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CURVE MODELING VIA INTERPOLATION BASED ON MULTIDIMENSIONAL REDUCED DATA

A Middle Bird's cable:

..."If in The Rainy Island of Every Day Struggle you cannot find a bird of paradise just get a wet hen."...

Nikita Khrushchev

Curve modeling via interpolation based on multidimensional reduced data

Summary

In this monograph we consider the problem of modeling curves together with the estimation of their length via various interpolation schemes (*i.e.* piecewise-polynomials) based on discrete *reduced data* $\mathcal{Q}_m = (q_0, q_1, \ldots, q_m)$. The latter term defines an ordered sequence of m input points in $\mathbb{R}^n$ stripped from the corresponding component of parameters. More precisely, reduced data are obtained by sampling a regular parametric curve (sufficiently smooth) $\gamma : [0,T] \to \mathbb{R}^n$ with $\gamma(t_i) = q_i$ (where $0 \le i \le m$) in arbitrary Euclidean space without provision of the corresponding tabular parameters $\{t_i\}_{i=0}^m \in [0,T]^{m+1}$, $t_0 = 0$ and $t_m = T$ - customarily coined in the literature as *interpolation knots*.

In this work, interpolation schemes based on reduced data are termed as *non-parametric*. On the other hand, fitting *non-reduced data* (*i.e.* the pair $(\mathcal{Q}_{i=0}^m, \{t_i\}_{i=0}^m)$ with interpolation knots $\{t_i\}_{i=0}^m$, assumed given, renders a classical *parametric interpolation* setting. The analysis of approximation orders for the trajectory of γ and the length estimation $d(\gamma)$ from stripped information encoded in reduced data (*the main topic of this monograph*) involves, in comparison with the parametric case, two new extra components. Firstly, for a given family of samplings, a proper guess of the discrete sequence of increasing tabular parameters $\{\hat{t}_i\}_{i=0}^m \in [0,\hat{T}]^{m+1}$, $\hat{t}_0 = 0$, $\hat{t}_m = \hat{T}$ (estimating the unknown $\{t_i\}_{i=0}^m$) is required for explicit derivation of a chosen interpolant $\tilde{\gamma} : [0,\hat{T}] \to \mathbb{R}^n$. Secondly, as now clearly both curves γ and $\tilde{\gamma}$ are defined over different domains (*i.e.* over $[0,T]$ and $[0,\hat{T}]$, respectively), a suitable choice of domains $[0,\hat{T}_m]$ and reparameterizations $\psi_m : [0,T] \to [0,\hat{T}_m]$ varying with m (having respective derivatives uniformly bounded) has to be made, so that built-in asymptotics of $\tilde{\gamma}_m$ can be studied and justified.

In addition, the important question of reaching *identical approximation orders* (for both γ and $d(\gamma)$ estimation) from reduced *versus* non-reduced data and based on the corresponding non-parametric and parametric interpolation is discussed throughout this monograph. In particular, we confirm the feasibility of compensating for the loss of information (stripped from the non-reduced data to form reduced sampling) upon construction of the appropriate non-parametric interpolation schemes. The issue of performance of the herein analyzed non-parametric curve modeling algorithms is addressed for both *dense* (m large) and *sporadic* (m small) reduced data.

The problem of fitting data, despite its analysis being limited here to parametric curves and interpolation, is *applicable* in *e.g.* geometric and terrain modeling, computer graphics, motion rendering and analysis, computer vision (image segmentation) and other applications such as medical image processing including automated diagnosis, monitoring or early detection of various diseases (*e.g.* of epilepsy, schizophrenia or glaucoma).

We present now core-issue characteristics of each chapter (labeled from 1 to 7) emphasizing the *new results* contained herein and formulated in the format of proved theorems and lemmas throughout this monograph. The Summary closes with a review of this work, collating main achievements and contributions of all chapters.

In *Chapter 1* we introduce the notion of reduced (and non-reduced) data supplemented with the definition of non-parametric (and parametric) interpolation. In sequel, the task of estimating the trajectory of γ and its length $d(\gamma)$ from reduced data is outlined in the context of existing results and applications. Next, potential difficulties in guessing the unknown knot parameters $\{t_i\}_{i=0}^m$ are briefly underlined.

Different families of samplings (*i.e.* admissible, uniform, ε-uniform and more-or-less uniform) are subsequently introduced and illustrated by examples. Orders of approximation for piecewise-r-degree Lagrange interpolation applied to uniform, ε-uniform and the general class of admissible samplings are established under the temporary assumption of modeling non-reduced data - see Theorems 1.1, 1.2 and 1.3. The uniform samplings are also analyzed for reduced data (upon slight adjustment of the non-reduced uniform case). The above results are needed later for comparisons with the case of non-parametric interpolation.

The sharpness of the above results, is confirmed by experiments for length estimation (at least for $n = 2$ and $r = 1, 2, 3, 4$).

Chapter 1 closes with discussion and motivation for Chapter 2. *The essential novel contribution* to the field analyzed herein is given by Theorem 1.3. The latter proves that, for the special subfamily (ε-uniform) of admissible non-reduced samplings, extra acceleration in convergence rates occurs, for $d(\gamma)$ estimation (see Theorems 1.2 and 1.3). The asymptotic analysis used herein relies *de facto* upon decomposing the ε-uniform case into the uniform (already analyzed in Theorem 1.1) and into a special non-uniform one (see (1.55)).

New results presented in this chapter (see Theorems 1.1, 1.2 and 1.3) are published in [32] and [46].

In *Chapter 2* we analyze the case of non-parametric interpolation, for $\{t_i\}_{i=0}^m$ guessed as equidistant $\hat{t}_i = i$, *i.e.* with no consideration whatsoever given to *the geometry of the distribution* of sampling points $\mathcal{Q}_m$. An important Example 2.1 compares the quality of both γ and $d(\gamma)$ approximation by *a piecewise-quadratic Lagrange interpolation* $\tilde{\gamma}_2$ used with either $\hat{t}_i = i$ or with true knots $\{t_i\}_{i=0}^m$. Contrary to parametric interpolation the last example indicates not only a substantial drop in convergence orders in $d(\gamma)$ estimation (with $\hat{t}_i = i$) but also points to the existence of a convergence-"divergence" duality. By "divergence" we mean $d(\tilde{\gamma}_2) \not\to d(\gamma)$ with $\delta \to 0$.

In the next step this undesirable situation (together with mentioned deceleration effect) is justified by two *novel results* (see Theorems 2.1 and 2.2) establishing the orders of convergence to estimate γ and $d(\gamma)$ by piecewise-quadratic Lagrange interpolation $\tilde{\gamma}_2$ used with $\hat{t}_i = i$ to fit ε-uniform and more-or-less uniform reduced data. More specifically, the resulting asymptotics show a substantial deceleration (to "divergence") in approximation orders to estimate γ (ranging from cubic to linear) and to estimate $d(\gamma)$ (ranging from quartic to zero). The proofs of Theorems 2.1 and 2.2 hinge on determining the asymptotics of the respective derivatives of $\tilde{\gamma}_2$ and of accordingly selected family of reparameterizations $\psi_m : [0, T] \to [0, \hat{T}_m]$ (varying with m). Subsequently, the asymptotic analysis for length and trajectory follows.

The sharpness (or nearly sharpness) of Theorems 2.1 and 2.2 is verified and confirmed experimentally (at least for $n = 2$ and for $d(\gamma)$ estimation).

Chapter 2 closes with discussion and motivation for Chapter 3. The fundamental conclusion stemming out of the discussion so-far clearly indicates that estimation of the unknown interpolation knots $t_i \approx \hat{t}_i$ should incorporate the geometry of the distribution of sampling points $\mathcal{Q}_m$. The results from this chapter extend also knowledge on the negative impact of blind choices of parameters $\{\hat{t}_i\}_{i=0}^m$ to estimate both γ and $d(\gamma)$ by non-parametric interpolation.

The new results from this chapter (see Theorems 2.1 and 2.2) are published in [46] and [47].

In *Chapter 3* a partial solution to the task of finding a correct set of knots $\{\hat{t}_i\}_{i=0}^m$ for given reduced data $\mathcal{Q}_m$ and of deriving an appropriate non-parametric interpolation scheme constructed from the pair $(\mathcal{Q}_m, \{\hat{t}\}_{i=0}^m)$ is proposed and studied. More specifically, *a piecewise-4-point quadratic* Q (a track-sum of $Q^i : [0, \beta_i] \to \mathbb{R}^2$) interpolating each consecutive four data points $\mathcal{Q}_m^{i,3} = (q_i, q_{i+1}, q_{i+2}, q_{i+3})$ is derived together with the estimates of the interpolation knots $(t_i, t_{i+1}, t_{i+2}, t_{i+3}) \approx (0, 1, \alpha_i, \beta_i)$, where $1 < \alpha_i < \beta_i$. On this occasion, the geometry of the distribution of $\mathcal{Q}_m^{i,3}$ impacts here on both Q^i and (α_i, β_i). The essential *limitation* of this interpolation procedure (together with the analysis of the convergence orders) stems from the fact that it is merely applicable to reduced data representing *more-or-less uniformly sampled, strictly convex planar curves*.

A non-trivial analysis exploiting the latter constraints proves the existence (in explicit form) of the interpolant Q and the above mentioned sequence of knots $\{\hat{t}_i\}_{i=0}^m$ estimating the unknown interpolation knots $\{t_i\}_{i=0}^m$. *The fundamental new result* of this chapter is claimed in Theorem 3.1, where fast *quartic orders of convergence* to estimate trajectory of γ and its length $d(\gamma)$ from reduced data (satisfying the above assumptions) are established. Again the proof of Theorem 3.1 involves an advanced analysis needed to determine the asymptotics of the respective derivatives of Q and of the accordingly selected family of reparameterizations $\psi_m : [0, T] \to [0, \hat{T}_m]$ (varying with m). The last step of the proof follows the previous pattern.

In the next part of this chapter, *the sharpness* of quartic order of convergence in length estimation and *the necessity of the assumptions* from Theorem 3.1 are experimentally confirmed. The examples illustrate also the excellent performance of Q in approximating both γ and $d(\gamma)$ on sporadic data. The latter is vital for practical application in *e.g.* modeling. Of course, the case of m small is not covered by the very nature of the asymptotic analysis in this monograph, which is relevant exclusively to reduced data $\mathcal{Q}_m$ with m sufficiently large. In addition, minor discontinuities of derivatives at the junction points $\{q_{3i}\}_{i=1}^{\frac{m}{3}}$ between two consecutive 4-point quadratics Q^{3i} and $Q^{3(i+1)}$ yield almost invisible corners in the trajectory of Q.

Finally, this chapter closes with discussion and motivation for Chapter 4. It should be emphasized that the piecewise-4-point quadratic interpolant Q outperforms, when applicable (for γ and $d(\gamma)$ estimation), the piecewise-quadratic Lagrange interpolation used with either $\{t_i\}_{i=0}^m$ known or $\hat{t}_i = i$ (if $\{t_i\}_{i=0}^m$ unknown) - see Theorems 1.2, 2.1, 2.2, and 3.1. The scheme in question also matches the performance of the piecewise-cubic Lagrange interpolation used with $\{t_i\}_{i=0}^m$ known (see Theorems 1.2 and 3.1).

Thus the non-parametric interpolation scheme Q yields a positive solution to the problem signaled in the third paragraph of this Summary referring to the possibility of *compensating for the loss of information carried by reduced data*. The latter is at least covered for the special case of more-or-less uniformly sampled strictly convex planar curves. Such limitations are waived in the next chapter, where the notion of Lagrange interpolation through *cumulative chord parameterization based on reduced data* is introduced. Subsequently, it is shown that all advantages of Q in fact extend to the case of regular curves sampled in $\mathbb{R}^n$ with no constraints imposed from above.

The whole chapter inputs essentially *a novel contribution* to the field of modeling reduced data generated by sampling planar curves.

The results from this chapter (see Theorems 2.1 and 2.2 together with auxiliary lemmas) are published in [43] and [44].

In *Chapter 4 cumulative chord piecewise-quadratics* and *piecewise-cubics* $\hat{\gamma}_k$ (where $k = 2, 3$) are examined in detail and compared with other low degree interpolants fitting reduced data $\mathcal{Q}_m$ from regular curves in $\mathbb{R}^n$, especially with piecewise-4-point quadratics. More precisely, the discrete sequence of unknown parameters $\{t_i\}_{i=0}^m$ is estimated here by the so-called cumulative chord parameterization $\{\hat{t}_i\}_{i=0}^m$ defined as follows: $\hat{t}_0 = 0$, $\hat{t}_{i+1} = \hat{t}_i + \|q_{i+1} - q_i\|$, for $0 \leq i \leq m-1$, where $\|\cdot\|$ denotes a standard norm in $\mathbb{R}^n$. Though similarly to the previous chapter the cumulative chord parameterization takes into account *the geometry of the distribution of sampling points* $\mathcal{Q}_m$, in addition it is also *applicable without the constraints imposed in Chapter 3*.

Orders of approximation of γ and $d(\gamma)$ with $\hat{\gamma}_k$ (for $k = 2, 3$) for arbitrary admissible (or ε-uniform) samplings are proved in *the main Theorem 4.1* to be cubic and quartic, for $k = 2, 3$, respectively (or $\min\{4, 3+\varepsilon\}$ for $k = 2$, $\varepsilon \geq 0$ and $d(\gamma)$ estimation). For the proof of Theorem 4.1 a different analysis, based on divided differences and Newton's Interpolation Formula, is invoked to determine the asymptotics of the respective derivatives of the interpolant $\hat{\gamma}_k$ ($k = 2, 3$) and of the accordingly selected family of reparameterizations $\psi_m : [0, T] \to [0, \hat{T}_m]$ (varying with m). Again the last part of the proof follows the previous pattern.

The sharpness of the estimates established in Theorem 4.1 together with *the necessity of the assumptions drawn herein* are confirmed by numerical experiments (at least for $n = 2, 3$ and $d(\gamma)$ approximation). Similarly to piecewise-4-point quadratics (for the relevant criteria see the Summary paragraph concerning Q), cumulative chord piecewise-quadratics and piecewise-cubics are also shown experimentally to *perform well* on *sporadic data*. The latter is not covered by the asymptotic analysis used in Theorem 4.1, applicable for sufficiently large m.

Again, this chapter closes with discussion and motivation for Chapter 5. As shown, for γ and $d(\gamma)$ estimation, cumulative chord piecewise-quadratics and piecewise-cubics approximate to the same order as the piecewise-quadratic and piecewise-cubic Lagrange interpolants used with non-reduced data (*i.e.* when $\{t_i\}_{i=0}^m$ is known) - see Theorems 1.2, 1.3 and 4.1. Thus the cumulative chord piecewise-quadratics and piecewise-cubics $\hat{\gamma}_k$ (for $k = 2, 3$) again yield the positive solution to the problem raised in the third paragraph of this Summary. This time, however, *the possibility of compensating for the loss of information carried by reduced data* (upon using $\hat{\gamma}_k$; $k = 2, 3$) is proved for *an arbitrary regular curve* in $\mathbb{R}^n$ (sufficiently smooth) and sampled according to *the general class of admissible samplings* $\mathcal{V}_G^m$.

The main novelty of this chapter, *i.e.* Theorem 4.1, contributes to the field of modeling regular curves in $\mathbb{R}^n$ from the general class of admissible samplings forming reduced data.

The next chapter discusses the issue of whether an extra acceleration to quintic orders, in γ and $d(\gamma)$ estimation, occurs for *cumulative chord piecewise-quartics* $\hat{\gamma}_4$. Recall that such an increment in convergence orders eventuates while interpolating non-reduced data $(\mathcal{Q}_m, \{t_i\}_{i=0}^m)$ by piecewise-r-degree Lagrange polynomial with knots $\{t_i\}_{i=0}^m$ assumed to be known (see Theorem 1.2).

The results of this chapter (see Theorem 4.1) are published in [42] and [45].

In *Chapter 5* we study the problem of estimating the trajectory of a regular curve γ in $\mathbb{R}^n$ and its length $d(\gamma)$ by *cumulative chord piecewise-quartics* $\hat{\gamma}_4$. In doing so, we extend results on cumulative chord piecewise-quadratics and piecewise-cubics γ_k (for $k = 2, 3$) analyzed in Chapter 4 (see Theorem 4.1). The corresponding convergence rates are established for different types of reduced data including ε-uniform and more-or-less uniform samplings (see Theorems 5.1 and 5.2).

As shown herein, further acceleration in convergence orders with cumulative chord piecewise-quartics $\hat{\gamma}_4$ follows only for the special subsamplings (*e.g.* for ε*-uniform* samplings - ranging from 4 to 5 or from 4 to 6 depending on smoothness of γ). On the other hand, approximation orders to estimate γ and $d(\gamma)$ with $\hat{\gamma}_4$ based on *more-or-less uniform* samplings coincide with those already established for the cumulative chord piecewise--cubics $\hat{\gamma}_3$ *i.e.* with order 4 (*no acceleration*).

The proof of Theorem 5.1 relies on the additional analysis of the fourth divided difference (in general not uniformly bounded) yielding the asymptotics of the respective derivatives of the interpolant $\hat{\gamma}_4$, and of the accordingly selected family of smooth reparameterizations $\psi_m : [0, T] \to [0, \hat{T}_m]$ (varying with m). The last part of the proof essentially follows the previous pattern.

Both Theorems 5.1 and 5.2 are experimentally confirmed to be *sharp* (at least in the case of $n = 2, 3$ and $d(\gamma)$ estimation). *The necessity* of the assumptions drawn in the above theorems is also verified.

The good performance of cumulative chord piecewise-quartics $\hat{\gamma}_4$ in γ and $d(\gamma)$ estimation (for the relevant criteria see the Summary paragraph concerning function Q) extends also to *sporadic data* not covered by asymptotic analysis presented in this work.

We close the chapter with discussion and motivation for Chapter 6. As shown herein, cumulative chord piecewise-quartics $\hat{\gamma}_4$ for the general class of admissible samplings do not approximate γ and $d(\gamma)$ to the same order as the piecewise-quartics Lagrange interpolants used with non-reduced data (*i.e.* when $\{t_i\}_{i=0}^m$ is known) - see Theorems 1.2 and 5.2. On the other hand when the sampling is ε-uniform then both rates coincide (see Theorems 1.3 and 5.1). Cumulative chord piecewise-quartics inherit convergence properties of cumulative chord-cubics (at least shown here for more-or-less uniform samplings). Thus in a search for a fast interpolation scheme, in general it suffices to interpolate reduced data with cumulative chord piecewise-quadratics or piecewise-cubics $\hat{\gamma}_k$ (for $k = 2, 3$) yielding cubic and quartic orders of γ and $d(\gamma)$ approximation, respectively. Note that, all so-far discussed interpolants are not smooth at the junction points where two local interpolants are glued together. In the next chapter we remove this blemish.

The main contributions of this part of the monograph are Theorems 5.1 and 5.2. Again the latter contributes to the field of fitting reduced samplings of a regular curve γ in $\mathbb{R}^n$.

The main results of this chapter (see 5.1 and 5.2) are published in [27], [28] and [29].

In *Chapter 6*, we construct and analyze the resulting asymptotics of a regular *cumulative chord C^1 piecewise-cubics* γ_H, for reduced samplings $\mathcal{Q}_m$ generated by regular sufficiently smooth curves γ in $\mathbb{R}^n$. The construction of $\gamma_H \in C^1$ is split into two steps. First the derivatives at sampling points $\mathcal{Q}_m$ are estimated from overlapping local cumulative chord cubics $\hat{\gamma}_3^i$ (introduced in Chapter 4). Next for each pair of points $(q_i, q_{i+1}) \in \mathcal{Q}_m \times \mathcal{Q}_m$ and the corresponding pair of tangent vectors (computed in the previous step) Hermite interpolation is invoked to generate a C^1 piecewise-cubic interpolant γ_H forming also (as shown) a regular curve. This ascertains the analytical and geometrical smoothness of the trajectory of γ_H with no potential cusps or corners (for sufficiently large m). The last property of γ_H is vital for modeling curves.

The main result of this chapter (see Theorem 6.1) yields quartic orders of convergence in estimating γ and $d(\gamma)$ by interpolant γ_H. The proof of Theorem 6.1 falls into three main components. Firstly, the asymptotics of the respective derivatives of γ_H and of the accordingly selected family of reparameterizations $\phi_m : [0,T] \to [0,\hat{T}_m]$ (upon some non-trivial steps) are calculated in terms of cumulative chord piecewise-cubic $\hat{\gamma}_3$ and respective reparameterizations $\psi_m : [0,T] \to [0,\hat{T}_m]$ (compare with the paragraphs above from Chapter 4). Next again upon some analysis the corresponding asymptotics of differences $\gamma - \gamma_H$ and $\dot{\gamma} - \gamma'_H$ are found in terms of asymptotics of $\gamma_H - \hat{\gamma}_3$ and of $\gamma'_H - \hat{\gamma}'_3$. The last easier step follows the previous pattern.

The sharpness of the main result (see Theorem 6.1) for estimation of $d(\gamma)$ and *the necessity of the assumptions adopted herein* are verified through numerical experiments (at least for $n = 2,3$). Good performance of the smooth interpolant γ_H (to estimate γ and $d(\gamma)$) is also confirmed experimentally on *sparse data* - for the relevant criteria see the pertinent paragraph of this Summary referring to the function Q.

Cumulative chord C^1 piecewise-cubics γ_H inherit all the advantages of cumulative chord piecewise-cubics $\hat{\gamma}_3$ over other schemes discussed in Chapters 2 and 3. Thus the scheme in question γ_H also provides a positive solution to the problem from the third paragraph of this Summary. Namely, compensation for the loss of information carried by reduced data $\mathcal{Q}_m$ (obtained by sampling a regular curve γ in $\mathbb{R}^n$) is possible upon using non-parametric interpolation γ_H. More importantly, from the point of view of applications, γ_H does even better than $\hat{\gamma}_k$ (for $k = 2,3,4$) by rendering a smooth trajectory.

This part of monograph forms essentially *a novel contribution* to the field of fitting reduced data with smooth non-parametric interpolation.

The results from this chapter (including Theorem 6.1) are published in [30] and [31].

In *Chapter 7* we recapitulate the main claims and back-bone results presented in this monograph. In addition, we outline possible future directions and open problems for the topic in question. Some hints and avenues to pursue the above problems are also given.

