ZESZYTY NAUKOWE POLITECHNIKI ŚLĄSKIEJ 2004

Seria: MATEMATYKA-FIZYKA z. 91 Nr kol. 1652

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ON REPRESENTATION OF A SEMIDIRECT PRODUCT OF CYCLIC GROUPS BY A 2-STATE TIME-VARYING MEALY AUTOMATON

Summary. We present the subgroup $K = [W, W] \cdot \mathbb{Z}$ of the wreath product $W = \mathbb{Z} \wr_{X_0} \mathbb{Z}$ relative to an n-element set X_0 as a group generated by a 2-state time-varying Mealy automaton. The action of K on the sets of finite and infinite words over the changing alphabet we study.

REPREZENTACJA PRODUKTU PÓŁPROSTEGO GRUP CYKLICZNYCH ZA POMOCĄ ZMIENNEGO W CZASIE AUTOMATU MEALY'EGO O DWÓCH STANACH

Streszczenie. W splocie $W = \mathbb{Z} \wr_{X_0} \mathbb{Z}$ względem n-elementowego zbioru X_0 rozważamy podgrupę $K = [W, W] \cdot \mathbb{Z}$ jako grupę generowaną przez zmienny w czasie automat Mealy'ego o dwóch stanach. Badamy działanie grupy K na zbiorach słów skończonych i nieskończonych nad zmiennym alfabetem.

1. Introduction

The theory of Mealy automata and groups generated by them have rapidly expanded in recent years and now it plays an important role in algebra and theory of dynamical systems. Even the simplest automata generate groups with complicated structure and extraordinary properties. An extensive presentation of the theory of automatic groups is included in [2]. The well known problem in this theory is whether any given automatic group is generated by a finite automaton. The known examples of such groups are free groups, some free products of finite groups, linear groups $GL_n(\mathbb{Z})$ for $n \geqslant 2$, the lamplighter group and others.

The idea of an automaton with a changing alphabet and a changing set of its internal states is a generalization which allows to represent groups acting on level homogenous rooted trees which may be not homogeneous. Let A be a given time-varying automaton. Any state q from the set Q_0 of its internal states defines a transformation f_q^A on the set of words over the changing alphabet. The (semi)group $\langle f_q^A : q \in Q_0 \rangle$ is called the (semi)group generated by automaton A or the automatic transformation (semi)group defined by A. If the sets of letters in the changing alphabet are finite any such group is residually finite. Conversely, any k-generated residually finite group can be realized as a group of a time-varying automaton with a k-element set of states [8]. One of the simplest group for which the problem of construction of such an automaton is still open is the wreath product $\mathbb{Z} \wr \mathbb{Z}$.

The aim of this paper is to present and study the subgroup

$$K = \{((z_0, z_1, \ldots, z_{n-1}), z) : \sum_{i=0}^{n-1} z_i = 0\}$$

of the wreath product $W = \mathbb{Z} \wr_{X_0} \mathbb{Z}$ relative to an n-element set X_0 as a group generated by a 2-state time-varying Mealy automaton. According to definition of the wreath product (see [4] for example) W is a semi-direct product $\mathbb{Z}^{X_0} \rtimes \mathbb{Z}$ with the action of $\mathbb{Z}$ on X_0 by a shift. Hence for the group K we have

$$K = [W, W] \cdot \mathbb{Z}.$$

The group K is a torsion free, metabelian, nonnilpotent group and it is isomorphic to the semi-direct product $\mathbb{Z}^{n-1} \rtimes \mathbb{Z}$ with the action of $\mathbb{Z}$ on

$\mathbb{Z}^{n-1}$ by linear transformations as follows

$$z \mapsto \begin{pmatrix} -1 & -1 & \dots & -1 & -1 & -1 \\ 1 & 0 & \dots & 0 & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 & 0 & 0 \\ 0 & 0 & \dots & 0 & 1 & 0 \end{pmatrix}^i \cdot z, \quad z \in \mathbb{Z}^{n-1}, \quad i \in \mathbb{Z}.$$

The center of K is isomorphic to $\mathbb{Z}$, the commutator subgroup K' is isomorphic to the cartesian product $\mathbb{Z}^{n-1}$ and the abelianization K/K' is isomorphic to $\mathbb{Z}_n \times \mathbb{Z}$.

