k \subseteq (S^{-1}S)^{k-1}S^{-1}S \overset{(4)}{=} (S^{-1}S)^2 \overset{(3)}{\subseteq} S^{-1}S.$$

Since for every $g \in G$ there exists $k \in N$ such that

$$g \in \underbrace{S^{(-1)^{2k}} \cdots S^{-1}SS^{-1}S}_{2k} \subseteq (S^{-1}S)^k$$

and we have $G = S^{-1}S$. Using left Ore condition we get $g \in SS^{-1}$ and finally, $G = S^{-1}S = SS^{-1}$.

$(ii) \Rightarrow (i)$ Let $G = S^{-1}S = SS^{-1}$. It means, that for all $s_1, s_2 \in S$ we have $s_1^{-1}s_2 \in SS^{-1}$. Therefore there exist such $t_1, t_2 \in S$, that $s_1^{-1}s_2 = t_1 t_2^{-1}$ and $s_1 t_1 = s_2 t_2$. Similarly we can find $u_1, u_2 \in S$ for which $u_1 s_1 = u_2 s_2$, so S satisfies Ore conditions.

$(iii) \Rightarrow (ii)$ We denote by $\varphi^{\#}$, a mapping $F \to G \cong F/N$, and by φ induced mapping $\mathcal{F} \to S \cong \mathcal{F}/\rho$. Then from (iii), $F = \mathcal{F}\mathcal{F}^{-1}N$, and hence:

$$G = F^{\varphi^{\#}} = \mathcal{F}^{\varphi^{\#}} \left(\mathcal{F}^{\varphi^{\#}}\right)^{-1} N^{\varphi^{\#}} = SS^{-1}.$$

Similarly, from $F = \mathcal{F}^{-1}\mathcal{F}N$ we get the equality $G = S^{-1}S$.

$(ii) \Rightarrow (iii)$ Let $G = SS^{-1}$. Let $a \in F$, it means $a \in G^{\varphi^{-1}} = (SS^{-1})^{\varphi^{-1}}$. Then $a \in \mathcal{F}\mathcal{F}^{-1}$ *mod* N and it follows that $a \in \mathcal{F}\mathcal{F}^{-1}N$ and $F = \mathcal{F}\mathcal{F}^{-1}N$. Similarly $G = S^{-1}S$ implies $F = \mathcal{F}^{-1}\mathcal{F}N$.

$(i) \Leftrightarrow (iv)$ It follows from definitions as we have noticed in Corollary 1.□

We can use Theorem 1. to prove the following.

Property 1. *If a cancellative semigroup S satisfies a nontrivial law, then S has a group of fractions $G = SS^{-1} = S^{-1}S$.*

Proof. In view of Theorem 1., it is enough to show that if a semigroup S satisfies a nontrivial law, then S satisfies Ore conditions.

Let S satisfy a nontrivial law $u(x_1, \ldots, x_n) = w(x_1, \ldots, x_n)$. We can assume that this law is balanced, that is every x_i occurs the same number of times in u and w. If the law is not balanced then it implies a law of finite exponent and then $G = S$ satisfies Ore conditions.

By using mapping $x_i \to xy^i$ to the law $u(x_1, \ldots, x_n) = w(x_1, \ldots, x_n)$, we get a nontrivial law with two variables. Hence S satisfied a cancelled binary law, which can be written in two ways:

$$u_1(x, y) \cdot x = w_1(x, y) \cdot y, \quad x \cdot u_2(x, y) = y \cdot w_2(x, y). \tag{5}$$

Under every mapping of x, y into S the expression (5) becomes the equalities in S which implies the Ore conditions in S.□

We shall need the following Lemma, which is a modification of the known modularity rule.