Summing up: the analysis presented herein tackles an important issue of constructing appropriate non-parametric interpolation schemes to fit reduced data $\mathcal{Q}_m$ in $\mathbb{R}^n$ (*i.e.* the data without provision of the corresponding interpolation knots $\{t_i\}_{i=0}^m$). Our discussion demonstrates that *cumulative chord piecewise-quadratics and piecewise-cubics* $\hat{\gamma}_k$ (for $k = 2,3$) yield fast and excellent γ and $d(\gamma)$ approximation, with γ assumed to define a regular and sufficiently smooth curve in $\mathbb{R}^n$, sampled according to the general class of admissible samplings. In particular, it is shown that *compensation for the loss of information,* upon passing from non-reduced to reduced data, is feasible when invoking pertinent non-parametric interpolation schemes (*e.g.* $\hat{\gamma}_k$, $k = 2,3$). For the latter the asymptotics in γ and $d(\gamma)$ estimation coincide with those established for the corresponding parametric interpolations, *i.e.* piecewise-r-degree polynomials (with $r = 2, 3$) used with knots $\{t_i\}_{i=0}^m$ assumed to be known. Another remarkable discovery shows that, contrary to the non-reduced data case (where orders for γ and $d(\gamma)$ estimation for piecewise-r-degree polynomials increment to $r+1$), the cumulative chord paramaterization does not necessarily accelerate convergence orders (with r incremented), *e.g.* up to quintic orders for $\hat{\gamma}_4$. In this monograph a smooth analogue of $\hat{\gamma}_3$, *i.e. cumulative chord C^1 piecewise-cubic γ_H*, is also introduced and analyzed accordingly. Excellent approximation properties (proved herein) of non-parametric interpolants Q, $\hat{\gamma}_k$ (for $k = 2, 3, 4$) and γ_H, are also retained (as verified experimentally) upon passing from *dense* to *sparse* reduced data. As also shown, the omission of *the geometry of the distribution of the reduced data $\mathcal{Q}_m$* in guessing the interpolation knots $\{t_i\}_{i=0}^m \approx \{\hat{t}_i\}_{i=0}^m$ (*e.g.* with $\hat{t}_i = i$) may have serious consequences for estimating γ and $d(\gamma)$ - upon applying "blind" non-parametric interpolation scheme. The results from this monograph having a general character with *no tight constraints imposed on curve γ* nor *on type and dimension of data* (*i.e.* on $\mathcal{V}_G^m$ and n), constitute *a new input into the field of fitting reduced data with non-parametric interpolation*.

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Modelowanie krzywych poprzez interpolację na bazie wielowymiarowych danych zredukowanych

Streszczenie

Niniejsza rozprawa dotyczy problemu modelowania krzywych wraz z estymacją ich długości poprzez zastosowanie różnych schematów interpolacyjnych (tzn. funkcji sklejanych wielomianowych) na bazie uporządkowanych dyskretnych *zredukowanych danych* $\mathcal{Q}_m = (q_0, q_1, \ldots, q_m)$ będących ciągiem m punktów w $\mathbb{R}^n$ okrojonych z komponenty odpowiadającego im ciągu parametrów. Precyzyjniej: zredukowane dane wejściowe $\mathcal{Q}_m$ otrzymuje się poprzez próbkowanie nieznanej krzywej regularnej (odpowiednio gładkiej) $\gamma : [0, T] \to \mathbb{R}^n$ w dowolnej przestrzeni euklidesowej, dla której $\gamma(t_i) = q_i$ (gdzie $0 \leq i \leq m$), przy założeniu braku znajomości odpowiadającego im rosnącego ciągu parametrów $\{t_i\}_{i=0}^m \in [0, T]^{m+1}$, $t_0 = 0$, $t_m = T$ $-$ zwanych w literaturze dotyczącej interpolacji *węzłami interpolacyjnymi*.

W przedłożonej monografii schematy interpolacyjne, skonstruowane na podstawie danych zredukowanych, określa się terminem *interpolacji nieparametrycznych*. W porównaniu z klasyczną *interpolacją parametryczną* $-$ gdzie dodatkowo zakłada się, iż węzły interpolacyjne $\{t_i\}_{i=0}^m$ są również zadane, a parę $(\mathcal{Q}_m, \{t_i\}_{i=0}^m)$ oznacza się mianem *danych pełnych* lub *niezredukowanych* $-$ przy tak okrojonej informacji analiza rzędów aproksymacji dla oszacowań trajektorii krzywej γ i jej długości $d(\gamma)$ (*co stanowi główny temat rozprawy*) zawiera dwa dodatkowe elementy składowe. Po pierwsze, dla zadanej rodziny próbkowań w celu skonstruowania krzywej interpolującej $\tilde{\gamma} : [0, \hat{T}] \to \mathbb{R}^n$ konieczne jest zdefiniowanie dziedziny $[0, \hat{T}]$ oraz rosnącego ciągu parametrów $\{\hat{t}_i\}_{i=0}^m \in [0, \hat{T}]^{m+1}$, $\hat{t}_0 = 0$, $\hat{t}_m = \hat{T}$, estymującego nieznany rozkład węzłów interpolacyjnych $\{t_i\}_{i=0}^m$. Po drugie, ponieważ obie krzywe γ i $\tilde{\gamma}$ są zdefiniowane w różnych dwóch dziedzinach (tzn. $[0, T]$ oraz $[0, \hat{T}]$), należy znaleźć (przy m rosnącym) właściwą rodzinę dziedzin $[0, \hat{T}_m]$ i reparametryzacji $\psi_m : [0, T] \to [0, \hat{T}_m]$ zmieniających się wraz z m (z odpowiednimi pochodnymi ograniczonymi jednostajnie), tak by można było dowieść asymptotyki zbieżności dla każdego schematu $\tilde{\gamma}_m$ rozważanego z osobna.

W niniejszej pracy szczegółowo omówiono istotną kwestię możliwości otrzymania identycznych rzędów zbieżności (dla oszacowań γ i $d(\gamma)$), zarówno na danych zredukowanych, jak i pełnych, poprzez zastosowanie analogicznych nieparametrycznych i parametrycznych schematów interpolacyjnych. Innymi słowy, wykazujemy, iż *kompensacja straty informacji* przy przejściu z danych pełnych do zredukowanych jest możliwa przy stosowaniu właściwych interpolacji nieparametrycznych. Ponadto efektywność rozważanych algorytmów przy modelowaniu krzywych omówiona została i w aspekcie zredukowanych danych *gęstych* (m duże), i danych *rzadkich* (m małe).

Problem analizowany w monografii, pomimo że ograniczony jedynie do krzywych parametrycznych i interpolacji, ma duże *znaczenie praktyczne* w modelowaniu geometrycznym, modelowaniu terenu, grafice komputerowej, analizie i edycji ruchu, wizji komputerowej (segmentacja obrazu) czy innych zastosowaniach, takich jak np. medyczne przetwarzanie obrazu $-$ w szczególności we wczesnej diagnozie, rozpoznawaniu i monitorowaniu różnych schorzeń (np. epilepsji, schizofrenii lub jaskry).

Poniżej przedstawiamy pokrótce charakterystykę każdego rozdziału niniejszej rozprawy (tj. $1-7$), uwypuklając *nowe wyniki badań*, sformułowane w postaci lematów i twierdzeń, które ściśle uzasadniono w przedlożonej monografii. Streszczenie zakończono krótkim podsumowaniem.

Rozdział 1 wprowadza pojęcia danych pełnych (niezredukowanych) oraz zredukowanych wraz z podaniem definicji interpolacji parametrycznej i nieparametrycznej. Następnie skrótowo omówiono w aspekcie już istniejących wyników i zastosowań zagadnienie oszacowania trajektorii γ i jej długości $d(\gamma)$ na podstawie zredukowanych danych. W kolejnym kroku zasygnalizowano potencjalne trudności (w kontekście aproksymacji γ i $d(\gamma)$) wynikające z estymacji „na ślepo" nieznanych węzłów interpolacyjnych $\{t_i\}_{i=0}^m$.

W dalszej części definiuje się różne rodziny próbkowań (tj. *dopuszczalne*, *równomierne*, ε*-równomierne* lub *mniej lub bardziej równomierne*), każdorazowo ilustrując je przykładami. W konsekwencji dla oszacowań γ i $d(\gamma)$ (przy chwilowym założeniu dostępności danych niezredukowanych) udowodnione zostają odpowiednie rzędy zbieżności dla interpolacji parametrycznej, zdefiniowanej na podstawie sklejanych wielomianów Lagrange'a stopnia r i próbkowań, takich jak: równomierne, ε-równomierne czy też należące do ogólnej klasy danych dopuszczalnych – por. Twierdzenia 1.1, 1.2 i 1.3. Przypadek próbkowania równomiernego został także przeanalizowany dla danych zredukowanych (stosując modyfikację dowodu dla danych pełnych równomiernych). Powyższa analiza okazuje się niezbędna dla późniejszych porównań z analogicznymi schematami interpolacji nieparametrycznej.

Ostrość oszacowań rzędów zbieżności dla estymacji długości krzywych $d(\gamma)$ (omawianych w tym rozdziale) została potwierdzona eksperymentalnie, przynajmniej w przypadku krzywych na płaszczyźnie oraz $r=1,2,3,4$.

Rozdział 1 podsumowuje dyskusja wraz z uzasadnieniem dla rozdziału 2. Główny *nowatorski wkład badawczy* w rozdziale wstępnym zawiera się w Twierdzeniu 1.3, które ściśle uzasadnia, iż dla wybranej podrodziny (ε-równomiernych) dopuszczalnych próbkowań zachodzi dodatkowe przyśpieszenie w rzędzie zbieżności dla oszacowania $d(\gamma)$ (zob. Twierdzenia 1.2 i 1.3). Zastosowana tu analiza rzędów zbieżności polega na dekompozycji przypadku próbkowań ε-równomiernych na równomierne (już wcześniej zbadane – zob. Twierdzenie 1.1) oraz specjalny przypadek próbkowań nierównomiernych (zob. (1.55)).

Nowe wyniki omówione w niniejszym rozdziale (zawarte w trzech Twierdzeniach 1.1, 1.2 i 1.3) zostały opublikowane wcześniej w artykułach [32] i [46].

Rozdział 2 rozważa przypadek interpolacji nieparametrycznej, dla której nieznane parametry $\{t_i\}_{i=0}^m$ estymuje się rozkładem równomiernym $\hat{t}_i=i$ bez uwzględnienia *geometrii rozrzutu punków próbkowania* $\mathcal{Q}_m$. Ważny poglądowo przykład 2.1 porównuje jakość aproksymacji γ i $d(\gamma)$ funkcjami sklejanymi (przedziałowo-kwadratowe wielomiany Lagrange'a $\hat{\gamma}_2$) dla dwóch przypadków, tj. gdy $\hat{t}_i=i$ lub węzłów $\{t_i\}_{i=0}^m$ zadanych. Powyższy przykład ujawnia przede wszystkim (dla danych zredukowanych z $\hat{t}_i=i$) nie tylko znaczny spadek rzędów zbieżności w estymacji $d(\gamma)$ (w porównaniu z interpolacją parametryczną), lecz wskazuje także na istnienie dualizmu zbieżność *versus* „rozbieżność" w oszacowaniu długości krzywych na bazie danych zredukowanych. Przez „rozbieżność" rozumiemy $d(\hat{\gamma}_2)\not\to d(\gamma)$ dla $\delta\to 0$.

W dalszej części rozdziału dualizm ten wraz z wymienionym efektem spowolnienia zbieżności uzasadniają dwa *nowe wyniki* (zob. Twierdzenia 2.1 i 2.2). Określają one asymptotykę rzędów zbieżności w oszacowaniu γ i $d(\gamma)$ dla przedziałowo-kwadratowych wielomianów Lagrange'a $\tilde{\gamma}_2$ na danych zredukowanych $\hat{t}_i = i$, próbkowanych ε-równomiernie i mniej lub bardziej równomiernie. Uściślając: powyższe asymptotyki wykazują spadek rzędów zbieżności (aż do „rozbieżności”) dla oszacowań γ (w zakresie od rzędu kubicznego do liniowego) oraz $d(\gamma)$ (w zakresie od rzędu czwartego do zerowego). Dowód Twierdzeń 2.1 i 2.2 opiera się na wyznaczeniu asymptotyk pochodnych interpolanta $\tilde{\gamma}_2$ oraz asymptotyk pochodnych odpowiednich rzędów właściwie dobranej rodziny reparametryzacji $\psi_m : [0,T] \to [0,\hat{T}_m]$ (zmieniających się wraz z m).

Następny krok dokonuje weryfikacji i potwierdza eksperymentalnie (przynajmniej dla $n = 2$) *ostrość* (lub prawie ostrość) oszacowań $d(\gamma)$ sformułowanych w Twierdzeniach 2.1 i 2.2.

Rozdział 2, podobnie jak pierwszy, zamyka dyskusja oraz uzasadnienie dla rozdziału 3. Fundamentalna konkluzja wynikająca z dotychczasowej analizy wskazuje jasno, że przy doborze właściwej estymacji nieznanych węzłów interpolacyjnych $\hat{t}_i \approx t_i$ należy uwzględnić *geometrię rozkładu próbkowanych danych* $\mathcal{Q}_m$. Ponadto wyniki omówione w tym rozdziale poszerzają stan wiedzy dotyczący negatywnego wpływu wyboru węzłów interpolacyjnych $\{\hat{t}_i\}_{i=0}^m$ „na ślepo” w przypadku interpolacji nieparametrycznej dla oszacowań γ i $d(\gamma)$.

Nowe rezultaty przedstawione w niniejszym rozdziale (sformułowane w Twierdzeniach 2.1 i 2.2) zostały już opublikowane w artykułach [46] i [47].

Rozdział 3 przeprowadza analizę częściowego rozwiązania problemu wyznaczenia (dla zredukowanych danych $\mathcal{Q}_m$) właściwego ciągu węzłów $\{\hat{t}_i\}_{i=0}^m$ oraz użycia stosownej interpolacji nieparametrycznej na bazie pary $(\mathcal{Q}_m, \{\hat{t}_i\}_{i=0}^m)$. Ściślej: dla rodziny kolejnych czwórek punktów $\mathcal{Q}_m^{i,3} = (q_i, q_{i+1}, q_{i+2}, q_{i+3})$ definiuje się *4-punktową przedziałowo--kwadratową interpolację* Q (sklejoną z poszczególnych $Q^i : [0,\beta_i] \to \mathbb{R}^2$) wraz z oszacowaniami na węzły interpolacyjne $(t_i, t_{i+1}, t_{i+2}, t_{i+3}) \approx (0, 1, \alpha_i, \beta_i)$ z $1 < \alpha_i < \beta_i$. Tym razem geometria rozkładu punktów $\mathcal{Q}_m^{i,3}$ została uwzględniona zarówno w formułach na funkcję Q^i, jak i na parę (α_i, β_i). Istotnym *ograniczeniem* przydatności powyższej procedury (wraz z przedstawioną analizą rzędów zbieżności) jest fakt, iż ma ona zastosowanie tylko do próbkowań *mniej lub bardziej równomiernych* oraz do krzywych *ściśle wypukłych na płaszczyźnie*.

Nietrywialna analiza uwzględniająca powyższe ograniczenia dowodzi istnienia funkcji interpolującej Q (określonej wzorem *explicite*) wraz ze wspomnianym ciągiem oszacowań $\{\hat{t}_i\}_{i=0}^m$ nieznanych węzłów interpolacyjnych $\{t_i\}_{i=0}^m$. Zasadniczy *rezultat tego rozdziału* sformułowany został w Twierdzeniu 3.1, które wykazuje (przy założeniu spełnienia powyższych ograniczeń) *czwarty rząd zbieżności* dla oszacowań zarówno trajektorii krzywej γ, jak i długości $d(\gamma)$. Uzasadnienie Twierdzenia 3.1 wymaga, podobnie jak poprzednio, ingerencji zaawansowanej analizy matematycznej dla wyznaczenia asymptotyk pochodnych Q i pochodnych właściwie dobranej rodziny reparametryzacji $\psi_m : [0,T] \to [0,\hat{T}_m]$ (zmieniających się wraz z m). Ostatni krok dowodu jest powieleniem wcześniej stosowanej argumentacji.

W dalszej części rozdziału 3 *ostrość* oszacowań na estymację długości krzywej $d(\gamma)$ wraz z *istotnością założeń* z Twierdzenia 3.1 zostaje zweryfikowana eksperymentalnie na różnych przykładach. Ilustrują one również zachowanie bardzo dobrych własności funkcji Q w oszacowaniu γ i $d(\gamma)$ na próbkowaniach *rzadkich*, co ma istotne znaczenie przy *zastosowaniach praktycznych*, np. w modelowaniu. Oczywiście, przypadek małych m nie jest objęty analizą rzędów zbieżności z Twierdzenia 3.1, odnoszącą się wyłącznie do danych zredukowanych $\mathcal{Q}_m$ dla dostatecznie dużych m. Dodatkową zaletą omawianego schematu (zarówno na danych gęstych, jak i rzadkich) jest minimalna nieciągłość pochodnych, pojawiająca się w punktach sklejania $\{q_{3i}\}_{i=1}^{\frac{m}{3}}$ pomiędzy sąsiadującymi interpolantami Q^{3i} oraz $Q^{3(i+1)}$. Własność ta oznacza niezauważalne nieciągłości w geometrycznej gładkości trajektorii Q.

Rozdział 3 podsumowuje dyskusja wraz z uzasadnieniem dla rozdziału 4. Warto podkreślić, iż 4-punktowa przedziałowo-kwadratowa interpolacja Q (tam gdzie można ją stosować) daje lepsze wyniki w oszacowaniu γ i $d(\gamma)$ niż przedziałowo-kwadratowa interpolacja Lagrange'a, zastosowana zarówno ze znanymi węzłami t_i, jak i z $\hat{t}_i = i$ (gdy $\{t_i\}_{i=0}^m$ są nieznane) – por. Twierdzenia 1.2, 2.1, 2.2 i 3.1 z rozdziałów 1, 2 i 3. Ponadto wykazane rzędy zbieżności w estymacji γ i $d(\gamma)$ funkcją Q (stanowiącej przykład interpolacji nieparametrycznej) pokrywają się z asymptotyką zbieżności określoną dla *przedziałowo-kubicznej parametrycznej interpolacji Lagrange'a* na bazie znanych węzłów $\{t_i\}_{i=0}^m$ (zob. Twierdzenia 1.2 i 3.1).

Tak więc badana w omawianym rozdziale metoda interpolacji nieparametrycznej daje pozytywne rozwiązanie problemu zasygnalizowanego w trzecim akapicie niniejszego streszczenia, dotyczące możliwości *kompensacji straty informacji zawartej w zredukowanych danych*, przynajmniej w przypadku ściśle wypukłych krzywych płaskich próbkowanych mniej lub bardziej równomiernie. Ograniczenia te zostaną usunięte w następnym rozdziale, gdzie wprowadza się pojęcie interpolacji Lagrange'a opartej na *skumulowanej parametryzacji długością cięciwy* dla danych zredukowanych, ekstrapolując tym samym omówione własności funkcji Q na przypadek interpolacji dowolnej krzywej regularnej γ (odpowiednio gładkiej), próbkowanej w $\mathbb{R}^n$ niekoniecznie mniej lub bardziej równomiernie.

Rozdział 3 wnosi zatem *nowatorski wkład* w zagadnienie modelowania zredukowanych danych pochodzących z próbkowań krzywych płaskich ściśle wypukłych.

Wyniki z rozdziału 3 (w tym Twierdzenia 2.1 i 2.2 wraz z pomocniczymi lematami) zostały opublikowane wcześniej w artykułach [43] i [44].

Rozdział 4 analizuje nieparametryczną, *przedziałowo-kwadratową i przedziałowo-kubiczną interpolację Lagrange'a* $\hat{\gamma}_k$ (dla $k = 2, 3$) na bazie *skumulowanej parametryzacji długością cięciwy*. Przede wszystkim porównuje się $\hat{\gamma}_k$ z już omówionymi interpolacjami parametrycznymi na bazie danych $\mathcal{Q}_m$ (dla dowolnej odpowiednio gładkiej krzywej regularnej w $\mathbb{R}^n$), w tym z 4-punktową przedziałowo-kwadratową interpolacją Q analizowaną w poprzednim rozdziale. Precyzyjniej: ciąg nieznanych węzłów $\{t_i\}_{i=0}^m$ zostaje oszacowany przez skumulowaną parametryzację długością cięciwy, dla której $\{\hat{t}_i\}_{i=0}^m$ definiuje się następująco: $\hat{t}_0 = 0$, $\hat{t}_{i+1} = \hat{t}_i + \|q_{i+1} - q_i\|$, dla $0 \leq i \leq m-1$, gdzie $\|\cdot\|$ oznacza standardową normę w $\mathbb{R}^n$. Chociaż podobnie jak w rozdziale poprzednim skumulowana parametryzacja długością cięciwy uwzględnia *geometrię rozkładu próbkowań* $\mathcal{Q}_m$, to jest także stosowalna *bez ograniczeń narzuconych w rozdziale 3*.

W *głównym twierdzeniu* (zob. Twierdzenie 4.1) tej części rozprawy dowodzi się, iż rzędy aproksymacji γ i $d(\gamma)$ funkcjami $\hat{\gamma}_k$ (dla $k = 2, 3$) na próbkowaniach dopuszczalnych (lub ε-równomiernych) są rzędu kubicznego lub stopnia czwartego, odpowiednio dla $k = 2, 3$ (lub rzędu $\min\{4, 3+\varepsilon\}$ dla $k = 2$ i $\varepsilon \geq 0$ w oszacowaniu $d(\gamma)$). Powyższy rezultat osiągnięto, zastosowawszy analizę bazującą na ilorazach różnicowych oraz na wzorze interpolacyjnym Newtona w celu wyznaczenia asymptotyk pochodnych funkcji $\hat{\gamma}_k$ (dla $k = 2, 3$) i asymptotyk odpowiednich pochodnych stosownie dobranej rodziny reparametryzacji $\psi_m : [0, T] \to [0, \hat{T}_m]$ (zmieniających się wraz z m). Ostatni krok dowodu oparto na poprzedniej argumentacji.