2. Time-varying automata and groups defined by them

A changing alphabet is an infinite sequence $X = (X_t)_{t \in \mathbb{N}_0}$ of nonempty finite sets X_t (sets of letters). A (finite) word over a changing alphabet X is a finite sequence $x_0 x_1 \dots x_l$, where $x_i \in X_i$ for $i = 0, 1, \dots, l$. Similarly we define an infinite word over X. We denote by X^* (X^ω) the set of all words (infinite words) over X.

Definition 1. *A time-varying Mealy automaton over the changing alphabet X is a quadruple*

$$A = (Q, X, \varphi, \psi),$$

where:

$Q = (Q_t)_{t \in \mathbb{N}_0}$ *(sequence of sets of internal states),*

$\varphi = (\varphi_t)_{t \in \mathbb{N}_0}$, $\varphi_t : Q_t \times X_t \to Q_{t+1}$ *(sequence of transition functions),*

$\psi = (\psi_t)_{t \in \mathbb{N}_0}$, $\psi_t : Q_t \times X_t \to X_t$ *(sequence of output functions).*

An automaton A with a fixed initial state $q \in Q_0$ is called the initial automaton and denoted by A_q. If A is a given automaton then for every state $q \in Q_0$ the initial automaton A_q defines a function $f_q^A : X^* \to X^*$ as follows:

$$f_q^A(x_0 x_1 \ldots x_l) = \psi_0(q_0, x_0)\psi_1(q_1, x_1)\ldots\psi_l(q_l, x_l),$$

where $q_0 = q$ and $q_i = \varphi_{i-1}(q_{i-1}, x_{i-1})$ for $i = 1, \ldots, l$. The function f_q^A is called the automaton function defined by the initial automaton A_q.

An automaton A is called permutational if the mappings

$$x \mapsto \psi_t(q, x)$$

are permutations on X_t for all $t \in \mathbb{N}_0$ and $q \in Q_t$. If A is a permutational automaton then the functions f_q^A are permutations on X^* for all $q \in Q_0$.

Time-varying permutational automata can be represented as labelled, directed locally finite graphs. The vertices of such a graph correspond to the states of the automaton and for every $t \in \mathbb{N}_0$ and every letter $x \in X_t$, an arrow labelled by x starts from every state $q \in Q_t$ to $\varphi_t(q, x)$; each vertex q_t is labelled by the corresponding element of the symmetric group $S(X_t)$. It is worth to see that different vertices of such a graph may correspond to the same internal state of the automaton.

It is known that the superposition of automaton functions defined by two initial automata over a common alphabet X is also automatic as well as an inverse to the automaton function defined by invertible automaton is automatic.

For every permutational automaton A we construct the group

$$G(A) = \langle f_q^A : q \in Q_0 \rangle.$$

The group $G(A)$ is called the group generated by automaton A.

3. The group K as a 2-state automatic group

Let $n > 1$ be a positive integer, $(m_i)_{i\in\mathbb{N}_0}$ – a sequence of integers with $m_i > 1$ $(i \in \mathbb{N}_0)$ and $m_0 = n$. Let X be a changing alphabet of the form

$$X_t = \{0, 1, \ldots, m_t - 1\} \text{ for } t = 0, 1, 2, \ldots .$$

We consider a time-varying automaton $A = (Q, X, \varphi, \psi)$ in which:

$$Q_t = \{q_0, q_1\},$$

$$\varphi_t(q_0, x) = q_1 \text{ for } x \neq m_t - 1,\ \varphi_t(q_0, m_t - 1) = q_0,$$

$$\varphi_t(q_1, x) = q_1 \text{ for } t \neq 0 \text{ or } x \neq 0,\ \varphi_0(q_1, 0) = q_0,$$

$$\psi_t(q_0, x) = x +_{m_t} 1,$$

$$\psi_t(q_1, x) = x \text{ for } t \neq 0,\ \psi_0(q_1, x) = x +_{m_0} 1,$$

where $+_t$ is an arithmetical addition (mod t). The automaton under this construction is presented in the figure (1 is the neutral element and σ_t is a cycle $(0, 1, \ldots, m_t - 1)$ in the symmetric group $S(X_t)$).