Lemma 2. *Let A be a subset of a group G ($A \ni e$). Let B, C, AB be subgroups of G and let $B \subseteq C$. Then*

$$AB \cap C = (A \cap C)B.$$

Proof. Inclusions $(A \cap C)B \subseteq AB$ and $(A \cap C)B \subseteq CB = C$ imply $(A \cap C)B \subseteq AB \cap C$. Let now $g \in AB \cap C$, it means, there exist $a \in A, b \in B$, that $ab \in C$. Hence $a \in Cb^{-1}$, so $a \in C$ and $a \in A \cap C$. Then we get $g = ab \in (A \cap C)B$ and inclusion $AB \cap C \subseteq (A \cap C)B$ is satisfied. □

3. On normal subgroups defining left (right) Ore congruences

It follows from Lemma 1. and Theorem 1., that each cancellative Ore congruence ρ defines the normal subgroup $N_\rho = ngp\ \mathcal{A}_\rho$ and conversely, each normal subgroup in F defines a cancellative congruence ρ_N on $\mathcal{F}$, such that

$$(a, b) \in \rho_N \quad \Leftrightarrow \quad ab^{-1} \in N \cap \mathcal{F}\mathcal{F}^{-1}. \tag{6}$$

The question, we consider is *whether the same cancellative congruence can be defined by different normal subgroups.*

If ρ is the identity congruence, the answer is positive, because all those normal subgroups, for which the intersection with $\mathcal{F}\mathcal{F}^{-1}$ is trivial define the identity congruence. For example this happens for normal subgroups satisfying $N \subseteq F''$ (where F'' is the second commutator subgroup in F), because in [7] Malcev proved that $F'' \cap \mathcal{F}\mathcal{F}^{-1} = e$. Another example is given in [1].

However we can show that if ρ is a cancellative right or left Ore congruence, then such situation is impossible. We note that the identity congruence is not an Ore congruence.

Theorem 2. *If a normal subgroup $M \lhd F$ defines a congruence ρ on $\mathcal{F}$ satisfying left or right Ore condition then $M = N_\rho = ngp\,\mathcal{A}_\rho$.*

Proof. Let $M \lhd F$ defines a left Ore congruence ρ on $\mathcal{F}$ (if ρ is right Ore congruence then the proof is similar). Then by (6)

$$\mathcal{F}\mathcal{F}^{-1} \cap M = \mathcal{A}_\rho.$$

By Lemma 1., the congruence ρ defines the normal subgroup $N_\rho = ngp\,\mathcal{A}_\rho$, such that

$$\mathcal{F}\mathcal{F}^{-1} \cap M = \mathcal{A}_\rho = \mathcal{F}\mathcal{F}^{-1} \cap N_\rho. \tag{7}$$

Since $N_\rho = ngp\,\mathcal{A}_\rho \subseteq M$, we have

$$M = F \cap M \stackrel{L.1.}{=} \mathcal{F}\mathcal{F}^{-1}N_\rho \cap M \stackrel{L.2.}{=}$$
$$= (\mathcal{F}\mathcal{F}^{-1} \cap M)N_\rho \stackrel{(7)}{=} (\mathcal{F}\mathcal{F}^{-1} \cap N_\rho)N_\rho = N_\rho,$$

which finishes the proof. $\square$

References

1. K. I. Appel, F. M. Djorup, *On the group generated by a free semigroup*, Proc. Amer. Math. Soc. **15** (1964), 838–840.
2. A. H. Clifford, G. B. Preston, *The algebraic theory of semigroups*, Amer. Math. Soc., R.I. Vol. I – 1961, vol. II – 1967.
3. P. Dubreil, *Contribution a la theorie des demi-groups*, Mem. Acad. Sci. Inst. France **63** (1941), 52.
4. P. Dubreil, *Sur les problemes d'immersion et la theorie des modules*, C. R. Acad. Sci. Paris **216** (1943), 625–627.
5. J. Lambek, *The immersibility of a semigroup into a group*, Canad. J. Math. **3** (1951), 34–43.
6. A. I. Malcev, *On the immersion of associative systems in groups*, Mat. Sbornik (N.S.) **6** (1939), 331–336.
7. A. I. Malcev, *Nilpotent semigroups*, Ivanov. Gos. Ped. Inst. Uč. Zap. Fiz.-Mat. Nauki **4** (1953), 107–111.
8. V. Ptak, *Immersibility of semigroups*, Acta Fac. Nat. Univ. Carol. Prague **192** (1949), 16.