Ostrość oszacowań dla $d(\gamma)$ wraz z *istotnością założeń* z Twierdzenia 4.1 potwierdzają liczne eksperymenty numeryczne (przynajmniej dla $n = 2, 3$). W dalszej części rozdziału, podobnie jak dla 4-punktowej przedziałowo-kwadratowej interpolacji Q, wykazuje się eksperymentalnie, iż $\hat{\gamma}_k$ (dla $k = 2, 3$) *zachowuje bardzo dobre własności estymacyjne* (por. odpowiednie kryteria z akapitu tego streszczenia dotyczącego funkcji Q) na *danych rzadkich*, co, podobnie jak poprzednio nie zostało objęte analizą dla Twierdzenia 4.1, stosowną tylko dla dostatecznie dużych m.

Rozdział 4, tak jak wcześniejsze, podsumowuje dyskusja wraz z uzasadnieniem dla rozdziału 5. Jak wykazano powyżej, rzędy zbieżności w estymacji γ i $d(\gamma)$ dla przedziałowo-kwadratowej i przedziałowo-kubicznej interpolacji Lagrange'a, opartej zarówno na skumulowanej parametryzacji długością cięciwy, jak i na bazie węzłów $\{t_i\}_{i=0}^m$ (o ile są one znane) pokrywają się (zob. Twierdzenia 1.2, 1.3 i 4.1). W konsekwencji obydwa omówione powyżej schematy interpolacji nieparametrycznej $\hat{\gamma}_k$ (dla $k = 2, 3$), oparte na skumulowanej parametryzacji długością cięciwy, dają ponownie pozytywne rozwiązanie dla problemu poruszonego w trzecim akapicie niniejszego streszczenia. Jednak tym razem *kompensacja straty informacji zawartej w danych zredukowanych* (stosując $\hat{\gamma}_k$) wykazana została dla *dowolnej krzywej regularnej* w $\mathbb{R}^n$ (odpowiednio gładkiej) próbkowanej w takt *ogólnej klasy dopuszczalnych próbkowań* $\mathcal{V}_G^m$.

Twierdzenie 4.1 stanowi *nowatorski wkład badawczy* w zakresie modelowania krzywych regularnych w $\mathbb{R}^n$ na bazie danych zredukowanych.

Następny rozdział odpowiada na pytanie, czy podwyższenie stopnia wielomianu funkcji sklejanej (do czwartego stopnia przedziałowo-wielomianowej interpolacji Lagrange'a na bazie parametryzacji długością cięciwy) daje przyśpieszenie zbieżności (zarówno w oszacowaniu γ, jak i $d(\gamma)$) do rzędu piątego – otóż taka sytuacja zachodzi, gdy np. węzły $\{t_i\}_{i=0}^m$ są zadane (por. Twierdzenie 1.2).

Wyniki zawarte w niniejszym rozdziale (włącznie z wynikami Twierdzenia 4.1) zostały już opublikowane w artykułach [42] i [45].

Rozdział 5 podejmuje problem oszacowania γ i $d(\gamma)$, próbkowanej w $\mathbb{R}^n$ z zastosowaniem przedziałowo-wielomianowej nieparametrycznej interpolacji Lagrange'a *stopnia czwartego* $\hat{\gamma}_4$ (na bazie skumulowanej parametryzacji długością cięciwy). W tym celu rozszerza się wyniki z rozdziału 4 (tj. z Twierdzenia 4.1) uzyskane dla $\hat{\gamma}_k$, gdzie $k = 2, 3$. W szczególności ustala się odpowiednie rzędy zbieżności dla $\hat{\gamma}_4$ i różnego typu danych zredukowanych $\mathcal{Q}_m$, włączając w to przypadki próbkowań ε-równomiernych i mniej lub bardziej równomiernych (zob. Twierdzenia 5.1 i 5.2).

Powyższe *nowe wyniki* wykazują, iż dalsze przyśpieszenie rzędu zbieżności dla $\hat{\gamma}_4$ jest osiągalne tylko dla specjalnego podzbioru dopuszczalnych próbkowań. I tak np. dla *danych* ε*-równomiernych* rzędy zbieżności w oszacowaniu długości $d(\gamma)$ (lub γ) zmieniają się wraz z $\varepsilon > 0$ w zakresie od 4 do 5 lub od 4 do 6 (od 4 do 5) w zależności od stopnia gładkości krzywej γ. Z drugiej strony odpowiednie rzędy zbieżności w estymacji krzywej

γ i jej długości $d(\gamma)$ dla $\hat{\gamma}_4$ na bazie *mniej lub bardziej równomiernych* próbkowań pokrywają się z ustalonym poprzednio wynikiem dla $\hat{\gamma}_3$, tzn. z rzędem 4 (*brak przyśpieszenia*).

Dowód Twierdzenia 5.1 opiera się na dodatkowej analizie asymptotyki czwartego ilorazu różnicowego (w ogólnym przypadku niekoniecznie ograniczonego jednostajnie), która prowadzi do uzyskania asymptotyk pochodnych $\hat{\gamma}_4$ oraz asymptotyk odpowiednich pochodnych właściwie dobranej rodziny reparametryzacji $\psi_m : [0,T] \to [0,\hat{T}_m]$ (zmieniających się stosownie wraz z m). Ostatnia część dowodu uwzględnia poprzednią argumentację.

Ostrość uzyskanych wyników dla estymacji $d(\gamma)$ oraz *istotność założeń* z Twierdzeń 5.1 i 5.2 zostały zweryfikowane eksperymentalnie na różnych przykładach (przynajmniej dla $n = 2, 3$).

W kolejnym kroku potwierdza się, również eksperymentalnie, przydatność $\hat{\gamma}_4$ w estymacji γ i $d(\gamma)$ na *zredukowanych danych rzadkich* (por. odpowiednie kryteria z akapitu tego streszczenia dotyczącego funkcji Q). Podobnie jak poprzednio analiza asymptotyczna z Twierdzeń 5.1 i 5.2 nie obejmuje przypadku małych m.

Rozdział 5 zamyka dyskusja oraz uzasadnienie dla rozdziału 6. Jak wykazano w tej części tezy (por. Twierdzenie 5.2), zwiększenie stopnia przedziałowo-wielomianowej interpolacji Lagrange'a do czwartego w zasadzie nie prowadzi – dla skumulowanej parametryzacji długością cięciwy na dopuszczalnych danych – do zwiększenia rzędów zbieżności z 4 do 5 (przy estymacji γ i $d(\gamma)$). Sytuacja taka jest możliwa dla próbkowań dopuszczalnych z zadanymi węzłami $\{t_i\}_{i=0}^m$ (por. Twierdzenie 1.2) lub dla danych zredukowanych, w których próbkowania mają specjalny charakter (por. Twierdzenia 5.1 i 5.2). W konsekwencji omówione wcześniej szybko zbieżne funkcje $\hat{\gamma}_k$ $(k = 2, 3)$ z rozdziału 4 w zupełności wystarczają jako narzędzia do interpolacji zredukowanych próbkowań krzywych regularnych w $\mathbb{R}^n$. Zauważmy, że jak dotąd krzywe interpolujące nie gwarantują *gładkości* na węzłach sklejania kolejnych lokalnych interpolantów. W następnym rozdziale powyższa wada zostanie skorygowana.

Najważniejsze wyniki tej części rozprawy (por. Twierdzenia 5.1 i 5.2) zostały wcześniej opublikowane w artykułach [27], [28] i [29].

Rozdział 6 omawia asymptotykę *przedziałowo-kubicznej gładkiej interpolacji* (oznaczonej symbolem γ_H) skonstruowanej na dopuszczalnych danych zredukowanych $\mathcal{Q}_m$, pochodzących z próbkowania krzywej regularnej γ (odpowiednio gładkiej) w $\mathbb{R}^n$. Konstrukcja interpolanta $\gamma_H \in C^1$ odbywa się w następujących *dwóch fazach*. Najpierw pochodne w punktach próbkowania $\mathcal{Q}_m$ estymuje się odpowiednio pochodnymi nakładających się kubicznych interpolacji Lagrange'a $\hat{\gamma}_3^i$ (zdefiniowanych w rozdziale 4 na bazie skumulowanej parametryzacji długością cięciwy). Następnie dla każdej pary kolejnych punktów $(q_i, q_{i+1}) \in \mathcal{Q}_m \times \mathcal{Q}_m$ oraz związanej z nią pary wektorów stycznych (wyznaczonych w poprzednim kroku) interpolacja Hermite'a definiuje analitycznie gładką funkcję $\gamma_H \in C^1$, której wykres dodatkowo opisuje tu krzywą regularną w $\mathbb{R}^n$ (dla dużych m). Powyższe własności gwarantują geometryczną gładkość trajektorii γ_H (dla dostatecznie dużych m), tzn. bez występowania „rogów” w trajektorii γ_H dla danych gęstych. Własność ta ma fundamentalne znaczenie w *modelowaniu krzywych gładkich*.

Najważniejszym wynikiem tej części monografii jest Twierdzenie 6.1, które udowadnia *zbieżność czwartego rzędu* dla estymacji γ i $d(\gamma)$ poprzez zastosowanie funkcji γ_H. Dowód Twierdzenia 6.1 składa się z trzech nietrywialnych komponentów. Najpierw określa się asymptotyki pochodnych γ_H oraz pochodnych odpowiednio dobranej rodziny reparametryzacji $\phi_m : [0,T] \to [0,\hat{T}_m]$ w terminach pochodnych $\hat{\gamma}_3$ i w terminach stosownych pochodnych reparametryzacji $\psi_m : [0,T] \to [0,\hat{T}_m]$ (por. wyżej akapity dotyczące rozdziału 4). Następnie, asymptotyki różnic $\gamma - \gamma_H$ oraz $\dot{\gamma} - \gamma'_H$ wyraża się w terminach odpowiadających sobie asymptotyk wyrażeń $\gamma_H - \hat{\gamma}_3$ oraz $\gamma'_H - \hat{\gamma}'_3$. Ostatni, najłatwiejszy krok podobny jest do stosowanego już wcześniej schematu dowodowego.

Ostrość uzyskanych wyników dla estymacji długości krzywej γ oraz *istotność założeń* z Twierdzenia 6.1, zostają zweryfikowane na różnych przykładach (przynajmniej dla przypadku $n = 2,3$). Analogicznie jak poprzednio wybrane eksperymenty ilustrują również przydatność γ_H w estymacji γ oraz $d(\gamma)$ na *zredukowanych danych rzadkich* (por. kryteria z akapitu tego streszczenia dotyczącego funkcji Q).

Przedziałowo-kubiczna gładka interpolacja γ_H dziedziczy wszystkie zalety $\hat{\gamma}_3$, w tym swoją wyższość nad innymi metodami interpolacji nieparametrycznej omówionymi w rozdziałach 2 i 3. Co więcej, daje również pozytywną odpowiedź w istotnej kwestii poruszonej w trzecim akapicie streszczenia, a dotyczącej kompensacji straty informacji zawartej w zredukowanych danych dla interpolacji nieparametrycznej i dowolnej krzywej regularnej w $\mathbb{R}^n$ (odpowiednio gładkiej) próbkowanej w takt ogólnej klasy dopuszczalnych próbkowań. Co ważniejsze, z punktu widzenia zastosowań γ_H może być bardziej przydatna (np. w modelowaniu krzywych) niż $\hat{\gamma}_k$ (dla $k = 2,3$), ponieważ definiuje gładką trajektorię (przynajmniej dla dużych m).

Cała tematyka niniejszego rozdziału stanowi *nowy wkład* w dziedzinę interpolacji zredukowanych danych z użyciem funkcji sklejanych gładkich.

Wyniki przedstawione w rozdziale 6 (włącznie z Twierdzeniem 6.1) zostały wcześniej opublikowane w artykułach [30] i [31].

Rozdział 7 systematyzuje i podsumowuje wyniki uzyskane w niniejszej monografii oraz krótko omawia nowe kierunki badań z zakresu tematyki związanej z przedłożoną rozprawą. Starano się także naświetlić otwarte i nie do końca jeszcze rozwiązane problemy dotyczące dyskutowanego zagadnienia wraz z zarysowaniem możliwych sposobów ich analizy.

Reasumując: analizowane w niniejszej rozprawie podejście do zagadnienia interpolacji nieparametycznej (opartej na *zredukowanych danych* $\mathcal{Q}_m$ bez zadanych węzłów interpolacyjnych $\{t_i\}_{i=0}^m$) wykazuje, iż *interpolacje przedziałowo-kwadratowe i przedziałowo--kubiczne* $\hat{\gamma}_k$ (dla $k = 2,3$) na bazie *skumulowanej parametryzacji długością cięciwy* dają bardzo dobre i szybko zbieżne estymacje trajektorii i długości $d(\gamma)$ krzywej regularnej γ w $\mathbb{R}^n$ (odpowiednio gładkiej) próbkowanej w takt ogólnej klasy dopuszczalnych danych. W szczególności wykazano, iż *kompensacja straty informacji* przy przejściu z danych pełnych do zredukowanych jest możliwa przy zastosowaniu właściwych interpolacji nieparametrycznych (np. $\hat{\gamma}_k$, $k = 2,3$), dla których dowodzi się, iż asymptotyki zbieżności dla oszacowań γ i $d(\gamma)$ są identyczne z tymi, jakie otrzymuje się przy odpowiadającej interpolacji parametrycznej zdefiniowanej jako funkcje sklejane wielomianowe stopnia $2,3$ na bazie znanych węzłów $\{t_i\}_{i=0}^m$. Co ważniejsze, w odróżnieniu od przypadku parametrycznego (gdzie rzędy zbieżności przy estymacji γ i $d(\gamma)$ dla funkcji sklejanych wielomianowych stopnia r wzrastają do $r+1$) skumulowana parametryzacja długością cięciwy w ogólności niekoniecznie daje przyśpieszenie o rząd 1, np. do rzędu 5 dla $\hat{\gamma}_4$. W rozprawie przeanalizowano również gładki odpowiednik γ_H funkcji $\hat{\gamma}_3$ na bazie *skumulowanej parametryzacji długością cięciwy*. Udowodnione dobre własności aproksymacyjne omawianych tu interpolantów, tj. funkcji Q, $\hat{\gamma}_k$ (dla $k = 2,3,4$) oraz γ_H, zachowane są także (jak pokazują eksperymenty) przy przejściu z danych zredukowanych *gęstych* do danych zredukowanych *rzadkich*. W rozprawie wykazano także, iż pominięcie *geometrii rozrzutu zredukowanych danych* $\mathcal{Q}_m$ w procesie estymacji węzłów interpolacyjnych $\{t_i\}_{i=0}^m$ (przybliżanymi węzłami $\{\hat{t}_i\}_{i=0}^m$, np. $\hat{t}_i = i$) może mieć poważne konsekwencje w jakości estymacji γ i $d(\gamma)$ przy zastosowaniu wybranego nieparametrycznego schematu interpolacyjnego. Wyniki z niniejszej monografii (mające charakter bardzo ogólny, *bez silnych ograniczeń na krzywą* γ oraz *na typ i wymiar danych*, tzn. bez ograniczeń na $\mathcal{V}_G^m$ i n) stanowią *nowy wkład w zakresie interpolacji nieparametrycznych opartych na danych zredykowanych*.

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Ryszard Kozera
grudzień 2004

Contents

Spis treści

Chapter 1

Introduction

Abstract

Fitting reduced $\mathcal{Q}_m$ and non-reduced ($\mathcal{Q}_m, \{t_i\}_{i=0}^m$) data via interpolation in $\mathbb{R}^n$ is introduced and the resulting differences are underlined in examples. The problem of estimating the trajectory and length of the unknown curve from reduced data is outlined in the context of existing results and potential applications. Different subfamilies of admissible samplings including uniform, ε-uniform and more-or-less uniform samplings are defined and illustrated on various curves. Orders of convergence for piecewise-r-degree Lagrange interpolation $\tilde{\gamma}_r$ for uniform, ε-uniform and the general class of admissible samplings are established under the temporary assumption of fitting non-reduced data (also in case of uniform samplings for reduced data) - see Theorems 1.1, 1.2 and 1.3. The sharpness of the latter results is confirmed by experiments (at least for $n = 2$ and length estimation). Part of this work is published in [32] and [46].

1.1 Problem formulation and examples

Let $\gamma : [0, T] \to \mathbb{R}^n$ (with $0 < T < \infty$) be *a smooth regular parametric curve*, namely γ is C^k for some $k \geq 1$ and $\dot{\gamma}(t) \neq \vec{0}$ for all $t \in [0, T]$. Our aim is to estimate *via interpolation* the *trajectory* of γ (*i.e.* $\gamma(t)$, where $t \in [0, T]$) and its *length*

$$\boxed{d(\gamma) = \int_0^T \|\dot{\gamma}(t)\| \, dt} \tag{1.1}$$

with some convergence orders from an *ordered* $m + 1$-tuple

$$\boxed{\mathcal{Q}_m = (q_0, q_1, \ldots, q_m)} \tag{1.2}$$

of points in $\mathbb{R}^n$, where $q_i = \gamma(t_i)$, and $0 = t_0 < t_1 < \ldots < t_m = T$, with the corresponding *knot parameters* (called *tabular parameters*) $\{t_i\}_{i=0}^m \in [0, T]^{m+1}$ assumed to be *unknown*. This task is coined as *fitting the reduced data (or sampling)* $\mathcal{Q}_m$ and any interpolation scheme based on such data is called *non-parametric interpolation*. On the other hand, a classical interpolation based on *non-reduced data (or sampling)* $(\mathcal{Q}_m, \{t_i\}_{i=0}^m)$ is termed as *parametric interpolation*. Before defining precisely the notion of *approximation orders* (called also convergence orders; see Definition 1.3) some comments are first made.

In general, depending on what is known about the $\{t_i\}_{i=0}^m$ the problem may be straightforward or unsolvable. For example, if none of the $\{t_i\}_{i=0}^m$ lie in $(0, T/2)$ the task becomes intractable, independently of whether $\{t_i\}_{i=0}^m$ are known or unknown. Of course, if $\mathcal{Q}_m$ are the only data available then γ can at most be approximated up to *reparameterizations*. Recall (see Chapter 1; Proposition $1.1.5$ of [26]), that since γ is *regular* one can assume (at least for the sake of proofs) that γ is parameterized by *arc-length i.e.* $\|\dot{\gamma}\| = 1$. Note that the assumption about a known order of sampling points in $\mathcal{Q}_m$ is also automatically satisfied by *parametric interpolation*, where $\{\hat{t}_i\}_{i=0}^m$, as given, define explicitly an ordered sequence of tabular parameters $t_0 < t_1 < \ldots < t_m$ yielding the respective set of successive sampling points in $\mathcal{Q}_m$. Note also that non-parametric curves (defined implicitly by some algebraic equation(s)) [60] are not considered in this work.

In order to apply any scheme based on non-parametric interpolation, a careful *guess of the distribution* of knots $\{\hat{t}_i\}_{i=0}^m \in [0, \hat{T}]^{m+1}$ needs to be made so that the resulting interpolant $\tilde{\gamma}$ approximates γ and $d(\gamma)$ with some, preferably fast orders of convergence. The potential pitfalls in choosing $\{\hat{t}_i\}_{i=0}^m \approx \{t_i\}_{i=0}^m$ *blindly* (*e.g.* uniformly $\hat{t}_i = i$) are illustrated in Figures 1.1, 1.2, at least for *sporadic data*, *i.e.* when m in $\mathcal{Q}_m$ is small. Not surprisingly, except the uniform case (see Theorem 1.1), the performance of piecewise-quadratic Lagrange interpolation with $\{t_i\}_{i=0}^m$ given outperforms the case with $\{\hat{t}_i\}_{i=0}^m$ guessed as uniform. Of course, since non-reduced data $(\mathcal{Q}_m, \{t_i\}_{i=0}^m)$ comprises more information than their reduced counterpart $\mathcal{Q}_m$, the experiments from Figures 1.1, 1.2 are somehow expected. In fact, as Example 2.1 illustrates, such difference in performance between parametric *versus* non-parametric interpolation extends further to $\mathcal{Q}_m$ with m large (*i.e.* for *data dense*). This ultimately is discussed in Theorems 2.1 and 2.2 for different types of samplings. Before addressing the issue of suitable choice of knots $\{\hat{t}_i\}_{i=0}^m \approx \{t_i\}_{i=0}^m$, we first establish some new convergence results for special families of *admissible samplings* (see Definition 1.2) with tabular parameters assumed to be temporarily known (see Theorem 1.3).

Recall that Runge example (see Chapter 2 of [12]) indicates the limitations of interpolation with polynomial of order $r = m$ in fitting non-reduced data $\mathcal{Q}_m$. More specifically, the interpolation error for non-reduced $(\mathcal{Q}_m, \{t_i\}_{i=0}^m)$ data may increase with m (and thus with $r = m$) getting larger. Of course, the same applies to the case of reduced data $\mathcal{Q}_m$. The remedy is to invoke *a piecewise-r-degree polynomial interpolation* (see [12]) for which r is fixed while m increases.

Definition 1.1. For a given $r+1$-tuple points $\mathcal{Q}_m^{i,r} = (q_i, q_{i+1}, \ldots, q_{i+r})$ one can define r-degree Lagrange polynomial $\tilde{\gamma}_{i,r}$ (see (1.8)) based on either $\{t_i\}_{i=0}^m$ known or $\{\hat{t}_i\}_{i=0}^m$ somehow guessed, accordingly. Then *piecewise-r-degree Lagrange polynomial* $\tilde{\gamma}_r$ interpolating $\mathcal{Q}_m$ is defined as *a track-sum*[1] of $\{\tilde{\gamma}_{ir,r}\}_{i=0}^{\frac{m}{r}-1}$ (see Remark 1.1).

[1] Without loss we can assume that m is divisible by r.