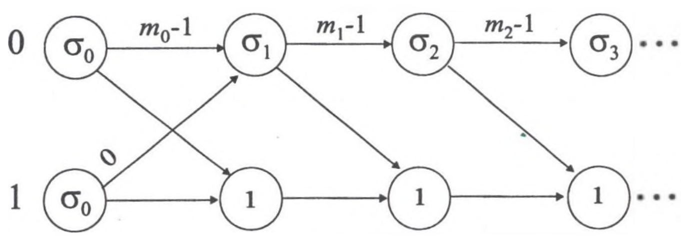

Fig. 1. An automaton that generates K
Rys. 1. Automat generujący grupę K

The automaton A is permutational by definition. We denote the generators $f_{q_0}^A$, $f_{q_1}^A$ of the group $G = G(A)$ by a and b respectively. The generators transform the word $xu \in X^*$ with $x \in X_0$ in the following way:

$$a(xu) = (x +_n 1)u \text{ for } x \neq n - 1,\ a((n-1)u) = 0\overline{a}(u),$$

$$b(xu) = (x +_n 1)u \text{ for } x \neq 0,\ b(0u) = 1\overline{a}(u).$$

where $\overline{a}$ is the so-called remainder of a corresponding to the letter $(n-1) \in X_0$. From the above formulas we have for any exponent s

$$a^s(xu) = (x +_n s)\overline{a}^{\left[\frac{x+s}{n}\right]}(u), \quad b^s(xu) = (x +_n s)\overline{a}^{\left[\frac{x+s}{n}\right]+\delta_n(x+s)-\delta_n(x)}(u), \tag{1}$$

where $\delta_n(t)$ equals to 0 if n divides t or to 1 in other case. Every element of G is defined by some word w in a, b of the form

$$w = a^{s_1}b^{s_2}\dots a^{s_{2k-1}}b^{s_{2k}}. \tag{2}$$

For the word (2) we consider a pair

$$D(w) = ((D_0, D_1, \dots, D_{n-1}), S)$$

in which S is the sum of all exponents on generators in w and

$$D_i = |\{j\colon S_{2j-1} \equiv_n -i\}| - |\{j\colon S_{2j} \equiv_n -i\}| \quad \text{for} \quad i = 0, 1, \dots, n-1,$$

where $S_i = \sum\limits_{j=1}^{i} s_j$ for $i = 1, 2, \dots, 2k$. In particular $\sum\limits_{i=0}^{n-1} D_i = 0$.

Lemma 1. *$w(xu) = (x +_n S)\overline{a}^E(u)$, where $E = \left[\frac{x+S}{n}\right] + D_x$.*

Proof. It follows from formulas (1) by induction on the length of w. □

Theorem 1. *The map $\phi : w \mapsto D(w)$ defines the isomorphism between G and K.*

Proof. Let w and w' be words in a, b with their corresponding pairs

$$D(w) = ((D_0, D_1, \dots, D_{n-1}), S) \quad \text{and} \quad D(w') = ((D'_0, D'_1, \dots, D'_{n-1}), S')$$

respectively. The words w and w' define the same element in G iff $D(w) = D(w')$. Indeed, if $D(w) = D(w')$, the thesis follows from Lemma 1. On the other hands, if w and w' define the same element in G, then $S \equiv_n S'$ from

Lemma 1. The remainder $\overline{a}$ is a transformation of infinite order. Thus again from Lemma 1 for any $x \in X_0$ we have

$$D_x - D'_x = \left[\frac{x+S'}{n}\right] - \left[\frac{x+S}{n}\right] = \left[\frac{S'}{n}\right] - \left[\frac{S}{n}\right].$$

But $\sum_{x=0}^{n-1}(D_x - D'_x) = 0$. In consequently $S = S'$ and $D_x = D'_x$ for any $x \in X_0$.