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Streszczenie

Niech F i $\mathcal{F}$ będą, odpowiednio, grupą wolną i półgrupą wolną o wspólnym zbiorze wolnych generatorów. Wtedy $\mathcal{F} \subset F$.

Kongruencja ρ w $\mathcal{F}$ jest skracalna, jeśli dla każdych $s_1, s_2, t_1, t_2 \in \mathcal{F}$, zachodzi następująca implikacja: jeżeli $(t_1s_1t_2, t_1s_2t_2) \in \rho$, to $(s_1, s_2) \in \rho$.

Grupa G jest grupą ułamków półgrupy S, jeśli $S \subseteq G$ oraz $G = SS^{-1} = S^{-1}S$. Kongruencja ρ w $\mathcal{F}$ spełnia prawostronny (lewostronny) warunek Ore, jeśli dla każdych $s_1, s_2 \in S$ istnieją $t_1, t_2 \in S$ takie, że $(t_1s_1, t_2s_2) \in \rho$ ($(s_1t_1, s_2t_2) \in \rho$). Mówimy, że ρ jest prawostronną (lewostronną) kongruencją Ore, jeśli ρ spełnia prawostronny (lewostronny) warunek Ore. Jeśli ρ spełnia obydwa warunki Ore, wtedy ρ nazywa się kongruencją Ore.

Półgrupa S spełnia prawostronny (lewostronny) warunek Ore, jeśli dla każdych $s_1, s_2 \in S$ istnieją $t_1, t_2 \in S$ takie, że $t_1s_1 = t_2s_2$ ($s_1t_1 = s_2t_2$).

Będziemy mówić, że podgrupa normalna N w F jest normalną podgrupą Ore, jeśli zachodzą równości $F = \mathcal{F}\mathcal{F}^{-1}N = \mathcal{F}^{-1}\mathcal{F}N$.

Oznaczmy przez $ngp\,A$ podgrupę normalną generowaną przez zbiór A. W Lemacie 1. pokażemy, że dla skracalnej kongruencji w F, spełniającej warunki Ore, półgrupę $\mathcal{F}/\rho$ można zanurzyć w grupę F/N_ρ i zachodzi następująca równość:

$$N_\rho = ngp(N_\rho \cap \mathcal{F}\mathcal{F}^{-1}).$$

W Twierdzeniu 1. wskażemy równoznaczność następujących warunków:

(i) Półgrupa S spełnia obydwa warunki Ore.

(ii) G jest grupą ułamków półgrupy S.

(iii) $G \cong F/N$, gdzie N jest normalną podgrupą Ore.

(iv) $S \cong \mathcal{F}/\rho$, gdzie ρ jest skracalną kongruencją Ore.

Z Twierdzenia 1. wynika fakt, że spełnienie pewnej tożsamości przez nieskracalną półgrupę S pociąga za sobą istnienie grupy G, dla której $G = SS^{-1} = S^{-1}S$. Każda skracalna kongruencja Ore ρ definiuje dzielnik normalny $N_\rho := ngp(ab^{-1} : (a,b) \in \rho)$. Również każdy dzielnik normalny N w F definiuje skracalną kongruencję ρ_N w $\mathcal{F}$, dla której

$$(a,b) \in \rho_N \quad \Leftrightarrow \quad ab^{-1} \in N \cap \mathcal{F}\mathcal{F}^{-1}.$$

Okazuje się, że jeżeli ρ jest skracalną kongruencją Ore, to może ona być zdefiniowana tylko przez jeden dzielnik normalny N w F, co pokażemy w Twierdzeniu 2. W jego dowodzie korzystamy ze zmodyfikowanej wersji prawa modularności, przedstawionej w Lemacie 2.