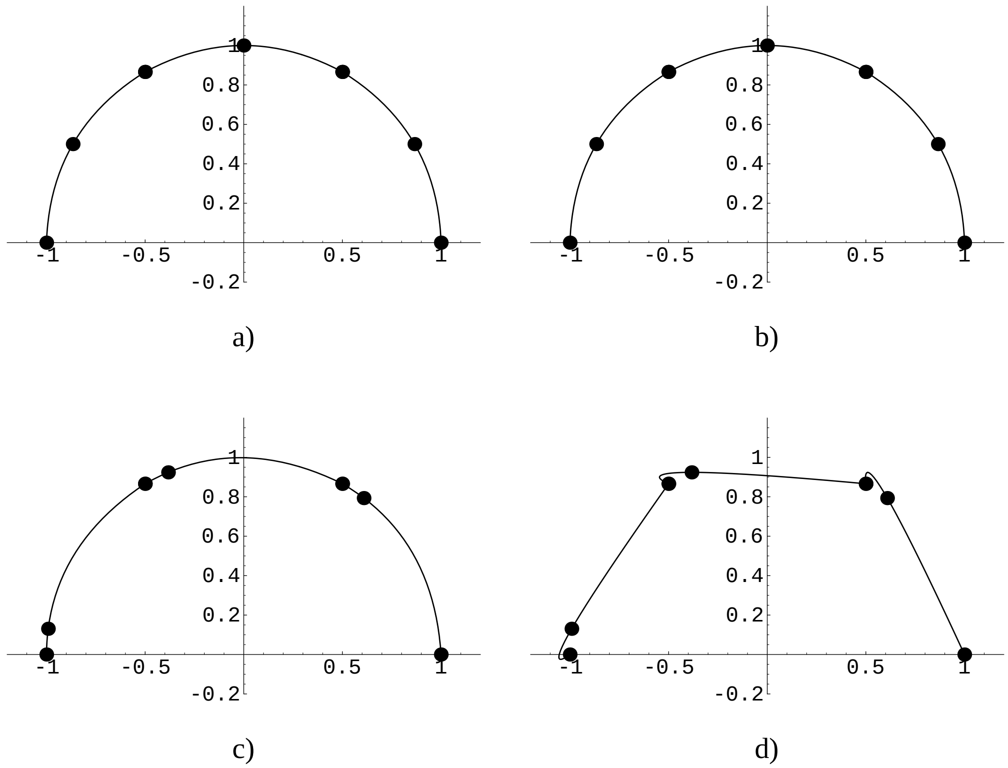


Fig. 1.1. Errors $E = |\pi - d(\tilde{\gamma}_2)|$ for *a semicircle* γ_{sc} (1.9) interpolated by *piecewise-quadratics* $\tilde{\gamma}_2$ (solid) for: a) known $t_i^u = i/m$ and $\mathcal{Q}_6^u$ (dotted): $E = 0.0036$, b) guessed $\hat{t}_i = i$ and $\mathcal{Q}_6^u$: $E = 0.0036$, c) known T_i^{nu}: $t_i^{nu} = \frac{i}{6}$ for i even, $t_i^{nu} = \frac{i}{6} - \frac{3}{24}$ for i odd (for $0 \leq i \leq 6$) and $\mathcal{Q}_6^{nu}$ (dotted): $E = 0.0067$, d) for guessed $\hat{t}_i = i$ and $\mathcal{Q}_6^{nu}$: $E = 0.1987$

Rys. 1.1. Błędy $E = |\pi - d(\tilde{\gamma}_2)|$ dla interpolacji *półokręgu* γ_{sc} (1.9) *przedziałowo-kwadratowymi funkcjami sklejanymi* $\tilde{\gamma}_2$ (linia ciągła) dla: a) zadanych $t_i^u = i/m$ i $\mathcal{Q}_6^u$ (wytłuszczone punkty): $E = 0.0036$, b) użytych węzłów $\hat{t}_i = i$ i $\mathcal{Q}_6^u$: $E = 0.0036$, c) zadanych T_i^{nu}: $t_i^0 = \frac{i}{6}$ dla i parzystych i $t_i^{nu} = \frac{i}{6} - \frac{3}{24}$ dla i nieparzystych (dla $0 \leq i \leq 6$), $\mathcal{Q}_6^{nu}$ (wytłuszczone punkty): $E = 0.0067$, d) użytych węzłów $\hat{t}_i = i$ i $\mathcal{Q}_6^{nu}$: $E = 0.1987$

Note that strictly speaking (1.8) yields a polynomial of at most degree r (for convenience the word “at most” is here omitted). The *track-sum* is understood as follows:

Remark 1.1. Let $f_i : [a_i, b_i] \to \mathbb{R}^n$ with $0 \leq a_i < b_i$, be given for $0 \leq i \leq k_0 - 1$. By the *track-sum* of the family of functions $\{f_i\}_{i=0}^{k_0-1}$ we understand the function $f : [0, \hat{T}] \to \mathbb{R}^n$, where $\hat{T} = \sum_{i=0}^{k_0-1}(b_i - a_i)$ satisfying:

$$\begin{array}{rcl} f(t) & = & f_0(t + a_0)\,, \quad t \in [0, \hat{T}_0]\,, \quad \hat{T}_0 \;=\; b_0 - a_0\;; \\ f(t) & = & f_{k+1}(t + a_{k+1} - T_k)\,, \quad t \in [\hat{T}_k, \hat{T}_{k+1}]\,, \quad \hat{T}_{k+1} \;=\; \hat{T}_k + b_{k+1} - a_{k+1}\;; \end{array}$$

for $0 \leq k \leq k_0 - 2$.

Surprisingly, as discussed throughout this monograph (see Chapter 3 onward), one can *compensate* for such stripped information encoded in reduced data $\mathcal{Q}_m$ by *adjusting choices* of $\{\hat{t}_i\}_{i=0}^m$ to the geometry of $\mathcal{Q}_m$. More specifically, the respective convergence orders established herein, to approximate γ and $d(\gamma)$ from reduced data $\mathcal{Q}_m$, not only outperform the uniform case with $\hat{t}_i = i$, but also match those rates established for corresponding parametric interpolation based on $(\mathcal{Q}_m\{t_i\}_{i=0}^m)$.

We begin now with the following two definitions:

Definition 1.2. The collection of reduced data $\mathcal{Q}_m$ (or non-reduced data $(\mathcal{Q}_m\{t_i\}_{i=0}^m)$) (for $m \geq 2$) is said to form *admissible class of samplings*, if the corresponding tabular parameters $\{t_i\}_{i=0}^m$ satisfy:

$$\boxed{\lim_{m\to\infty}\delta_m \to 0^+ \ , \ \text{ for } \ \delta_m \ = \ \max_{1\leq i\leq m}\{t_i - t_{i-1} : i = 1, 2, \ldots, m\} \ .} \tag{1.3}$$

The class of the corresponding set of knot parameters $\{t_i\}_{i=0}^m$ is denoted herein as $\mathcal{V}_G^m$.

For the sake of convenience, from now on the subscript m appearing in δ_m is suppressed. Furthermore, recall the following:

Definition 1.3. A family $\{f_\delta, \delta > 0\}$ of functions $f_\delta : [0,T] \to \mathbb{R}$ is said to be of *order* $O(\delta^p)$ when there is a constant $K > 0$ such that, for some $\delta_0 > 0$, $|f_\delta(t)| < K\delta^p$ for all $\delta \in (0,\delta_0)$ and all $t \in [0,T]$. In such a case write $f_\delta = O(\delta^p)$. For a family of vector-valued functions $F_\delta : [0,T] \to \mathbb{R}^n$, write $F_\delta = O(\delta^p)$ when $\|F_\delta\| = O(\delta^p)$, where $\|\cdot\|$ denotes the Euclidean norm. An approximation $\tilde{\gamma} : [0,T] \to \mathbb{R}^n$ to γ determined by $\mathcal{Q}_m$ is said to have *order* p when

$$\boxed{\tilde{\gamma} - \gamma \ = \ O(\delta^p) \ .} \tag{1.4}$$

A similar comparison can be made between the lengths of γ and that of $\tilde{\gamma}$.

Note that the formula (1.4) assumes both *domains* of $\tilde{\gamma}$ and γ to coincide. For example, if Lagrange parametric interpolation is invoked [33], this condition is automatically satisfied as both $\gamma, \tilde{\gamma} : [0,T] \to \mathbb{R}^n$. Thus (1.4) can be directly used for proving respective convergence rates in trajectory estimation if $\{t_i\}_{i=0}^m$ are given. On the other hand the non-parametric interpolation yields the interpolant $\tilde{\gamma}$ defined over *different domain i.e.* $\tilde{\gamma} : [0,\hat{T}] \to \mathbb{R}^n$. Therefore, for comparison reasons in order to use (1.4), one needs to reparameterize $\tilde{\gamma}$ to $\tilde{\gamma} \circ \psi$ with some $\psi : [0,T] \to [0,\hat{T}]$. Though $\tilde{\gamma} \circ \psi$ defines a new function, the *trajectories* of $\tilde{\gamma}$ *i.e.* the set of points

$$\mathcal{G}_{\tilde{\gamma}} \ = \ \{p \in \mathbb{R}^n : p = \tilde{\gamma}(s) \ , \quad \text{for } s \in [0,\hat{T}]\} \tag{1.5}$$

and of $\gamma \circ \psi$

$$\mathcal{G}_{\tilde{\gamma}\circ\psi} \ = \ \{p \in \mathbb{R}^n : p = (\tilde{\gamma} \circ \psi)(t) \ , \quad \text{for } t \in [0,T]\}$$

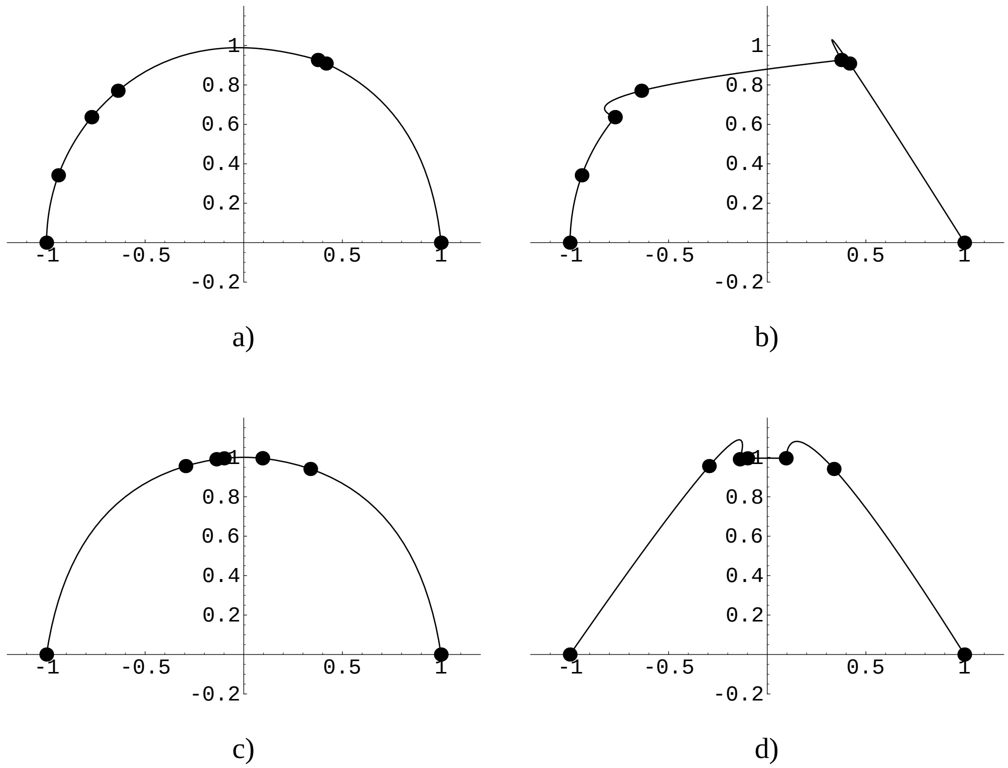


Fig. 1.2. Errors $E = |\pi - d(\tilde{\gamma}_2)|$ for *a semicircle* γ_{sc} (1.9) interpolated by *piecewise-quadratics* $\tilde{\gamma}_2$ (solid) for: a) known randomly ordered $T_1 = \{t_i^{r_1}\}_{i=0}^{6} \in [0,1]^7$ and $\mathcal{Q}_6^{r_1}$ (dotted): $E = 0.0103$, b) guessed $\hat{t}_i = i$ and $\mathcal{Q}_6^{r_1}$: $E = 0.2058$, c) known randomly ordered $T_2 = \{t_i^{r_2}\}_{i=0}^{6} \in [0,1]^7$ and $\mathcal{Q}_6^{r_2}$ (dotted): $E = 0.0202$, d) guessed $\hat{t}_i = i$ and $\mathcal{Q}_6^{r_2}$: $E = 0.1070$

Rys. 1.2. Błędy $E = |\pi - d(\tilde{\gamma}_2)|$ dla interpolacji *półokręgu* γ_{sc} (1.9) *przedziałowo-kwadratowymi funkcjami sklejanymi* $\tilde{\gamma}_2$ (linia ciągła) dla: a) zadanych, losowo uporządkowanych węzłów $T_1 = \{t_i^{r_1}\}_{i=0}^{6} \in [0,1]^7$ i $\mathcal{Q}_6^{r_1}$ (wytłuszczone punkty): $E = 0.2058$, b) użytych węzłów $\hat{t}_i = i$ i $\mathcal{Q}_6^{r_1}$: $E = 0.2058$, c) zadanych, losowo uporządkowanych węzłów $T_2 = \{t_i^{r_2}\}_{i=0}^{6} \in [0,1]^7$ i $\mathcal{Q}_6^{r_2}$ (wytłuszczone punkty): $E = 0.0202$, d) użytych węzłów $\hat{t}_i = i$ i $\mathcal{Q}_6^{r_2}$: $E = 0.1070$

form the same geometrical set in $\mathbb{R}^n$. One can now naturally extend the Definition 1.3 to reduced data $\mathcal{Q}_m$. Namely, we say that an approximation $\tilde{\gamma} : [0, \hat{T}] \to \mathbb{R}^n$ to γ determined by $\mathcal{Q}_m$ and chosen $\psi : [0, T] \to [0, \hat{T}]$, is said to have *order* p when

$$\boxed{\tilde{\gamma} \circ \psi - \gamma \; = \; O(\delta^p) \; .} \tag{1.6}$$

Note that with $\delta = \delta_m$ varying, functions ψ, $\tilde{\gamma}$ and domain's upper bound $\hat{T}$ change accordingly. If in addition the approximation is to be of *class* C^r, then one also requires $\tilde{\gamma}, \psi \in C^r$. Recall also that $d(\tilde{\gamma} \circ \psi) = d(\tilde{\gamma})$ if $\psi \in C^1$ is a reparameterization (*i.e.* either $\dot{\psi} > 0$ or $\dot{\psi} < 0$). The second case $\dot{\psi} < 0$ is however excluded, since $\hat{t}_i < \hat{t}_{i+1}$ must hold. Thus for length estimation to be invariant with respect to the choice of ψ one assumes that $\psi \in C^k$ (where $k \geq 1$) defines an *order-preserving reparameterization* ($\dot{\psi} > 0$) of $\tilde{\gamma}$. This class can be slightly relaxed to the *piecewise-C^1 reparameterizations* since

$$d(\tilde{\gamma}) - d(\gamma) \;=\; \sum_{i=0}^{m-1} (d(\tilde{\gamma}_i) - d(\gamma_i)) \;=\; \sum_{i=0}^{m-1} (d(\tilde{\gamma}_i \circ \psi_i) - d(\tilde{\gamma}_i)) \;, \tag{1.7}$$

where for each $0 \leq i \leq m-1$, $\gamma_i \equiv \gamma|_{[t_i, t_{i+1}]}$, $\tilde{\gamma}_i \equiv \tilde{\gamma}|_{[\hat{t}_i, \hat{t}_{i+1}]}$ and $\psi_i [t_i, t_{i+1}] \to [\hat{t}_i, \hat{t}_{i+1}]$ is C^1 with $\dot{\psi}_i > 0$.

Remark 1.2. Note that for *non-parametric interpolation*, the task of finding parameterization has the following double meaning. In the first step, *a discrete set of parameters* $\{\hat{t}_i\}_{i=0}^m$ is to be found which is vital for the derivation of an explicit interpolant $\tilde{\gamma} : [0, \hat{T}] \to \mathbb{R}^n$. The latter should be done for m varying, if additionally the asymptotic analysis is also in question. In the second step, for comparison with γ (see (1.6)), the quality of the selected $\{\hat{t}_i\}_{i=0}^m$ is verified by finding (for all m) a suitable family of domains $[0, \hat{T}_m]$ of $\tilde{\gamma}_m$ and a proper family of C^∞ reparameterization[2] $\psi_m : [0, T] \to [0, \hat{T}_m]$. For convenience we omit often m in the subscripts of ψ_m, $\tilde{\gamma}_m$ and $\hat{T}_m$.

This subsection closes with an example which not only again underlines the *importance of appropriate choices* of $\{\hat{t}_i\}_{i=m}^m$ but also provides some *geometrical clues* guiding suitable choices of $\{\hat{t}_i\}_{i=0}^m$. Although these two issues are illustrated in Example 1.1 only for sporadic data (*i.e.* with m small), they also recur for samplings $\mathcal{Q}_m$ with m large (see Example 2.1). Before passing to the example recall that for sampling points $(\gamma(t_i), \gamma(t_{i+1}), \ldots, \gamma(t_{i+r}))$ Lagrange r-degree polynomial interpolation formula reads (see e.g. [33]):

$$\tilde{\gamma}_{i,r}(s) = \sum_{j=0}^{r} \tilde{\gamma}(t_{i+j}) l_j(s) \;, \quad \text{for } \; s \in [t_i, t_{i+r}] \;, \tag{1.8}$$

where for $j = 0, \ldots, r$

$$l_j(s) = \frac{(s - t_i) \ldots (s - t_{i+j-1})(s - t_{i+j+1}) \ldots (s - t_{i+r})}{(t_{i+j} - t_i) \ldots (t_{i+j} - t_{i+j-1})(t_{i+j} - t_{i+j+1}) \ldots (t_{i+j} - t_{i+r})} \;.$$

Example 1.1. Consider *a semicircle* $\gamma_{sc} : [0, 1] \to \mathbb{R}^2$ defined as

$$\gamma_{sc}(t) \;=\; (\cos(\pi(1-t)), \sin(\pi(1-t))) \tag{1.9}$$

with two triples of sampling points

$$\mathcal{Q}_3^u \;=\; (\gamma_{sc}(0), \gamma_{sc}(0.5), \gamma_{sc}(1)) \quad \text{and} \quad \mathcal{Q}_3^{nu} \;=\; (\gamma_{sc}(0), \gamma_{sc}(0.68), \gamma_{sc}(1)) \;,$$

see dashed curve γ_{sc} and dotted points in Figure 1.3. Let $\tilde{\gamma}_{0,2}^1, \tilde{\gamma}_{0,2}^2, \tilde{\gamma}_{0,2}^3, \tilde{\gamma}_{0,2}^4 : [0, 2] \to \mathbb{R}^2$ define the Lagrange quadratics (see (1.8)) interpolating $\mathcal{Q}_3^u$ and $\mathcal{Q}_3^{nu}$ with $\{\hat{t}_i\}_{i=0}^2$ first

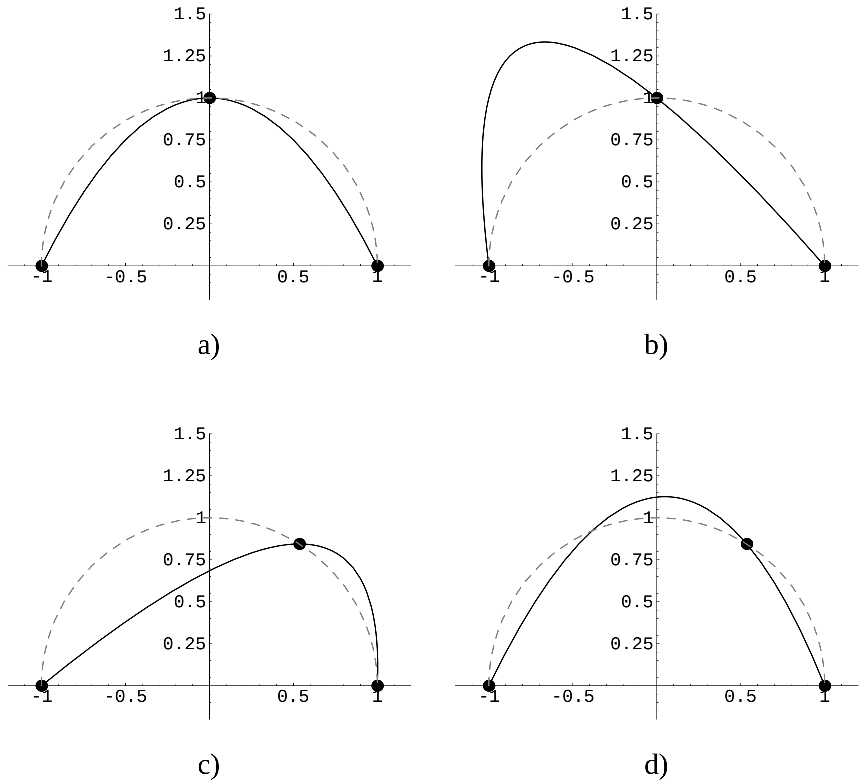


Fig. 1.3. Errors $E = |\pi - d(\tilde{\gamma}^i_{0,2})|$ for *a semicircle* γ_{sc} (1.9) (dashed) interpolated (see Example 1.1) by *quadratic curves* $\tilde{\gamma}^i_{0,2}$ (solid): a) $\tilde{\gamma}^1_{0,2}$ for $\mathcal{Q}^u_3$ (dotted) with guessed $\hat{t}_i \in \mathcal{T}_1$: $E = 0.1837$, b) $\tilde{\gamma}^2_{0,2}$ for $\mathcal{Q}^u_3$ with guessed $\hat{t}_i \in \mathcal{T}_2$: $E = 0.5502$, c) $\tilde{\gamma}^3_{0,2}$ for $\mathcal{Q}^{nu}_3$ (dotted) with guessed $\hat{t}_i \in \mathcal{T}_1$: $E = 0.2694$, d) $\tilde{\gamma}^4_{0,2}$ for $\mathcal{Q}^{nu}_3$ with guessed $\hat{t}_i \in \mathcal{T}_2$: $E = 0.0119$