Let, now S_i $(i = 1, 2, \dots, 2k)$ and S'_i $(i = 1, 2, \dots, 2l)$ be sums corresponding to words w and w'. Let S''_i $(i = 1, 2, \dots, 2k + 2l)$ be adequate sums for the word ww'. From the equalities

$$S''_i = \begin{cases} S_i & \text{for} \quad i = 1, 2, \dots, 2k, \\ S + S'_{i-2k} & \text{for} \quad i = 2k+1, \dots, 2k+2l \end{cases}$$

we have

$$|\{j \colon S''_{2j-\epsilon} \equiv_n -i\}| = |\{j \colon S_{2j-\epsilon} \equiv_n -i\}| + |\{j \colon S'_{2j-\epsilon} \equiv_n -i - S\}|$$

for $\epsilon \in \{0, 1\}$. Hence for the pair

$$D(ww') = ((D''_0, D''_1, \dots, D''_{n-1}), S'')$$

we have $S'' = S''_{2k+2l} = S + S'$ and $D''_i = D_i + D'_{i+_nS}$ for $i = 0, 1, \dots, n-1$. In consequently $\phi(ww') = \phi(w)\phi(w')$.

Let, now $((D_0, \dots, D_{n-1}), S) \in K$ and let D_{i_j} $(j = 0, \dots, r)$ be all non-negative numbers in the sequence (D_i). We build a sequence $(S_i)_{i \in \{1,2,\dots,2k\}}$ of the length $2k = 2 + \sum_{i=0}^{n-1}|D_i|$, in which $S_{2k} = S_{2k-1} = S$ and

$$(S_1, S_3, \dots, S_{2k-3}) = (\underbrace{-i_0 \dots - i_0}_{D_{i_0}}, \dots, \underbrace{-i_r \dots - i_r}_{D_{i_r}}),$$

$$(S_2, S_4, \dots, S_{2k-2}) = (\underbrace{-i_{r+1} \dots - i_{r+1}}_{|D_{i_{r+1}}|}, \dots, \underbrace{-i_{n-1} \dots - i_{n-1}}_{|D_{i_{n-1}}|}).$$

Then $\phi(w) = ((D_0, D_1, \dots, D_{n-1}), S)$ for the word 2, in which $s_1 = S_1$, $s_i = S_i - S_{i-1}$ for $i = 2, 3, \dots, 2k$. Hence ϕ is onto. □

The generators a and b are mapped via ϕ as follows

$$a \mapsto ((1, -1, 0, \dots, 0), 1), \quad b \mapsto ((-1, 1, 0, \dots, 0), 1).$$

Proposition 1. *The group K is finitely presented and its finite presentation is of the form*

$$\left\langle a, b\colon a^n = b^n,\ \left[a^i b^{-i}, a^j b^{-j}\right] = 1,\ i, j \in \{0, 1, \dots, n-1\}\right\rangle.$$

Proof. The equalities in the above presentation are relations in G. Let, now assume that the word 2 defines the neutral element and let $(S_i)_{i\in\{1,\dots,2k\}}$ be its corresponding sequence. We have

$$w = a^{S_1} b^{S_2 - S_1} a^{S_3 - S_2} b^{S_4 - S_3} \dots a^{S_{2k-1} - S_{2k-2}} b^{S_{2k} - S_{2k-1}}.$$

Let r_i be the remainder from the division of S_i by n. From the relation $a^n = b^n$ and from the equality $S = S_{2k} = 0$ we may from w arrive at

$$a^{r_1} b^{r_2 - r_1} a^{r_3 - r_2} b^{r_4 - r_3} \dots a^{r_{2k-1} - r_{2k-2}} b^{r_{2k} - r_{2k-1}} a^{-r_{2k}} \sim$$
$$\sim a^{r_1} b^{-r_1} \left(a^{r_2} b^{-r_2}\right)^{-1} \dots a^{r_{2k-1}} b^{-r_{2k-1}} \left(a^{r_{2k}} b^{-r_{2k}}\right)^{-1}.$$