Rys. 1.3. Błędy $E = |\pi - d(\tilde{\gamma}^i_{0,2})|$ interpolacji *półokręgu* γ_{sc} (1.9) (linia przerywana) *funkcjami kwadratowymi* $\tilde{\gamma}^i_{0,2}$ (linia ciągła; patrz Przykład 1.1): a) $\tilde{\gamma}^1_{0,2}$ z $\mathcal{Q}^u_3$ (wytłuszczone punkty) oraz $\hat{t}_i \in \mathcal{T}_1$: $E = 0.1837$, b) $\tilde{\gamma}^2_{0,2}$ z $\mathcal{Q}^u_3$ oraz $\hat{t}_i \in \mathcal{T}_2$: $E = 0.5502$, c) $\tilde{\gamma}^3_{0,2}$ z $\mathcal{Q}^{nu}_3$ (wytłuszczone punkty) oraz $\hat{t}_i \in \mathcal{T}_1$: $E = 0.2694$, d) $\tilde{\gamma}^4_{0,2}$ z $\mathcal{Q}^{nu}_3$ oraz $\hat{t}_i \in \mathcal{T}_2$: $E = 0.0119$

guessed as $\mathcal{T}_1 = (\hat{t}_{10}, \hat{t}_{11}, \hat{t}_{12}) = (0, 1, 2)$ and then as $\mathcal{T}_2 = (\hat{t}_{20}, \hat{t}_{21}, \hat{t}_{22}) = (0, 1.5, 2)$, respectively, *i.e.*:

[2] In fact the differentiability orders can be lower depending on the particular case.

$$\begin{aligned}
\tilde{\gamma}^1_{0,2}(\hat{t}) &= \gamma_{sc}(0)\frac{(\hat{t}-1)(\hat{t}-2)}{2} - \gamma_{sc}(0.5)\hat{t}(\hat{t}-2) + \gamma_{sc}(1)\frac{\hat{t}(\hat{t}-1)}{2} , \\
\tilde{\gamma}^2_{0,2}(\hat{t}) &= \gamma_{sc}(0)\frac{(\hat{t}-1.5)(\hat{t}-2)}{3} - 4\gamma_{sc}(0.5)\hat{t}\frac{(\hat{t}-2)}{3} + \gamma_{sc}(1)\hat{t}(\hat{t}-1.5) , \\
\tilde{\gamma}^3_{0,2}(\hat{t}) &= \gamma_{sc}(0)\frac{(\hat{t}-1)(\hat{t}-2)}{2} - \gamma_{sc}(0.68)\hat{t}(\hat{t}-2) + \gamma_{sc}(1)\frac{\hat{t}(\hat{t}-1)}{2} , \\
\tilde{\gamma}^4_{0,2}(\hat{t}) &= \gamma_{sc}(0)\frac{(\hat{t}-1.5)(\hat{t}-2)}{3} - 4\gamma_{sc}(0.68)\hat{t}\frac{(\hat{t}-2)}{3} + \gamma_{sc}(1)\hat{t}(\hat{t}-1.5) .
\end{aligned}$$

Note that $\mathcal{T}_1$ and $\mathcal{T}_2$ differ only in $\hat{t}_{11}$ and $\hat{t}_{21}$. Figure 1.3a,b shows that for $\mathcal{Q}^u_3$, the performance of $\tilde{\gamma}^1_{0,2}$ (derived with uniform choice of $\hat{t}_i \in \mathcal{T}_1$) in estimating γ_{sc} and $d(\gamma_{sc})$ is better than of $\tilde{\gamma}^2_{0,2}$ (derived for $\hat{t}_i \in \mathcal{T}_2$). On the other hand, for $\mathcal{Q}^{nu}_3$, as illustrated in Figure 1.3c,d the opposite holds, *i.e.* the performance of $\tilde{\gamma}^3_{0,2}$ (with guessed $\hat{t}_i \in \mathcal{T}_1$) is outperformed by $\tilde{\gamma}^4_{0,2}$ (with guessed $\hat{t}_i \in \mathcal{T}_2$). $\square$

1.2 Literature and some applications

As illustrated in Example 1.1, contrary to the case of non-reduced data $(\mathcal{Q}_m, \{t_i\}_{i=0}^m)$, a particular $r+1$-tuple of reduced data points $\mathcal{Q}^{i,r}_m = (q_i, q_{i+1}, \ldots, q_{i+r})$ with different choices of knot parameters $\{\hat{t}_{i+r}\}_{i=0}^r$ yields different r-degree Lagrange polynomials. Evidently, this in turn, impacts on approximation performance of non-parametric piecewise--r-degree Lagrange interpolation. As shown later, the only *exception* to above *ambiguity* results for either $r=1$ (*i.e.* for *a piecewise-linear interpolation* - Remark 5.2) or when sampling $\{t_i\}_{i=0}^m$ is uniform (see Remark 1.6). In both cases the trajectory of the resulting r-degree Lagrange interpolant is uniquely determined by sampling points $\mathcal{Q}^{i,r}_m$ and fixed r. Another transparent hint stemming from Example 1.1 indicates that the choice of knot parameters $\{\hat{t}_i\}_{i=0}^m$ (in the absence of the real $\{t_i\}_{i=0}^m$) should somehow compensate for *the geometry of distribution of sampling points* $\mathcal{Q}_m$. In particular, as Example 1.1 shows, if such distribution is *regular* then setting $\{\hat{t}_i\}_{i=0}^m$ as uniform or almost equidistant might be appropriate. Otherwise, one should resort to more subtle choices of $\{\hat{t}_i\}_{i=0}^m$. In geometrical modeling design, commonly used parameterizations are: *a cumulative chord parameterization* Chapter 14 of [12], [16], Chapter 9 of [17], Chapter 11 of [33] and [36]

$$\boxed{\boxed{\hat{t}_0 = 0 , \quad \hat{t}_{i+1} = \hat{t}_i + \|q_{i+1} - q_i\| ,}} \qquad 1 \le i \le m-1 \tag{1.10}$$

a centripetal parameterization [35]

$$\boxed{\hat{t}_0 = 0 , \quad \hat{t}_{i+1} = \hat{t}_i + \|q_{i+1} - q_i\|^{\frac{1}{2}} ,} \qquad 1 \le i \le m-1$$

or more generally *an exponential parameterization* (see Chapter 11 of [33])

$$\boxed{\hat{t}_0 \;=\; 0\,, \quad \hat{t}_{i+1} \;=\; \hat{t}_i + \|q_{i+1} - q_i\|^e\,,} \qquad 0 \leq e \leq 1\,, \qquad 1 \leq i \leq m-1\,,$$

where for $e = 0, 0.5$ and 1 the latter renders uniform, centripetal and cumulative chord parameterization, respectively. Yet other parameterizations like *monotonicity preserving parameterization* (see Chapter 11 of [33]) designed to preserve shape for curves in $\mathbb{R}^2$ or *affine invariant parameterization* [18] and [41] designed to cope with wild set of data $\mathcal{Q}_m$, can be invoked. The interpolating nodes can also be derived through *non-linear optimization techniques* (see *e.g.* [20] or [37]) but those techniques are expensive and it is not clear what objective function should be used. Most of the above methods serve different tasks in modeling the incomplete data and often assume special cases like: $n = 2, 3$, sporadic data, special samplings or require to solve a non-linear system of equations to compute $\{\hat{t}_i\}_{i=0}^m$ (see also [13], [34], [40], [50], [54] and [55]).

The *cumulative chord parameterization* (1.10) offers *a fast and explicit method* of computing the knot parameters $\{\hat{t}_i\}_{i=0}^m$ designed to approximate the unknown $\{t_i\}_{i=0}^m$. There is so-far an incomplete analysis (see *e.g.* [16], [35] or [36]) on cumulative chord parameterization combined with various interpolation schemes. This *monograph extends the existing results* to arbitrary regular curves in $\mathbb{R}^n$ and interpolated by piecewise Lagrange and some non-Lagrange based schemes. In particular we analyze the asymptotics of the invoked interpolants in the context of *trajectory and length estimation*.

This work is specifically applicable in *range* and *image segmentation and classification* (*e.g.* in *medical image processing*), *tracking rigid body, computer aided geometrical design* and all problems, where fitting incomplete reduced data $\mathcal{Q}_m$ arises as an issue. *E.g.* research has shown (see [9], [51] and [53]) that accurately segmenting and subsequently estimating volume of the hippocampus plays an important role in *medical diagnosis* of Alzheimer, schizophrenia, and epilepsy. Upon finding the boundary of the hippocampus with non-parametric interpolation (sparse interpolation points marked by clinician) in each image-section (obtained from Magnetic Resonance Imaging) the area of each cross-section and the volume of the hippocampus can be computed. The quality of boundary estimation is vital for accurate rendering of the latter. Another possible application provide *radiotherapy treatment* (see [63]). Computer Tomography (or Magnetic Resonance) cross-section images are to be processed to build the 3D model (see [6]) and subsequently allow clinician to view the tumor from arbitrary directions in a form of parallel cross-sections (orthogonal to a chosen viewing axis). In both input cross-sections, where clinician manually marks some boundary points for each image (displayed in directions: axial - parallel to $XY-$plane, coronal - parallel to $XZ-$plane and saggital - $YZ-$plane), and in output cross-sections (along arbitrary axis) the non-parametric interpolation is needed to correctly segment each particular section of the tumor. For each output section (upon 3D model intersects with the particular viewing cross-plane) the tumor cross-section area and length of tumor's cross-section boundary (indicating the local dynamics of the disease) both require accurate boundary interpolation. Next possible application is the *detection of the progress of glaucoma*. By measuring the perimeter (over the time) of the destroyed part of the eye nerve (see [19]) the clinician can assess the current status and progress and of this eye disease (non-parametric interpolants together with their lengths can be used here as an option).

There are many other possible applications in engineering (*e.g.* robotics or data visualization) or physics (*e.g.* high-energy particle path reconstruction) where interpolation based on reduced data is vital.

More generally, the results presented in this monograph (*published* in [27], [28], [29], [30], [31], [32], [42], [43], [44], [45], [46] and [47]) maybe also be of interest in *computer vision* and *digital image processing e.g.* [3], [7], [15], [22], [23] or [25], *computer graphics* and *geometric modeling e.g.* [2], [5], [12], [17], [20], [21], [33], [48] or [56], *approximation* and *complexity theory e.g.* [4], [11], [10], [33], [39], [49], [59], [61] or [62] and *digital* and *computational geometry e.g.* [3], [24], [57] or [58].

1.3 Different samplings and basic results

We begin this section with the analysis of the simplest case *i.e. a uniform one*. In the next step, different *non-equidistant families of samplings* are introduced and the corresponding convergence orders for approximation of γ and $d(\gamma)$ are discussed under temporary assumption of accessing the *non-reduced data* $(\mathcal{Q}_m, \{t_i\}_{i=0}^m)$ (see Theorems 1.2 and 1.3). Ultimately, these results are compared (see Chapter 2 onward) with the corresponding ones based on *reduced data* and *cumulative chord parameterization*. Evidently, if sampling $\{t_i\}_{i=0}^m \notin \mathcal{V}_G^m$ *e.g.*

$$t_0 = 0\,, \quad t_1 = 0.5 \quad \text{and} \quad t_i = \frac{i-1}{2(m-1)} + 0.5\,, \quad \text{for} \quad i = 2, \ldots, m \tag{1.11}$$

then, in general one cannot expect good results for approximation of γ and $d(\gamma)$. In particular, by previous examples we know that if $\{t_i\}_{i=0}^m$ are not given then with $\hat{t}_i = i$ we may face difficulties. Indeed, Figure 1.4 shows again that for two ordered *random samplings* of $\{t_i\}_{i=0}^m \in [0,T]^{m+1}$ or for the so-called 0-uniform sampling (see Definition 1.4) the piecewise-quadratic Lagrange interpolation with $\hat{t}_i = i$ in most cases gives poor estimates for $d(\gamma)$ (and indeed for γ). In Figure 1.4 only the uniform data yield reasonable approximations for γ_{sc} defined as in (1.9). As before, the latter approximation properties illustrated for sporadic data extend also to m large (see Example 2.1).

1.3.1 Uniform samplings

We start our analysis for the special subfamily of $\mathcal{V}_G^m$. Namely, the problem is easiest when the unknown $\{t_i\}_{i=0}^m$ are sampled in *a perfectly uniform* manner, namely $t_i = \frac{iT}{m}$ (denoted $\{t_i\}_{i=0}^m \in \mathcal{V}_U^m \subset \mathcal{V}_G^m$) (*e.g.* see also [39] or [59]). Assume however that $\{t_i\}_{i=0}^m$ is not given. In such a case it seems natural to approximate γ (and thus $d(\gamma)$) by another curve $\tilde{\gamma}$ that is piecewise-polynomial of degree $r \geq 1$ with $\hat{t}_i = i \approx t_i$. The following holds (for proof see Section 1.4 or [46]):

Theorem 1.1. *Let γ be C^{r+2} and let $\{t_i\}_{i=0}^m$ be sampled perfectly uniformly. Then there exists piecewise-r-degree polynomial $\tilde{\gamma}_r$[3] $: [0,\hat{T}] \to \mathbb{R}^n$, determined by $\mathcal{Q}_m$ and $\hat{t}_i = i$*

[3] See Section 1.4 for details or Definition 1.1.

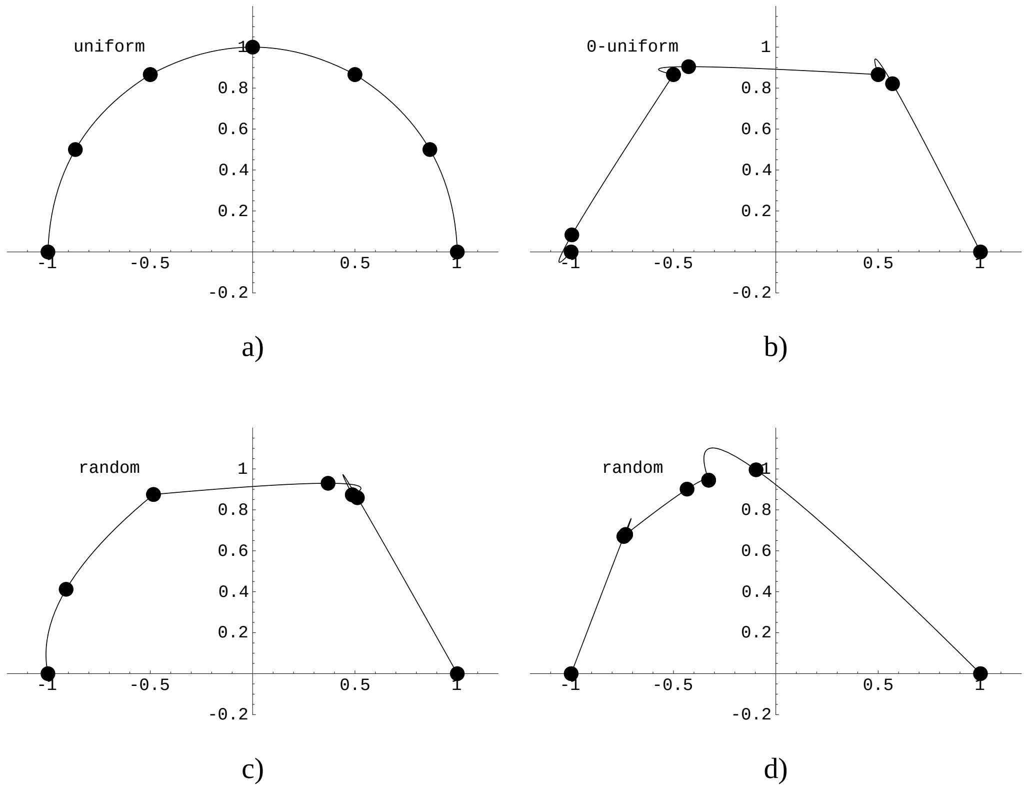


Fig. 1.4. Errors $E = |\pi - d(\tilde{\gamma}_2)|$ for *a semicircle* γ_{sc} (1.9) interpolated by *uniform piecewise--quadratics* $\tilde{\gamma}_2$ (solid) with $\hat{t}_i = i$, for samplings $\mathcal{Q}_6$ (dotted): a) uniform: $E = 0.0036$, b) 0-uniform (where $t_i = \frac{i}{6}$ for i even and $t_i = \frac{i}{6} - \frac{1}{12}$ for i odd; $0 \leq i \leq 6$): $E = 0.3232$, c) random: $E = 0.2299$, d) another random: $E = 0.1539$

Rys. 1.4. Błędy $E = |\pi - d(\tilde{\gamma}_2)|$ dla interpolacji *półokręgu* γ_{sc} (1.9) *przedziałowo-kwadratowymi funkcjami sklejanymi* $\tilde{\gamma}_2$ (linia ciągła) z $\hat{t}_i = i$, dla próbkowań $\mathcal{Q}_6$ (wytłuszczone punkty): a) równomiernego: $E = 0.0036$, b) ε-równomiernego (gdzie $t_i = \frac{i}{6}$ dla i parzystych i $t_i = \frac{i}{6} - \frac{1}{12}$ dla i nieparzystych; $0 \leq i \leq 6$): $E = 0.3232$, c) losowego: $E = 0.2299$, d) innego losowego: $E = 0.1539$

and piecewise-C^∞ *reparameterization* $\psi^4 : [0,T] \to [0,\hat{T}]$ *such that*

$$\boxed{\tilde{\gamma}_r \circ \psi \;=\; \gamma \;+\; O(\delta^{r+1}) \quad and \quad d(\tilde{\gamma}_r) \;=\; d(\gamma) \;+\; O(\delta^{r+p})\,,} \tag{1.12}$$

where p *is* 1 *or* 2 *according as* r *is odd or even.*

The orders of convergence in (1.12) are *sharp*. By *sharpness of convergence order*, we understand the existence of at least one curve $\gamma \in C^{r+2}$ which when sampled uniformly and interpolated by piecewise-r-degree Lagrange polynomial with time guessed

[4]See Section 1.4 for details.

$\hat{t}_i = i$, yields the convergence orders not faster that $O(\delta^{r+1})$. This notion of sharpness extends naturally to other choices of $\{\hat{t}_i\}_{i=0}^m$ (or $\{t_i\}_{i=0}^m$ if known) and arbitrary interpolation schemes.

The sharpness of (1.12), at least for length estimation and $r = 1, 2, 3, 4$, is experimentally confirmed for planar curves in Subsection 1.6.2 (see Tables 1.1 and 1.2). Testing sharpness of trajectory estimation is here abandoned as for each $min_r \leq m \leq max_r$ (see (1.65)) a non-linear global optimization problem (1.6) should be solved (*e.g.* with computationally expensive simulated annealing algorithm or as the problem is one dimensional gradient ascent-descent algorithm). Thus the collection of $max_r - min_r$ non-linear optimization tasks brings even heavier computational burden.

Note that Theorem 1.1 holds also for non-reduced data $\mathcal{Q}_m$ *i.e.* when $\tilde{\gamma}$ is defined by $\mathcal{Q}_m$ and tabular points $t_i = \frac{iT}{m}$ assumed here to be given (see Remark 1.6).

1.3.2 General class of admissible samplings

Of course, we are principally concerned with *non-uniform samplings*. In particular the principal question arises for any interpolation scheme based on *reduced data* $\mathcal{Q}_m$ forming admissible samplings:

Question I: Can the $\{\hat{t}_i\}_{i=0}^n$, estimating $\{t_i\}_{i=0}^n \in \mathcal{V}_G^m$, be chosen so that the corresponding convergence orders obtained for non-parametric interpolation match those established for piecewise-r-degree Lagrange polynomial interpolation used with reduced data?

A positive answer to this question would extend the case of uniform samplings. The next chapters discuss different approaches to Question I involving both Lagrange (see Remark 4.3) and non-Lagrange (see Remark 3.5) interpolation schemes.

We briefly discuss now some results established for piecewise-r-degree Lagrange polynomial interpolation based on non-reduced samplings $(\mathcal{Q}_m, \{t_i\}_{i=0}^m)$ with $\{t_i\}_{i=0}^m \in \mathcal{V}_G^m$. We begin with the result holding for $\mathcal{V}_G^m$ (for proof see *e.g.* Remark 1.7 or [32]):

Theorem 1.2. *Let $\gamma \in C^{r+1}$ be a regular curve $\gamma : [0, T] \to \mathbb{R}^n$. Assume that the knot parameters $\{t_i\}_{i=0}^m \in \mathcal{V}_G^m$ are known. Then a piecewise-r-degree Lagrange polynomial interpolation $\tilde{\gamma}_r$[5] used with $\{t_i\}_{i=0}^m$ known, yields sharps estimates*

$$\boxed{\tilde{\gamma}_r = \gamma + O(\delta^{r+1}) \quad and \quad d(\tilde{\gamma}_r) = d(\gamma) + O(\delta^{r+1}).} \tag{1.13}$$

The second formula requires an additional assumption, namely $m\delta = O(1)$.

The estimates for length estimation in (1.13) are here experimentally confirmed to be sharp at least for $n = 2$ and $r = 1, 2, 3, 4$ (see Table 1.1 and 1.2 in Subsection 1.6.2 and Table 1.3 in Subsection 1.6.3).

The analysis for convergence orders for interpolation with reduced data $\mathcal{Q}_m$ formed by general admissible samplings $\{t_i\}_{i=0}^m \in \mathcal{V}_G^m$ is performed in Chapters 4 and 6.

[5] See Remark 1.7 for details or Definition 1.1.

1.3.3 $\mathcal{E}$-uniform samplings

The claim of Theorem 1.2 can be strengthen for some subfamilies $\mathcal{V}^m \subset \mathcal{V}_G^m$. For example, this clearly occurs for uniform samplings $\mathcal{V}_U^m \subset \mathcal{V}_G^m$ - see Theorem 1.1 and Remark 1.6. In a search for other subfamilies of $\mathcal{V}^m \subset \mathcal{V}_G^m$ for which formulas (1.13) are sharper we discuss now various ways of forming ordered knot parameters

$$0 = t_0 < t_1 < \ldots < t_m = T \tag{1.14}$$

of variable size $m+1$ from the interval $[0,T]$ (where $T < \infty$). As mentioned before the simplest procedure is to consider *uniform sampling*, where $t_i = \frac{iT}{m}$ (with $0 \le i \le m$); of course $\mathcal{V}_U^m \subset \mathcal{V}_G^m$. Uniform sampling is not invariant with respect to *reparameterizations*, namely order-preserving C^∞ diffeomorphisms

$$\phi : [0,T] \to [0,T]\,, \quad \text{where} \quad k \ge 1\,.$$

A small perturbation of uniform sampling is no longer uniform, but may approach uniformity in some asymptotic sense, at least after some suitable reparameterization. The possible example of such perturbation is the following family of ε*-uniform samplings* $\mathcal{V}_\varepsilon^m \subset \mathcal{V}_G^m$:

Definition 1.4. For[6] $\varepsilon > 0$, the knot parameters $\{t_i\}_{i=0}^m$ are said to be ε*-uniformly sampled* when there is an order-preserving C^∞ reparameterization $\phi : [0,T] \to [0,T]$,*i.e.* $\dot\phi > 0$, such that

$$\boxed{t_i = \phi(\tfrac{iT}{m}) + O(\tfrac{1}{m^{1+\varepsilon}})\,.} \tag{1.15}$$

As usual, $O(g(m))$ means a quantity whose absolute value is bounded by some multiple of $g(m)$ as $m \to \infty$. Before passing to some examples some comments should be made first:

- Note that any ε*-uniform sampling* arises from *two types of perturbations of uniform sampling*: first *via* a diffeomorphic distortion $\phi : [0,T] \to [0,T]$ combined subsequently with added extra distortion term $O(1/m^{1+\varepsilon})$. The perturbation *via* ϕ has no effect on both $d(\gamma)$ and geometrical representation of γ. The only potential nuisance stems from the second perturbation term $O(1/m^{1+\varepsilon})$.

- Up to a reparameterization, the bigger ε, the more $\{t_i\}_{i=0}^m$ represent the uniform sampling. In fact as shown later in Theorems 1.3, 2.1, 4.1 and 5.1 the corresponding convergence rates for estimation of both and γ and $d(\gamma)$ with $\{t_i\}_{i=0}^m \in \mathcal{V}_{\varepsilon\ge1}^m$, coincide with those obtained for uniform case. Note that, in the case of reduced data one obtains the latter in conjunction with cumulative chord reparameterization (see (1.10)). On the other hand, the smaller ε is the bigger distortion of uniform case eventuates (see Figures 1.5, 1.6, 1.9).

[6] $\varepsilon = 0$ involves extra condition - see explanation in itemized comments on Definition 1.4.

- $\varepsilon > 0$ and $\dot\phi > 0$ combined with Taylor's Theorem yield

$$t_{i+1} - t_i \;=\; \dot\phi(\frac{iT}{m})\frac{T}{m} \;+\; O(\frac{1}{m^2}) \;+\; O(\frac{1}{m^{1+\varepsilon}}) \;>\; 0\,, \tag{1.16}$$

  holding for m large as $\ddot\phi$ is bounded over a compact set $[0,T]$ independently from m. Hence (1.14) is satisfied asymptotically.

- An extension of Definition 1.4 to 0-*uniform samplings* imposes an extra condition on $\{t_i\}_{i=0}^m \in \mathcal{V}_0^m \subset \mathcal{V}_G^m$, namely (1.14) should also be satisfied (at least asymptotically).

- By (1.16) we have that (1.3) holds and thus $\mathcal{V}_\varepsilon^m \subset \mathcal{V}_G^m$ (for $\varepsilon \geq 0$).

- Note that ϕ and asymptotic constants in (1.15) are chosen independently of $m \geq 1$ and that ε-uniform implies δ-uniform for $0 \leq \delta < \varepsilon$.

- Of course, a uniform sampling with $\phi = id$ and $O(1/m^{1+\varepsilon})$ term vanishing is also a ε-uniform sampling, for and arbitrary $\varepsilon \geq 0$. Thus $\mathcal{V}_U^m \subset \mathcal{V}_\varepsilon^m$.

- Note, that any family $\{t_i\}_{i=0}^m \in \mathcal{V}_\varepsilon^m$ is invariant with respect to any order-preserving reparameterization $\psi : [0,T] \to [0,T]$. The latter means $\{\psi(t_i)\}_{i=0}^m \in \mathcal{V}_\varepsilon^m$ and $\psi(t_i) < \psi(t_{i+1})$ for $i = 0,1,\ldots,m-1$. Indeed, Taylor's Theorem combined with (1.16) yields

$$\psi(t_i) \;=\; \psi(\phi(\frac{iT}{m})) \;+\; \psi'(\xi)O(\frac{1}{m^{1+\varepsilon}}) \;=\; \chi(\frac{iT}{m}) \;+\; O(\frac{1}{m^{1+\varepsilon}})\,,$$

  where $\chi = \psi \circ \phi$. Also ψ is strictly increasing as $\dot\psi > 0$.

Before giving some examples of ε-uniform samplings we show the existence of the knots $\{t_i\}_{i=0}^m \in \mathcal{V}_G^m$ which are *not* ε*-uniform*. The latter proves the strong inclusion $\mathcal{V}_\varepsilon^m \subset \mathcal{V}_G^m$.

Example 1.2. Consider the following sampling $\{t_i\}_{i=0}^m \in [0,1]^{m+1}$ (here $T = 1$)

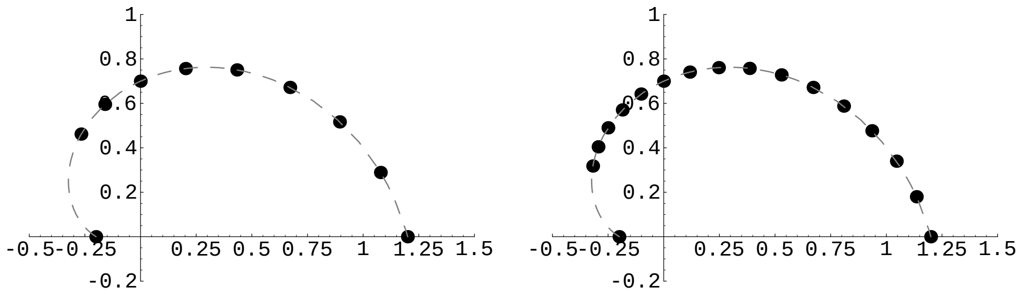
$$t_0 \;=\; 0\,, \quad \text{and} \quad t_i \;=\; \frac{1}{\sqrt{m}} \;+\; \frac{(i-1)(\sqrt{m}-1)}{(m-1)\sqrt{m}}\,, \quad 1 \leq i \leq m\,. \tag{1.17}$$

Of course, by (1.3) and (1.17) we have $\delta = t_1 - t_0 = 1/\sqrt{m}$. Thus since $\delta \to 0$ with $m \to \infty$ we have $\{t_i\}_{i=0}^m \in \mathcal{V}_G^m$. Assume now that (1.17) is ε-uniform (here $\varepsilon \geq 0$). Then, for some $\phi \in C^\infty$ satisfying $\phi : [0,1] \to [0,1]$ and $\varepsilon \geq 0$, Taylor's Theorem combined with (1.15) and $\dot\phi = O(1)$ render

$$t_1 - t_0 \;=\; \frac{\dot\phi(\xi)}{m} \;+\; O(\frac{1}{m^{1+\varepsilon}}) \;=\; O(\frac{1}{m})\,.$$

By (1.17), the latter stands in a clear contradiction with $t_1 - t_0 = 1/\sqrt{m}$. For *a spiral curve* $\gamma_{sp} : [0,1] \to \mathbb{R}^2$ defined as

$$\gamma_{sp}(t) \;=\; ((t+0.2)\cos(\pi(1-t)), (t+0.2)\sin(\pi(1-t))) \tag{1.18}$$



a) b)

Fig. 1.5. *A spiral* γ_{sp} (1.18) (dashed) with non-ε-uniform sampling (1.17) yielding interpolation points (dotted): a) $\mathcal{Q}_9$ with $t_1 - t_0 = 1/3$ and $t_{i+1} - t_i = 1/12$ for $1 \leq i \leq 8$, b) $\mathcal{Q}_{16}$ with $t_1 - t_0 = 1/4$ and $t_{i+1} - t_i = 1/15$ for $1 \leq i \leq 15$

Rys. 1.5. *Spirala* γ_{sp} (1.18) (linia przerywana) próbkowana nie ε-równomiernie (1.17) z punktami pomiarowymi (wytłuszczone): a) $\mathcal{Q}_9$, gdzie $t_1 - t_0 = 1/3$ i $t_{i+1} - t_i = 1/12$ dla $1 \leq i \leq 8$, b) $\mathcal{Q}_{16}$, gdzie $t_1 - t_0 = 1/4$ i $t_{i+1} - t_i = 1/15$ dla $1 \leq i \leq 15$

the distribution of sampling points $\mathcal{Q}_m$ (dotted) with knot $\{t_i\}_{i=0}^m$ as in (1.17), for $m = 9$ or $m = 16$ together with the graph of γ_{sp} (dashed) are shown in Figure 1.5. □

At the other extreme are examples, where sampling increments $t_i - t_{i-1}$ are neither large nor small, considering m, and yet sampling is not ε-uniform for any $\varepsilon > 0$:

Example 1.3. Set $\{t_i\}_{i=0}^m$ to be $\frac{i}{m}$ or $\frac{2i-1}{2m}$ according as i is even or odd. Clearly $\{t_i\}_{i=0}^m$ are 0-uniform sampled and satisfy

$$\frac{1}{2m} \leq t_i - t_{i-1} \leq \frac{3}{2m}, \tag{1.19}$$

for all $1 \leq i \leq m$ and all $m \geq 1$. Thus $\{t_i\}_{i=0}^m \in \mathcal{V}_G^m$. To see that $\{t_i\}_{i=0}^m \notin \mathcal{V}_{\varepsilon>0}^m$ assume the opposite. Then, for some C^∞ reparameterization $\phi : [0,1] \to [0,1]$,

$$t_{i+1} - t_i = \frac{1}{2m} = \phi\left(\frac{i+1}{m}\right) - \phi\left(\frac{i}{m}\right) + O(\frac{1}{m^{1+\varepsilon}}) \tag{1.20}$$

and

$$t_{i+2} - t_{i+1} = \frac{3}{2m} = \phi\left(\frac{i+2}{m}\right) - \phi\left(\frac{i+1}{m}\right) + O(\frac{1}{m^{1+\varepsilon}}) . \tag{1.21}$$

Taylor's Theorem coupled with (1.20) and (1.21) yields

$$\frac{1}{2m} = \frac{\phi'(\xi_{1i}^{(m)})}{m} + O(\frac{1}{m^{1+\varepsilon}}), \qquad \frac{3}{2m} = \frac{\phi'(\xi_{2i}^{(m)})}{m} + O(\frac{1}{m^{1+\varepsilon}}) , \tag{1.22}$$

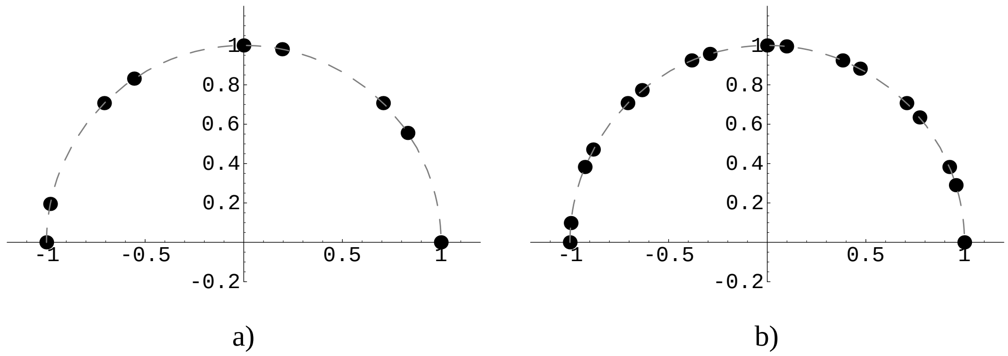


Fig. 1.6. *A semicircle* γ_{sc} (1.9) (dashed) with 0-uniform sampling (1.19) yielding sample points (dotted): a) $\mathcal{Q}_8$, b) $\mathcal{Q}_{16}$

Rys. 1.6. *Półokrąg* γ_{sc} (1.9) (linia przerywana) próbkowany 0-równomiernie (1.19) z punktami pomiarowymi (wytłuszczone): a) $\mathcal{Q}_8$, b) $\mathcal{Q}_{16}$

for some $\xi_{1i}^{(m)} \in (\frac{i}{m}, \frac{i+1}{m})$ and $\xi_{2i}^{(m)} \in (\frac{i+1}{m}, \frac{i+2}{m})$. Fixing i and increasing m, yields $\phi'(\xi_{1i}^{(m)}) \to \phi'(0)$ and $\phi'(\xi_{2i}^{(m)}) \to \phi'(0)$. On the other hand, by (1.22) and $\varepsilon > 0$, $\phi'(\xi_{1i}^{(m)}) \to 1/2$ and $\phi'(\xi_{2i}^{(m)}) \to 3/2$: a contradiction. For a semicircle γ_{sc} defined as in (1.9) a distribution of sampling points $\mathcal{Q}_m$ (dotted) with $\{t_i\}_{i=0}^m$ as above, for $m = 8$ or $m = 16$ together with the graph of γ_{sc} (dashed) are shown in Figure 1.6. □

The next example introduces special families of ε-uniform samplings used later to verify some of the results established in this monograph.

Example 1.4. For $l = 1, 2, 3$, let $\phi_l : [0, 1] \to [0, 1]$ define diffeomorphisms given by

$$\phi_1(t) = t\,, \qquad \phi_2(t) = \frac{t(\pi t + 1)}{\pi + 1}\,, \qquad \text{and} \qquad \phi_3(t) = \frac{\exp(\pi t) - 1}{\exp(\pi) - 1}\,.$$

With these functions we first introduce ε-uniform *random samplings* (see Figure 1.7)

$$t_i = \phi_l(\frac{i}{m}) + (Random[\ ] - 0.5)\frac{1}{m^{1+\varepsilon}}\,, \tag{1.23}$$

where $Random[\ ]$ takes the pseudorandom values from the interval $[0, 1]$ and $0 \le i \le m$.

In addition, we define two other families of *skew-symmetric* ε-uniform samplings, with ϕ_1 and $0 \le i \le m$ (see Figures 1.8, 1.9):

$$(i)\ t_i = \frac{i}{m} + \frac{(-1)^{i+1}}{3m^{1+\varepsilon}}\,, \qquad (ii)\ t_i = \begin{cases} \frac{i}{m} & \text{if } i \text{ even}\,, \\ \frac{i}{m} + \frac{1}{2m^{1+\varepsilon}} & \text{if } i = 4k+1\,, \\ \frac{i}{m} - \frac{1}{2m^{1+\varepsilon}} & \text{if } i = 4k+3\,. \end{cases} \tag{1.24}$$

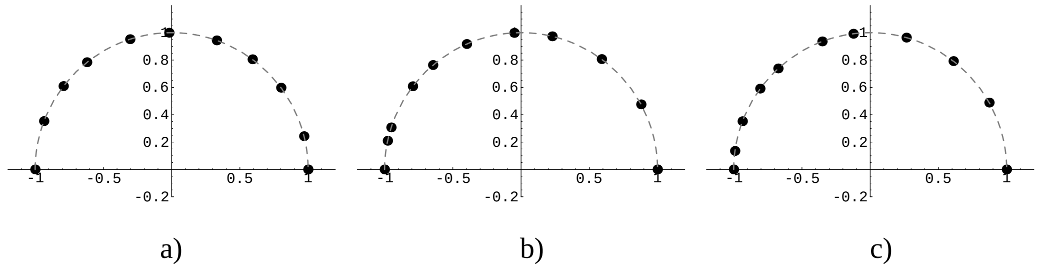


Fig. 1.7. *A semicircle* γ_{sc} (1.9) (dashed) with ε-uniform sampling (1.23) yielding $\mathcal{Q}_{10}$ (dotted) with $\varepsilon = 0.33$ for (see Example 1.4): a) ϕ_1, b) ϕ_2, c) ϕ_3

Rys. 1.7. *Półokrąg* γ_{sc} (1.9) (linia przerywana) próbkowany ε-równomiernie (1.23) dla punktów pomiarowych $\mathcal{Q}_{10}$ (wytłuszczone) z $\varepsilon = 0.33$ i (patrz Przykład 1.4): a) ϕ_1, b) ϕ_2, c) ϕ_3

To illustrate sampling (1.24)(i) *a cubic* $\gamma_c : [0,1] \to \mathbb{R}^2$ defined as

$$\gamma_c(t) \ = \ (\pi t, (\pi t+1)^3(\pi+1)^{-3}) \tag{1.25}$$

is chosen. □

Since $\mathcal{V}^m_\varepsilon \subset \mathcal{V}^m_G$, for $\{t_i\}_{i=0}^m$ *given,* clearly Theorem 1.2 applies also to ε-uniform samplings. It turns out that a tighter result can be proved for $\{t_i\}_{i=0}^m \in \mathcal{V}^m_\varepsilon$, at least for r even (for proof see Section 1.5 or [32]).

Theorem 1.3. *If sampling is ε-uniform, $\varepsilon \geq 0$ and $\gamma \in C^{r+2}$ then with the knot parameters $\{t_i\}_{i=0}^m$ known explicitly the piecewise-r-degree Lagrange interpolation $\tilde{\gamma}_r$[7] yields*

$$\boxed{d(\tilde{\gamma}_r) - d(\gamma) \ = \ \begin{cases} O(\delta^{r+1}) & \textit{if } r \geq 1 \textit{ is odd}, \\ O(\delta^{r+1+\min\{1,\varepsilon\}}) & \textit{if } r \geq 2 \textit{ is even}, \end{cases}} \tag{1.26}$$

and

$$\boxed{\tilde{\gamma}_r - \gamma \ = \ O(\delta^{r+1})\,.} \tag{1.27}$$

The convergence orders in (1.26) are experimentally confirmed to be sharp at least for $n = 2$ and $r = 2, 3, 4$ (see Table 1.3 in Subsection 1.6.3). Note that for $\varepsilon \geq 1$ (a minor perturbation) convergence rates in Theorem 1.3 coincide with those from Theorem 1.1. Thus small perturbation of uniform sampling (by the latter we understand here $\{\phi(iT/m)\}_{i=0}^m$) has no impact on convergence rates to estimate γ and $d(\gamma)$. ε-uniform samplings are also discussed for reduced data - see Chapters 2, 4, 5, and 6.

[7]See Definition 1.1.

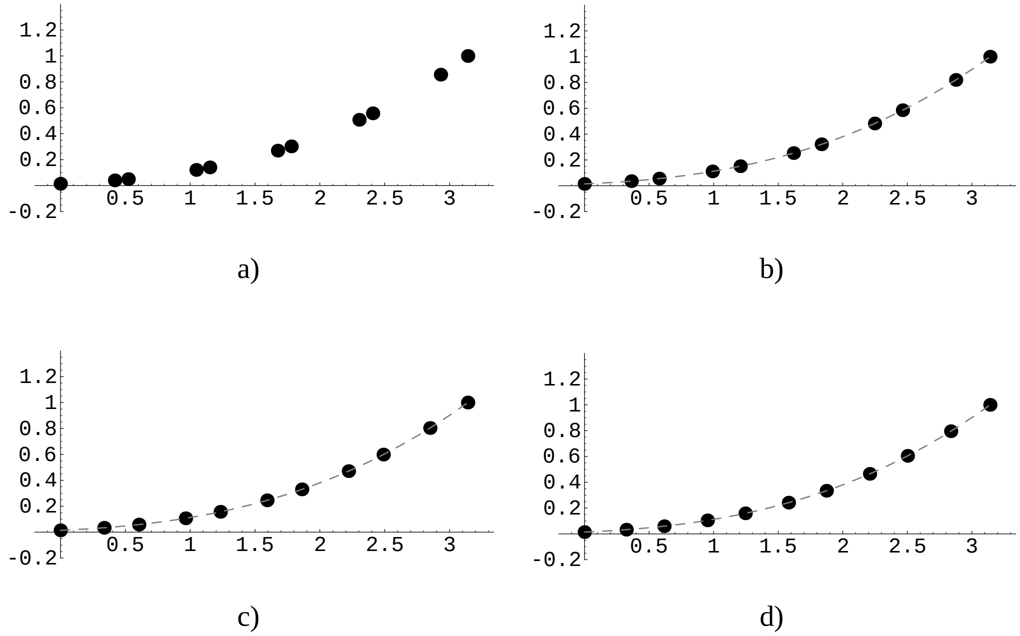


Fig. 1.8. *A cubic* γ_c (1.25) (dashed) with ε-uniform sampling (1.24)(i) yielding $\mathcal{Q}_{10}$ (dotted) for: a) $\varepsilon = 0$, b) $\varepsilon = 0.33$, c) $\varepsilon = 0.66$, d) $\varepsilon = 1$

Rys. 1.8. *Krzywa kubiczna* γ_c (1.25) (linia przerywana) próbkowana ε-równomiernie (1.24)(i) dla punktów pomiarowych $\mathcal{Q}_{10}$ (wytłuszczone), gdzie: a) $\varepsilon = 0$, b) $\varepsilon = 0.33$, c) $\varepsilon = 0.66$, d) $\varepsilon = 1$

1.3.4 More-or-less uniform samplings

The next subfamily $\mathcal{V}_{mol}^m \subset \mathcal{V}_G^m$ we consider herein, are the so-called *more-or-less uniform samplings*:

Definition 1.5. Sampling $\mathcal{Q}_m$ is said to be *more-or-less uniform*, if for the corresponding knot parameters $\{t_i\}_{i=0}^m$ there are constants $0 < K_l \leq K_u$ such that, for any sufficiently large integer m, and any $1 \leq i \leq m$,

$$\boxed{\boxed{\frac{K_l}{m} \;\leq\; t_i - t_{i-1} \;\leq\; \frac{K_u}{m}\;.}} \tag{1.28}$$

Remark 1.3. It is easy to see that (1.28) can be replaced by the equivalent condition:

$$\boxed{\boxed{\lambda\delta \;\leq\; t_i - t_{i-1} \leq \delta\;,}} \tag{1.29}$$

where δ is defined as in (1.3) and $0 < \lambda \leq 1$. Indeed if (1.29) is satisfied then as the following holds $\sum_{i=1}^m (t_i - t_{i-1}) = T$ we have $\delta \leq \frac{T}{\lambda m}$. Thus by (1.3) it suffices to set