The numbers of elements $\equiv_n r$ in $(S_i)_{i\in\{1,3,\dots,2k-1\}}$ and $(S_i)_{i\in\{2,4,\dots,2k\}}$ coincide for any r. Thus there is a bijection

$$\sigma\colon \{1, 3, \dots, 2k-1\} \to \{2, 4, \dots, 2k\}$$

for which $r_i = r_{\sigma(i)}$ for $i = 1, 3, \dots, 2k-1$. Using these equalities as well as relations $\left[a^i b^{-i}, a^j b^{-j}\right] = 1$ we may from the last word arrive at the empty word. □

4. The action of K on X^* and X^ω

By Theorem 1 the group K acts naturally on the sets X^* and X^ω. The action of the element $g = ((D_0, \dots, D_{n-1}), S) \in K$ may be described by formula

$$g(x_0 x_1 x_2 \dots) = (x_0 +_{m_0} r_0)(x_1 +_{m_1} r_1)(x_2 +_{m_2} r_2) \dots,\quad x_t \in X_t, \qquad (3)$$

where

$$r_0 = S, \quad r_1 = \left[\frac{x_0 + S}{n}\right] + D_{x_0}, \quad r_{t+1} = \left[\frac{x_t + r_t}{m_t}\right], \quad t = 1, 2, \ldots. \qquad (4)$$

The action on the set of words over the changing alphabet X is called spherically transitive if it is transitive on the set $X^{(t)}$ of words of the length t for $t = 0, 1, \ldots$.

For any infinite word u we denote by u^K its orbit $\{g(u) \colon g \in K\}$ and by u_t – the rest of u from the set X_t^ω of infinite words over the changing alphabet $(X_i)_{i \geqslant t}$.

With the group $G = G(A)$ we associate the subgroups: $St_G(x^*)$ which is the stabilizer of the word $x^* \in X^*$, $St_G(k) = \bigcap_{x^* \in X^{(k)}} St_G(x^*)$ which is the stabilizer of the k-th level, i.e. the intersection of the stabilizers of the words from $X^{(k)}$ and finally the subgroup $St_G(u)$ which is the stabilizer of the infinite word u. The groups $P_u = St_G(u)$ are called parabolic.

Theorem 2. *Let $k \in \mathbb{N}$, $w \in X^{(k)}$ and $u, v \in X^\omega$. Then*

(i) $St_K(k) \cong St_K(w) \cong P_u \cong \mathbb{Z}^n$,

(ii) the action of K on X^* is spherically transitive,

(iii) $u^K = v^K$ iff there is $t \geqslant 0$ such that $u_t = v_t$ or

$$\{u_t, v_t\} = \{000\ldots, (m_t - 1)(m_{t+1} - 1)(m_{t+2} - 1)\ldots\}.$$

Proof. (i) Let $g = ((D_0, \ldots, D_{n-1}), S) \in K$. From (3) and (4) the element $g \in St_K(k)$ iff for any $x^* \in X^*$ the number $r_t \equiv_{m_t} 0$ for $t = 0, 1, \ldots, |x^*| - 1$. Hence $g \in St_K(k)$ iff $S \equiv_n 0$ and $S/n + D_i \equiv_I 0$ for $i = 0, 1, \ldots, n-1$, where $I = \prod_{j=1}^{k-1} m_j$. From the equality $\sum_{i=0}^{n-1} D_i = 0$ we have $S \equiv_J 0$ for $J = lcm(n, I)$. In consequently the mapping

$$g \mapsto ((nD_0 + S)/nI, (nD_1 + S)/nI, \ldots, (nD_{n-2} + S)/nI, S/J)$$

defines the required isomorphism for $St_K(k)$. Similarly we have: $g \in St_K(w)$ iff $S + nD_x \equiv_{nI} 0$, where $x \in X_0$ is the first letter of the word w. In this case the mapping

$$g \mapsto (D_0, D_1, \ldots, D_{n-2}, (nD_x + S)/nI)$$

describes the required isomorphism. Finally for the parabolic subgroup $g \in P_u$ iff $S + nD_x = 0$, where x is the first letter of u. In this case the mapping $g \mapsto (D_0, D_1, \ldots, D_{n-1})$ is a required isomorphism.