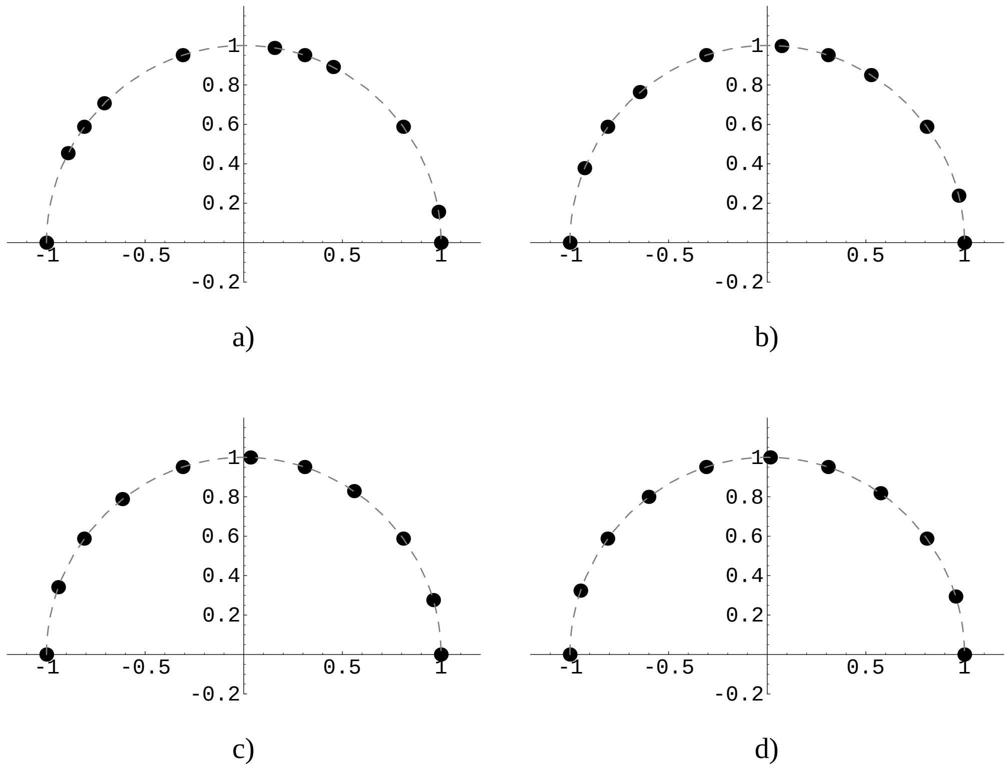


Fig. 1.9. *A semicircle* γ_{sc} (1.9) (dashed) with ε-uniform sampling (1.24)(ii) yielding $\mathcal{Q}_{10}$ (dotted) for: a) $\varepsilon = 0$, b) $\varepsilon = 0.33$, c) $\varepsilon = 0.66$, d) $\varepsilon = 1$

Rys. 1.9. *Półokrąg* γ_{sc} (1.9) (linia przerywana) próbkowany ε-równomiernie jak w (1.24)(ii) dla punktów pomiarowych $\mathcal{Q}_{10}$ (wytłuszczone), gdzie: a) $\varepsilon = 0$, b) $\varepsilon = 0.33$, c) $\varepsilon = 0.66$, d) $\varepsilon = 1$

$K_u = \frac{T}{\lambda}$. Similarly, by (1.29) and as by (1.3) $\sum_{i=1}^{m}(t_i - t_{i-1}) \leq m\delta$ we have

$$\frac{\lambda T}{m} = ((t_1 - t_0) + (t_2 - t_1) + \ldots + (t_m - t_{m-1}))\frac{\lambda}{m} \leq \lambda\delta \leq t_i - t_{i-1} .$$

Hence we set $K_l = \lambda T$. Of course here $0 < K_l \leq K_u$. Conversely, if (1.28) is satisfied, the putting $\lambda = \frac{K_l}{K_u} \in (0,1]$ we have $\frac{\lambda K_u}{m} \leq t_i - t_{i-1}$. The latter combined again with (1.28) yields

$$\lambda(t_1 - t_0) \leq t_i - t_{i-1} , \quad \lambda(t_2 - t_1) \leq t_i - t_{i-1} , \ldots , \lambda(t_m - t_{m-1}) \leq t_i - t_{i-1} .$$

Thus by (1.3) we arrive at $\lambda\delta \leq t_i - t_{i-1}$. Inequality $t_i - t_{i-1} \leq \delta$ follows directly from Definition 1.2.

Again, before passing to examples some comments should be made first:

- From Definition 1.5 it is clear that for $\{t_i\}_{i=0}^m \in \mathcal{V}_{mol}^m$ the increments between successive t_i 's are neither large nor small in proportion to $\frac{T}{m}$.
- Of course, by (1.28) $\{t_i\}_{i=0}^m$ satisfy (1.3) and thus $\mathcal{V}_{mol}^m \subset \mathcal{V}_G^m$.
- Clearly $\mathcal{V}_U^m \subset \mathcal{V}_{mol}^m$ as then both constants from (1.28) read $K_l = K_u = T$.
- Note that any family $\{t_i\}_{i=0}^m \in \mathcal{V}_{mol}^m$ is invariant with respect to the order-preserving reparameterization $\psi : [0,T] \to [0,T]$. By the latter we mean $\{\psi(t_i)\}_{i=0}^m \in \mathcal{V}_{mol}^m$ and $\psi(t_i) < \psi(t_{i+1})$ for $i = 0, 1, \ldots, m-1$. Indeed by Mean Value Theorem

  $$\psi(t_i) - \psi(t_{i-1}) \;=\; \dot{\psi}(\xi_i)(t_i - t_{i-1}) \,, \tag{1.30}$$

  for some $\xi_i \in [t_{i-1}, t_i]$. As $\psi \in C^1$, $\dot{\psi} > 0$ and $[0,T]$ is compact, both constants $L_{max} = sup_{t\in[0,T]}\dot{\psi}(t)$ and $L_{min} = inf_{t\in[0,T]}\dot{\psi}(t)$ are finite and positive. This combined with (1.28) and (1.30) yields

  $$\frac{M_l}{m} \;\le\; \psi(t_i) \;-\; \psi(t_{i-1}) \;\le\; \frac{M_u}{m} \,,$$

  where $M_u = K_u L_{max}$ and $M_l = K_l L_{min}$. Also ψ is strictly increasing as $\dot{\psi} > 0$.
- Note that $\mathcal{V}_\varepsilon^m \not\subset \mathcal{V}_{mol}^m$. Indeed, take *e.g.* any 0-uniform sampling with one pair satisfying

  $$t_i \;=\; \frac{iT}{m} + \frac{T}{m} - \frac{T}{m^2} \quad \text{and} \quad t_{i+1} \;=\; \frac{(i+1)T}{m} \,. \tag{1.31}$$

  Clearly, $t_{i+1} - t_i = 1/m^2$ and hence the left inequality from (1.28) is not satisfied.
- Note, however that asymptotically $\mathcal{V}_{\varepsilon>0}^m \subset \mathcal{V}_{mol}^m$. Indeed by (1.16), there exists $N > 0$ such that for $m > m_0$

  $$\dot{\phi}(\frac{iT}{m})\frac{T}{m} \;-\; \frac{N}{m^{\min\{2,1+\varepsilon\}}} \;\le\; t_{i+1} - t_i \;\le\; \dot{\phi}(\frac{iT}{m})\frac{T}{m} \;+\; \frac{N}{m^{\min\{2,1+\varepsilon,2\}}} \,.$$

  The latter combined with $\varepsilon > 0$, $\phi \in C^1$, and compactness of $[0,T]$ yields (1.28), asymptotically.
- Recall that sampling from Example 1.3 satisfies (1.19) and therefore it represents a more-or-less uniform sampling. As concluded still in Example 1.3 it cannot define an ε-uniform sampling for any $\varepsilon > 0$. Thus the following holds $\mathcal{V}_{mol}^m \not\subset \mathcal{V}_{\varepsilon>0}^m$.

We pass now to some examples:

Example 1.5. Evidently, samplings introduced in (1.11) or in (1.17) are not more-or-less uniform. As the second sampling is admissible (see Definition 1.2) this combined with (1.28) renders a strong inclusion $\mathcal{V}_{mol}^m \subset \mathcal{V}_G^m$. □

Example 1.6. On the other hand Example 1.3 introduces a more-or-less uniform sampling. For special distributions of sampling points see Figure 1.5. Similarly, formula (1.24)(i) with $\varepsilon = 0$ defines another more-or-less uniform sampling with $K_l = \frac{T}{3}$ and $K_u = \frac{5T}{3}$. Again the distribution of sampling points for special cases is illustrated in Figure 1.6. □

Example 1.7. For $0 < i < m$ let $\{t_i\}_{i=0}^m$ be random numbers (according to some distribution) in the interval $[\frac{(3i-1)T}{3m}, \frac{(3i+1)T}{3m}]^{m+1}$. Then sampling is more-or-less uniform, with K_u, K_l as in Example 1.6. □

Example 1.8. Choose $\theta > 0$ and $0 < L_l < L_u$. Set $s_0 = 0$. For $1 \leq i \leq m$ choose $\delta_i \in [\frac{L_l}{m}, \frac{L_u}{m}]$ independently from (say) the uniform distribution. Define $s_i = s_{i-1} + \delta_i$ for $i = 1, 2, \ldots, m$. The expectation of s_m is $\frac{L_u+L_l}{2}$ and the standard deviation $\frac{L_u-L_l}{2\sqrt{3m}}$. So if m is large $s_m \approx \frac{L_u+L_l}{2}$ with high probability. For $0 \leq i \leq m$, define $t_i = \frac{s_i T}{s_m}$. Set

$$K_l = \frac{2L_l T}{L_u + L_l} - \theta, \quad K_u = \frac{2L_u T}{L_u + L_l} + \theta .$$

Then with high probability for m large, the sampling $\{t_i\}_{i=0}^m \in [0, T]^{m+1}$ is more-or-less uniform with constants K_l, K_u. □

Remark 1.4. The *convergence orders* for *non-reduced data* $(\mathcal{Q}_m, \{t_i\}_{i=0}^m)$ which are sampled *more-or-less uniformly* coincide with those established in Theorem 1.2 for the class of *admissible samplings* $\mathcal{V}_G^m$ (see Subsection 1.6.3). Thus, contrary to $\mathcal{V}_\varepsilon^m \subset \mathcal{V}_G^m$, there is no extra acceleration in asymptotics for $\mathcal{V}_{mol}^m \subset \mathcal{V}_G^m$, if non-reduced data $(\mathcal{Q}_m, \{t_i\}_{i=0}^m)$ are used.

More-or-less uniform samplings are more useful in the context of reduced data $\mathcal{Q}_m$ for the discussion in Chapter 3 and justification of Theorems 2.2 and 5.2. In particular the proofs of Theorems 2.1 and 5.1 fail for $\varepsilon = 0$. They can be remedied, however, by extending the asymptotics derived for $\varepsilon > 0$ to the limiting case $\varepsilon \to 0^+$, provided that 0-uniform sampling satisfies $\{t_i\}_{i=0}^m \in \mathcal{V}_{mol}^m$. As by (1.31) there are 0-uniform samplings which are not more-or-less uniform, additional assumption about more-or-less uniformity is therefore needed.

1.4 Uniform and general non-reduced data - asymptotics

In this section we prove first the Theorem 1.1 and then conclude the Theorem 1.2 (see also [32] and [46]).

We begin with the easiest case, where the $\{t_i\}_{i=0}^m$ are *sampled perfectly uniformly i.e.* $t_i = \frac{iT}{m}$ (with $0 \leq i \leq m$). Of course, for uniform sampling $\delta = T/m$. Suppose $\gamma \in C^{r+2}$ and $r \geq 1$. Without loss of generality let m be a multiple of r. Then $\mathcal{Q}_m$ gives $\frac{m}{r}$ $r+1$-tuples of the form

$$(q_0, q_1, \ldots, q_r) , \quad (q_r, q_{r+1}, \ldots, q_{2r}) , \quad \ldots , \quad (q_{m-r}, q_{m-r+1}, \ldots, q_m) .$$

The j-th $r+1$-tuple is interpolated by the r-*degree Lagrange polynomial* $P_r^j : [0,r] \to \mathbb{R}^n$ (see (1.8)), here $1 \leq j \leq \frac{m}{r}$ satisfying

$$\boxed{P_r^j(0) \;=\; q_{(j-1)r} \;, \quad P_r^j(1) \;=\; q_{(j-1)r+1} \;, \quad \dots \;, \quad P_r^j(r) \;=\; q_{jr} \;.} \tag{1.32}$$

Note that P_r^j is defined in terms of a local variable $s \in [0,r]$ and estimated times $\hat{t}_i = i$. Define a piecewise-r-degree polynomial $\tilde{\gamma}_r : [0,m] \to \mathbb{R}^n$ (here $\hat{T} = m$) as a track-sum of $\{\mathcal{P}_r^j\}_{j=1}^{\frac{m}{r}}$ (see also Definition 1.1 and Remark 1.1). Recall Lemma 2.1 of Part 1 of [38] (the so-called Hadamard's Lemma):

Lemma 1.1. *Let* $f : [a,b] \to \mathbb{R}^n$ *be* C^l, *where* $l \geq 1$ *and assume that* $f(t_0) = \vec{0}$, *for some* $t_0 \in (a,b)$. *Then there exists a* C^{l-1} *function* $g : [a,b] \to \mathbb{R}^n$ *such that* $f(t) = (t-t_0)g(t)$.

Proof. We outline now the proof of Lemma 1.1 as it is vital for future arguments used in the monograph. For the i-th component of $f = (f_1, f_2, \dots, f_n)$ consider $F_i : [a,b] \to \mathbb{R}$ defined as $F_i(u) = f_i(tu + (1-u)t_0)$. By the Fundamental Theorem of Calculus

$$f_i(t) \;=\; F_i(1) - F_i(0) \;=\; (t-t_0)\int_0^1 f_i'(tu + (1-u)t_0)\,du \;.$$

Take $g = (g_1, g_2, \dots, g_n)$, where

$$g_i(t) \;=\; \int_0^1 f_i'(tu + (1-u)t_0)\,du \;.$$

The proof is complete. ☐

Remark 1.5. The proof of Lemma 1.1 shows also that $g = O(\frac{df}{dt})$. If f has multiple zeros $t_0 < t_1 < \dots < t_k$ then $k+1$ applications of Lemma 1.1 give

$$\boxed{f(t) \;=\; (t-t_0)(t-t_1)(t-t_2)\dots(t-t_k)h(t) \;,} \tag{1.33}$$

where h is $C^{l-(k+1)}$ and $h = O(\frac{d^{k+1}f}{dt^{k+1}})$.

We are ready now to prove Theorem 1.1:

Proof. Assuming that γ is C^{r+2} (where $r \geq 1$) we are going now to prove Theorem 1.1, where estimation of γ and $d(\gamma)$ is based on *piecewise-r-degree polynomial interpolation* $\tilde{\gamma}_r$ already defined in this section. For each j-th $r+1$-tuple consider the interpolating polynomial P_r^j. In order to compare γ and P_r^j we need to *reparameterize* P_r^j so that their domains coincide. In doing so let

$$\psi_j : [t_{(j-1)r}, t_{jr}] \to [0,r]$$

be the *affine mapping* given by $\psi_j(t_{(j-1)r}) = 0$ and $\psi_j(t_{jr}) = r$, namely

$$\boxed{\psi_j(t) \ = \ \tfrac{mt}{T} \ - \ (j-1)r \ .}$$

Thus $\dot{\psi}_j(t) = m/T$ and hence it is a diffeomorphism. Again by $\psi : [0, T] \to [0, \hat{T}]$ (here $\hat{T} = m$) we mean a track-sum of $\{\psi_j\}_{j=1}^{\frac{m}{r}}$ being a piecewise-linear and piecewise-C^∞ polynomial. Note that since both intervals $[t_{(j-1)r}, t_{jr}]$ and $[0, r]$ are *uniformly sampled*, ψ maps the t_i's to the corresponding grid points $\hat{t}_i$ in $[0, r]$. Define now

$$\boxed{\tilde{\gamma}_j \ = \ P_r^j \circ \psi_j : [t_{(j-1)r}, t_{jr}] \to \mathbb{R}^n \ .} \tag{1.34}$$

Then as ψ_j is affine, $\tilde{\gamma}_j$ is a polynomial of degree at most r. Note that

$$\boxed{f \ = \ \tilde{\gamma}_j - \gamma : [t_{(j-1)r}, t_{jr}] \to \mathbb{R}^n} \tag{1.35}$$

is C^{r+2} and satisfies $f(t_{(j-1)r}) = f(t_{(j-1)r+1}) = \ldots = f(t_{jr}) = \vec{0}$. By (1.33)

$$f(t) \ = \ (t - t_{(j-1)r})(t - t_{(j-1)r+1}) \ldots (t - t_{jr}) h(t) \ , \tag{1.36}$$

where $h : [t_{(j-1)r}, t_{jr}] \to \mathbb{R}^n$ by Lemma 1.1 is C^1. Still by proof of Lemma 1.1

$$h(t) \ = \ O(\frac{d^{r+1} f}{dt^{r+1}}) \ = \ O(\frac{d^{r+1}\gamma}{dt^{r+1}}) \ = \ O(1) \ , \tag{1.37}$$

because $\deg(\tilde{\gamma}_j) \leq r$ and $\frac{d^{r+1}\gamma}{dt^{r+1}}$ is $O(1)$; in fact all derivatives of γ, up to $r+2$-order are $O(1)$ since they are continuous functions over compact domain $[0, T]$. Thus by (1.36), (1.37), and $\delta = T/m$ we have

$$\boxed{f(t) \ = \ O(\tfrac{1}{m^{r+1}}) \ = \ O(\delta^{r+1}) \ ,} \tag{1.38}$$

for $t \in [t_{(r-1)j}, t_{rj}]$. This completes the proof of the first claim of Theorem 1.1.

Differentiating function h (defined as a $r+1$-multiple integral of $f^{(r+1)}$ over the compact cube $[0, 1]^{r+1}$; see proof of Lemma 1.1) yields

$$\dot{h}(t) \ = \ O(\frac{d^{r+2} f}{dt^{r+2}}) \ = \ O(\frac{d^{r+2}\gamma}{dt^{r+2}}) \ = \ O(1) \ , \tag{1.39}$$

as $\deg(\tilde{\gamma}_j) \leq r$. By (1.36) and (1.39) $f = O(\frac{1}{m^r})$ and hence for $t \in [t_{(j-1)r}, t_{jr}]$

$$\dot{\tilde{\gamma}}_j(t) - \dot{\gamma}(t) \ = \ \dot{f}(t) \ = \ O(\frac{1}{m^r}) \ . \tag{1.40}$$

Let $V_{\dot{\gamma}}^{\perp}(t)$ be the orthogonal complement of the line spanned by $\dot{\gamma}(t)$. Since $\|\dot{\gamma}(t)\| = 1$,

$$\dot{\tilde{\gamma}}_j(t) \;=\; \langle \dot{\tilde{\gamma}}_j(t), \dot{\gamma}(t)\rangle \dot{\gamma}(t) \;+\; v(t)\;, \tag{1.41}$$

where $v(t)$ is the orthogonal projection of $\dot{\tilde{\gamma}}_j(t)$ onto $V_{\dot{\gamma}}^{\perp}(t)$. Since $\dot{\tilde{\gamma}}_j(t) = \dot{f}(t) + \dot{\gamma}(t)$ and $\|\dot{\gamma}(t)\| = 1$, we have

$$\dot{\tilde{\gamma}}_j(t) \;=\; (1 + \langle \dot{f}(t), \dot{\gamma}(t)\rangle)\dot{\gamma}(t) \;+\; v(t)\;. \tag{1.42}$$

Furthermore, by (1.40),

$$v(t) \;=\; \dot{f}(t) \;-\; \langle \dot{f}(t), \dot{\gamma}(t)\rangle \dot{\gamma}(t) \tag{1.43}$$

and $v = O(\frac{1}{m^r})$. Combining the latter with (1.42), $\langle \dot{\gamma}(t), v(t)\rangle = 0$, and Taylor's Theorem applied to $g(x) = \sqrt{1+x} = 1 + 0.5x + 0.5g''(\xi)x^2$ (with $\xi \in (0,x)$ or $\xi \in (x,0)$) we have

$$\boxed{\|\dot{\tilde{\gamma}}_j(t)\| \;=\; \sqrt{1 + 2\langle \dot{f}(t), \dot{\gamma}(t)\rangle + O(\tfrac{1}{m^{2r}})} \;=\; 1 + \langle \dot{f}(t), \dot{\gamma}(t)\rangle + O(\tfrac{1}{m^{2r}})\;.} \tag{1.44}$$

Note that by (1.40) the expression $x = 2\langle \dot{f}(t), \dot{\gamma}(t)\rangle + O(\frac{1}{m^{2r}})$ is asymptotically separated from -1, yielding g'' uniformly bounded. Again $\|\dot{\gamma}\| = 1$ combined with (1.44), $f(t_{(j-1)r}) = f(t_{jr}) = \vec{0}$ and integration by parts applied to

$$\mathcal{I}_0 = \int_{t_{(j-1)r}}^{t_{jr}} (\|\dot{\tilde{\gamma}}_j(t)\| - \|\dot{\gamma}(t)\|)\, dt \tag{1.45}$$

leads to

$$\begin{aligned}
\mathcal{I}_0 \;&=\; \int_{t_{(j-1)r}}^{t_{jr}} \langle \dot{f}(t), \dot{\gamma}(t)\rangle\, dt \;+\; O(\frac{1}{m^{2r+1}}) \\
&=\; \sum_{k=1}^{n} \left((f_k(t_{jr})\dot{\gamma}_k(t_{jr}) - f_k(t_{(j-1)r})\dot{\gamma}_k(t_{(j-1)r})) - \int_{t_{(j-1)r}}^{t_{jr}} f_k(t)\ddot{\gamma}_k(t)\, dt \right) \\
&\qquad + O(\frac{1}{m^{2r+1}}) \\
&=\; -\int_{t_{(j-1)r}}^{t_{jr}} \langle f(t), \ddot{\gamma}(t)\rangle\, dt \;+\; O(\frac{1}{m^{2r+1}})\;,
\end{aligned} \tag{1.46}$$

where $f = (f_1, f_2, \ldots, f_n) \in \mathbb{R}^n$ and $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_n) \in \mathbb{R}^n$.