(ii) The automorphism a acts on X^ω by adding unity to an Q-adic number, where $Q = (m_t)_{t \in \mathbb{N}_0}$ (the so called Q-adic adding machine, see [1]). The above interpretation implies that K acts spherically transitive.

(iii) Let $u = x_0x_1x_2\ldots$ and $v = g(u)$ for some $g \in K$. From the inequality $0 \leqslant x_t < m_t$ and formulas (3), (4) there is $t \geqslant 1$ such that

$$|r_1| > |r_2| > \ldots > |r_t| \leqslant 1.$$

Now, we have three possibilities:

(a) $v_{t_0} = u_{t_0}$ for some $t_0 \geqslant t$, or

(b) $v_{t+1} = 000\ldots$ and $u_{t+1} = (m_{t+1} - 1)(m_{t+2} - 1)(m_{t+3} - 1)\ldots$, or

(c) $v_{t+1} = (m_{t+1} - 1)(m_{t+2} - 1)(m_{t+3} - 1)\ldots$ and $u_{t+1} = 000\ldots$.

Conversely, let $u = x_0x_1x_2\ldots$ and $v = z_0z_1z_2\ldots$ be infinite word with the condition such that (a) for $t_0 = t + 1$ or or (b), or (c). Then $g(u) = v$ for the element $g = ((0, 0, \ldots, 0), S)$, where $S = n\rho_t + z_0 - x_0$ and ρ_t is defined recurrently $\rho_{i+1} = \rho_i m_{t-i} + z_{t-i} - x_{t-i}$ for $i = 0, 1, \ldots, t-1$, where $\rho_0 = 0$ if (a), $\rho_0 = 1$ if (b) and $\rho_0 = -1$ if (c). □

Since $St_G(k)$ is a subgroup of finite index we obtain from point (ii)

Corollary 1. *The group G is of polynomial growth and the growth function $\gamma_G(m)$ is of order m^n.*

One of key problems in the theory of groups generated by Mealy automata is the problem of embeddability of other known classes of groups into these groups. One of the simplest group for which the construction of a suitable automaton is still unknown is a discrete wreath product of infinite cyclic groups $\mathbb{Z} \wr \mathbb{Z}$. The aim of this paper was to present the subgroup $K = [W, W] \cdot \mathbb{Z}$ of the wreath product $W = \mathbb{Z} \wr_{X_0} \mathbb{Z}$ relative to an n-element set X_0 as a group generated by a 2-state time-varying Mealy automaton. We showed that K acts spherically transitive on the set X^* of finite words over the changing alphabet X. The orbits of the action on the set X^ω of infinite words over X were described.

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Streszczenie

Kluczowym problemem w teorii grup generowanych przez automaty Mealy'ego jest zbadanie, czy dana grupa ze znanej klasy grup jest automatowa. Jedną z najprostszych grup, dla której konstrukcja odpowiedniego automatu Mealy'ego nie jest znana, jest dyskretny splot nieskończonych grup cyklicznych $\mathbb{Z} \wr \mathbb{Z}$. Celem niniejszej pracy było przedstawienie podgrupy $K = [W, W] \cdot \mathbb{Z}$ splotu $W = \mathbb{Z} \wr_{X_0} \mathbb{Z}$ względem n-elementowego zbioru X_0 jako grupy generowanej przez zmienny w czasie automat Mealy'ego o dwóch stanach. Opisano działanie grupy K na zbiorach słów skończonych i nieskończonych nad zmiennym alfabetem. Pokazano, że K działa sferycznie tranzytywnie na zbiorze X^* skończonych słów. Scharakteryzowano orbity działania na zbiorze X^ω słów nieskończonych.