Since γ is compact and C^3 by (1.37) and $h = O(1)$ we have $\langle h(t), \ddot{\gamma}(t)\rangle = O(1)$ and $\langle h(t), \gamma^{(3)}(t)\rangle = O(1)$. Similarly, by (1.39) we have $\langle \dot{h}(t), \ddot{\gamma}(t)\rangle = O(1)$. Hence, by

(1.36) and Taylor's Theorem applied to $r(t) = \langle h(t), \ddot{\gamma}(t)\rangle$ at $t = t_{(j-1)r}$, we get

$$\langle f(t), \ddot{\gamma}(t)\rangle = (t - t_{(j-1)r})(t - t_{(j-1)r+1}) \dots (t - t_{jr})(a + O(\frac{1}{m})) , \qquad (1.47)$$

where a is constant in t and $O(1)$. Since sampling is uniform the integral

$$\boxed{\int_{t_{(j-1)r}}^{t_{jr}} (t - t_{(j-1)r})(t - t_{(j-1)r+1}) \dots (t - t_{jr})\, dt = 0 ,} \qquad (1.48)$$

when r is even (as then upon substitution $s = t - (t_{jr} + t_{(j-1)r})/2$ one integrates the odd function over the interval $[-(t_{jr} + t_{(j-1)r})/2, (t_{jr} + t_{(j-1)r})/2]$). So by (1.47) and (1.48) $\int_{t_{(j-1)r}}^{t_{jr}} \langle f(t), \ddot{\gamma}(t)\rangle\, dt$ is either $O(\frac{1}{m^{r+2}})$ or $O(\frac{1}{m^{r+3}})$, according as r is odd or even. Hence as $2r + 1 \geq r + 3$ (for $r \geq 2$) and $2r + 1 \geq r + 2$ (for $r \geq 1$), by (1.46)

$$\boxed{\int_{t_{(j-1)r}}^{t_{jr}} (\|\dot{\tilde{\gamma}}_j(t)\| - \|\dot{\gamma}(t)\|)\, dt = \begin{cases} O(\frac{1}{m^{r+2}}) & \text{if } r \geq 1 \text{ is odd} \\ O(\frac{1}{m^{r+3}}) & \text{if } r \geq 2 \text{ is even .} \end{cases}} \qquad (1.49)$$

Take $\tilde{\gamma}_r$ to be a track-sum of the $\{\tilde{\gamma}_j\}_{j=0}^{\frac{m}{r}-1}$ (see Remark 1.1), *i.e.*

$$\boxed{d(\tilde{\gamma}_r) = \Sigma_{j=0}^{\frac{m}{r}-1} d(\tilde{\gamma}_j) = d(\gamma) + O(\frac{1}{m^{r+p}}) ,} \qquad (1.50)$$

where p is 1 or 2 according as r is odd or even. Upon recalling that $\delta = T/m$ the latter proves Theorem 1.1. □

Remark 1.6. Not surprisingly, inspection of proof of Theorem 1.1 from formula (1.36) onward, shows that if $t_i = (iT/m)$ are *known* then (1.12) prevails. Thus guessing times as $\hat{t}_i = i$ (which preserves the uniform distribution of $\{t_i\}_{i=0}^m$) matches the asymptotics for piecewise Lagrange interpolation $\tilde{P}_r^i : [t_i, t_{i+r}] \to \mathbb{R}^n$ used with $\{t_i\}_{i=0}^m$ known and uniform for which $\tilde{P}_r^i(t_{i+k}) = q_{i+k}$ (for $k = 0, 1, \dots, r$). The reason underpinning the above stems from the observation that here $\psi : [t_i, t_{i+r}] \to [\hat{t}_i, \hat{t}_{i+r}]$ satisfying $\psi(t_{i+k}) = \hat{t}_{i+k}$ (for $0 \leq k \leq r$) is *affine*. Thus since $\deg(P_r^i \circ \psi^i) = \deg(\tilde{P}_r^i)$ and $(P_r^i \circ \psi^i)(t_{i+k}) = \tilde{P}_r^i(t_{i+k})$, for $k = 0, 1, \dots, r$, the uniqueness of Lagrange interpolation renders $P_r^i \circ \psi^i \equiv \tilde{P}_r^i$ over $[t_i, t_{i+r}]$. Hence reparameterized P_r^i to $P_r^i \circ \psi^i$ inherit the δ-asymptotics established for $\tilde{P}_r^i$ (*i.e.* for non-reduced data $(\mathcal{Q}_m, \{t_i\}_{i=0}^m)$).

Remark 1.7. Note that if $\{t_i\}_{i=0}^m \in \mathcal{V}_G^m$ are *known* then again upon repeating the proof of Theorem 1.1 from formula (1.36) onward one obtains

$$\boxed{\tilde{\gamma} = \gamma + O(\delta^{r+1}) \quad \text{and} \quad d(\tilde{\gamma}) = d(\gamma) + O(\delta^{r+1}) ,}$$

for $\tilde{\gamma}_r = P_r : [0, \hat{T} = m]$, where P_r is a track-sum (see Remark (1.1)) of the Lagrange r-degree polynomials $P_r^i : [t_i, t_{i+r}] \to \mathbb{R}^n$ (see Remark 1.1) satisfying $P_r^i(t_{i+k}) = q_{i+k}$, where $0 \leq k \leq r$. Obviously for an $\{t_i\}_{i=0}^m \in \mathcal{V}_G^m \setminus \mathcal{V}_U^m$, contrary to the uniform case $\mathcal{V}_U^m \subset \mathcal{V}_G^m$, the integral (1.48) does not generically vanish. Hence no further acceleration to $O(\delta^{r+2})$ term (for r even) is achievable. Note also that for $\{t_i\}_{i=0}^m \in \mathcal{V}_G^m$ the formula (1.49) is expressed in terms of δ-asymptotics and thus passing to formula (1.50) requires an extra condition

$$\boxed{m\delta \ = \ O(1)\ .} \tag{1.51}$$

The latter, though satisfied by $\mathcal{V}_\varepsilon^m \subset \mathcal{V}_G^m$ and $\mathcal{V}_{mol}^m \subset \mathcal{V}_G^m$ may not hold for the whole class of admissible samplings $\mathcal{V}_G^m$ (see *e.g.* Example 1.2). This proves a Theorem 1.2.

1.5 $\mathcal{E}$-uniform non-reduced samplings - asymptotics

In this section we prove the Theorem 1.3 (see also [32]).

Proof. The first formula from (1.26) and (1.27) result directly from Theorem 1.2 upon recalling that $\mathcal{V}_\varepsilon^m \subset \mathcal{V}_G^m$. Without loss, we may assume that $\gamma : [0,1] \to \mathbb{R}^n$. Indeed originally parameterized $\gamma : [0,T] \to \mathbb{R}^n$ by arc-length (with $T = d(\gamma)$) can be reparameterized by an affine mapping $\psi : [0,1] \to [0,T]$ to $\tilde{\gamma} = \gamma \circ \psi$ with $\|\tilde{\gamma}'\| = T$. Thus we may assume that $\gamma : [0,1] \to \mathbb{R}^n$ and $\|\dot{\gamma}(t)\| = d(\gamma)$. Furthermore, repeating the argument from Theorem 1.1 up to (1.47) we obtain

$$\boxed{\langle f(t), \ddot{\gamma}(t)\rangle \ = \ (t - t_{(j-1)r}) \ldots (t - t_{jr})(a \ + \ O(\tfrac{1}{m}))\ ,} \tag{1.52}$$

where a is constant in t and $O(1)$. It should be mentioned that in reaching (1.47) as now $\|\dot{\gamma}(t)\| = d(\gamma)$ formula (1.41) reads

$$\dot{\tilde{\gamma}}_j(t) \ = \ \frac{\langle \dot{\tilde{\gamma}}_j(t), \dot{\gamma}(t)\rangle}{d(\gamma)^2}\dot{\gamma}(t) \ + \ v(t)\ ,$$

which modifies (1.44) to

$$\|\dot{\tilde{\gamma}}_j(t)\| \ = \ (1 + \frac{\langle \dot{f}(t), \dot{\gamma}(t)\rangle}{d(\gamma)^2})\|\dot{\gamma}(t)\| \ + \ O(\frac{1}{m^{2r}})$$

yielding (1.46) changed to

$$\mathcal{I}_0 \ = \ -\int_{t_{(j-1)r}}^{t_{jr}} \frac{\langle f(t), \ddot{\gamma}(t)\rangle}{d(\gamma)}\, dt \ + \ O(\frac{1}{m^{2r+1}})\ . \tag{1.53}$$

Upon substitution $(t_{(j-1)r}, t_{(j-1)r+1}, \ldots, t_{jr}) = (t_0, t_1, \ldots, t_r)$ let $\chi_i : \mathbb{R}^{r+1} \to \mathbb{R}$ be defined as

$$\boxed{\chi_i(\vec{h}) \;=\; \int_{t_0}^{t_r} (t - t_0) \ldots (t - t_r)\, dt \;,} \tag{1.54}$$

where $i = (j-1)r$, $t_k = \phi(\frac{i+k}{m}) + h_k$ (for $0 \le k \le r$) with $\vec{h} = (h_0, h_1, \ldots, h_r) \in \mathbb{R}^{r+1}$ satisfying $h_k = O(\frac{1}{m^{1+\varepsilon}})$, for each $0 \le k \le r$. By Taylor's Theorem and ε-uniformity there exists $\delta > 0$ such that for each $\vec{h} \in \bar{B}(0, \delta) \subset \mathbb{R}^{r+1}$

$$\boxed{\chi_i(\vec{h}) \;=\; \chi_i(\vec{0}) \;+\; D_h \chi_i(\xi(\vec{h}))(\vec{h}) \;,} \tag{1.55}$$

with $\xi(\vec{h}) = (\xi_0(\vec{h}), \xi_1(\vec{h}), \ldots, \xi_r(\vec{h})) \in \bar{B}(0, \delta)$ positioned on the line between two vectors $\vec{0} \in \mathbb{R}^{r+1}$ and $\vec{h} = O(\frac{1}{m^{1+\varepsilon}})$ (and thus here $\delta = O(\frac{1}{m^{1+\varepsilon}})$). Furthermore, the integral (1.54) at $\vec{h} = \vec{0}$ upon integration by substitution reads

$$\chi_i(\vec{0}) \;=\; \int_{\frac{i}{m}}^{\frac{i+r}{m}} (\phi(s) - \phi(\frac{i}{m})) \ldots (\phi(s) - \phi(\frac{i+r}{m})) \dot{\phi}(s)\, ds \;. \tag{1.56}$$

Again, Taylor's Theorem applied to each factor of the integrand of (1.56) combined with compactness of $[0, 1]$ and ϕ being a diffeomorphism yields

$$\chi_i(\vec{0}) \;=\; b \int_{\frac{i}{m}}^{\frac{i+r}{m}} (s - \frac{i}{m} + \tilde{h}_0) \ldots (s - \frac{i+r}{m} + \tilde{h}_r)(\dot{\phi}(0) \;+\; O(\frac{1}{m}))\, ds \;,$$

where $b = \prod_{k=0}^{r} \dot{\phi}(\frac{i+k}{m})$ is constant in s and $O(1)$ and $\tilde{h}_k = O(\frac{1}{m^2})$ (for $0 \le k \le r$). Furthermore,

$$\chi_i(\vec{0}) \;=\; c \int_{\frac{i}{m}}^{\frac{i+r}{m}} (s - \frac{i}{m}) \ldots (s - \frac{i+r}{m})\, ds \;+\; O(\frac{1}{m^{r+3}}) \;, \tag{1.57}$$

where $c = b\dot{\phi}(0)$ is constant in s and $O(1)$. Again, as previously, it is vital that both b and c are of order $O(1)$, since they vary with m. A simple verification shows that the integral in (1.57) either vanishes for r even or otherwise is of order $O(\frac{1}{m^{r+2}})$. Hence

$$\boxed{\chi_i(\vec{0}) \;=\; \begin{cases} O(\frac{1}{m^{r+2}}) & \text{if } r \ge 1 \text{ is odd ,} \\ O(\frac{1}{m^{r+3}}) & \text{if } r \ge 2 \text{ is even .} \end{cases}} \tag{1.58}$$

In order to determine the asymptotics of the second term in (1.55) let

$$\tilde{f}_i(t, h_0, \ldots, h_r) \;=\; (t - \phi(\frac{i}{m}) - h_0) \ldots (t - \phi(\frac{i+r}{m}) - h_r) \;. \tag{1.59}$$

As $[\phi(\frac{i}{m})+h_0, \phi((\frac{i+r)}{m})+h_r]$ is compact and $\tilde{f}_i(t,\vec{h})$ is C^1 we have

$$\frac{\partial\chi_i}{\partial h_k}(\vec{h}) \;=\; \int_{\phi(\frac{i}{m})+h_0}^{\phi(\frac{i+r}{m})+h_r} \frac{\partial\tilde{f}_i}{\partial h_k}(t,\vec{h})\,dt\;,\;\; \text{for}\;\; 1\le k\le r-1\;. \tag{1.60}$$

Similarly, with $\vec{h}=(h_0,\vec{h}_{1r})$ for $\mathcal{I}=\frac{\partial\chi_i}{\partial h_0}(\vec{h})$ we obtain

$$\begin{aligned}
\mathcal{I} \;&=\; \lim_{h_0^h\to 0} \frac{\int_{\phi(\frac{i}{m})+h_0+h_0^h}^{\phi(\frac{i+r}{m})+h_r} f_i(t,h_0+h_0^h,\vec{h}_{1r})\,dt - \int_{\phi(\frac{i}{m})+h_0}^{\phi(\frac{i+r}{m})+h_r} f_i(t,h_0,\vec{h}_{1r})\,dt}{h_0^h}\\
&=\; \lim_{h_0^h\to 0}\int_{\phi(\frac{i}{m})+h_0}^{\phi(\frac{i+r}{m})+h_r} \frac{f_i(t,h_0+h_0^h,\vec{h}_{1r})-f_i(t,h_0,\vec{h}_{1r})}{h_0^h}\,dt\\
&\qquad - \lim_{h_0^h\to 0}\frac{\int_{\phi(\frac{i}{m})+h_0}^{\phi(\frac{i}{m})+h_0+h_0^h} f_i(t,h_0+h_0^h,\vec{h}_{1r})\,dt}{h_0^h}\\
&=\; \int_{\phi(\frac{i}{m})+h_0}^{\phi(\frac{i+r}{m})+h_r} \frac{\partial\tilde{f}_i}{\partial h_0}(t,\vec{h})\,dt - \tilde{f}_i(\phi(\frac{i}{m})+h_0,\vec{h})\;.
\end{aligned} \tag{1.61}$$

The latter step for the second term uses Taylor's Theorem applied to $f_i(t,h_0+h_0^h,\vec{h}_{1r})$ at $t=\phi(\frac{i}{m})+h_0$. Note that by (1.59) the second term in (1.61) vanishes. Thus formulas (1.60) extend to $k=0$ and similarly to $k=r$. Hence by the Mean Value Theorem the second term in (1.55) satisfies

$$\begin{aligned}
D_h\chi_i(\xi(\vec{h}))(\vec{h}) \;&=\; \sum_{k=0}^{r} h_k \int_{\phi(\frac{i}{m})+\xi_0(\vec{h})}^{\phi(\frac{i+r}{m})+\xi_r(\vec{h})} \frac{\partial\tilde{f}_i}{\partial h_k}(t,\xi(\vec{h}))\,dt\\
&=\; \sum_{k=0}^{r} O(h_k)O(\phi(\frac{i+r}{m})-\phi(\frac{i}{m})+\xi_r(\vec{h})-\xi_0(\vec{h}))O(\frac{\partial\tilde{f}_i}{\partial h_k}(t,\xi(\vec{h})))\;,
\end{aligned} \tag{1.62}$$

with $t\in\mathcal{I}_\xi = [\phi(\frac{i}{m})+\xi_0(\vec{h}), \phi(\frac{i+r}{m})+\xi_r(\vec{h})]$ and, where as in (1.55) $\vec{h}\in\bar{B}(0,\delta)$ and $\xi(\vec{h})\in\bar{B}(0,\delta)$ is positioned on the line between $\vec{0},\vec{h}\in\mathbb{R}^{r+1}$. By Taylor's Theorem $\phi(\frac{i+r}{m})-\phi(\frac{i}{m})=O(\frac{1}{m})$ and

$$|\xi_r(\vec{h})-\xi_0(\vec{h})| \;\le\; 2\|h\| \;=\; O(\frac{1}{m^{1+\varepsilon}})\;.$$

Similarly, for each $0\le l\le r$ we have $t-\phi(\frac{i+l}{m})-\xi_l(\vec{h})=O(\frac{1}{m})$ and thus as $t\in\mathcal{I}_\xi$ by (1.59) we have $\frac{\partial\tilde{f}_i}{\partial h_k}(t,\xi(\vec{h}))=O(\frac{1}{m^r})$. Hence the asymptotics in (1.62) coincides with

$$\boxed{D_h\chi_i(\xi(\vec{h}))(\vec{h}) \;=\; \textstyle\sum_{k=0}^{r} O(\frac{1}{m^{1+\varepsilon}})O(\frac{1}{m})O(\frac{1}{m^r}) \;=\; O(\frac{1}{m^{r+2+\varepsilon}})\;.} \tag{1.63}$$

Coupling (1.58) and (1.63) with (1.55) renders

$$\boxed{\chi_i(\vec{h}) \;=\; \begin{cases} O(\frac{1}{m^{r+2}}) & \text{if } r \geq 1 \text{ is odd ,} \\ O(\frac{1}{m^{r+2+\min\{1,\varepsilon\}}}) & \text{if } r \geq 2 \text{ is even .} \end{cases}} \tag{1.64}$$

Thus putting (1.64) into (1.54) and combining the latter with (1.52), we obtain (similarly to the uniform case; see also (1.53)) the following

$$\begin{aligned} \int_{t_{(j-1)r}}^{t_{jr}} (\|\dot{P}_r^j(t)\| - \|\dot{\gamma}(t)\|)\,dt \;&=\; \int_{t_{(j-1)r}}^{t_{jr}} \frac{\langle \dot{f}(t), \dot{\gamma}(t)\rangle}{d(\gamma)}\,dt \;+\; O(\frac{1}{m^{2r+1}}) \\ &= \;-\int_{t_{(j-1)r}}^{t_{jr}} \frac{\langle f(t), \ddot{\gamma}(t)\rangle}{d(\gamma)}\,dt + O(\frac{1}{m^{2r+1}}) \\ &= \;\begin{cases} O(\frac{1}{m^{r+2}}) & \text{if } r \geq 1 \text{ is odd ,} \\ O(\frac{1}{m^{r+2+\min\{1,\varepsilon\}}}) & \text{if } r \geq 2 \text{ is even .} \end{cases} \end{aligned}$$

Hence as $d(\tilde{\gamma}_r) = \Sigma_{j=0}^{\frac{m}{r}-1} d(P_j^r)$, we finally obtain

$$\boxed{d(\gamma) - d(\tilde{\gamma}_r) \;=\; \begin{cases} O(\frac{1}{m^{r+1}}) & \text{if } r \geq 1 \text{ is odd ,} \\ O(\frac{1}{m^{r+1+\min\{1,\varepsilon\}}}) & \text{if } r \geq 2 \text{ is even .} \end{cases}}$$

Clearly (1.3) combined with $T = \sum_{i=1}^{m}(t_i - t_{i-1})$ yield $\frac{1}{m} \leq \frac{\delta}{T}$. The latter substituted into so far calculated trajectory and length estimation expressed in $1/m$- asymptotics translates easily into δ-asymptotics. This completes the proof of Theorem 1.3. □

1.6 Experiments

All experiments in this work are performed in *Mathematica* on a 700 MHZ Pentium III with 384 MB RAM. Recall that as $T = \sum_{i=1}^{m}(t_i - t_i) \leq m\delta$ to verify asymptotics in terms of $O(\delta^{\alpha})$ (for $\alpha > 0$) it is sufficient to accomplish such task in terms of $O(\frac{1}{m^{\alpha}})$ asymptotics. For a given collection of interpolants $\tilde{\gamma}_{r,m}$ of $\mathcal{Q}_m$ (interpolant $\tilde{\gamma}_r$ depends also on m with m varying) the convergence order in length approximation is found as follows. From the set of *absolute errors*

$$E_m \;=\; |d(\gamma) - d(\tilde{\gamma}_{r,m})| \tag{1.65}$$

the estimate of convergence rate α is computed by applying *linear regression* to the pair of points $(\log(m), -\log(E_m))$, with $min_r \leq m \leq max_r$ and $m = rn$, where $n \geq 1$. Note that for $\mathcal{Q}_m$ number of interpolation points is $m+1$.

1.6.1 Pseudocode

We comment on pseudocode (in *Mathematica*) for implementing a piecewise-r-degree Lagrange interpolation scheme based on non-reduced data $(\mathcal{Q}_m, \{t_i\}_{i=0}^m)$.

Initialize $list1$ and $list2$ (containing $r+1$-tuples of points and knots from non-reduced data $(\mathcal{Q}_m^{i,r}, \{t_i\}_i^{i+r})$), where $i = lr$ to:

$$list1 := \{q_i, q_{i+1}, \ldots, q_{i+r}\}; \quad \text{here} \quad /* \, list1[k] = q_{i+k} \, */$$
$$list2 := \{t_i, t_{i+1}, \ldots, t_{i+r}\}; \quad \text{here} \quad /* \, list2[k] = t_{i+k} \, */$$

where $0 \leq k \leq r$. Note that for maintaining the *uniformity* with loop counters (used in herein outlined pseudocodes), we adopt in this monograph the convention of indexing any list (as well as the list of outputs returned by a procedure) starting from 0 instead of 1. In addition in order to avoid brackets' cluttering we strip one square bracket $[\ ]$ while writing a code for accessing a particular element of the list. In fact, upon lists' initialization from above, the accessing of q_{i+k} and of t_{i+k} is achieved with the following *Mathematica* code: $list1[[k+1]]$ and $list2[[k+1]]$, for $0 \leq k \leq r$. Note also that whenever "underscore" in the right-hand sides of $:=$ assignment is used, we do not resort to the *Mathematica* built-in functions, and it is left to the reader to complete a relevant simple code executing a particular task. The procedure for derivation a local Lagrange r-degree polynomial $\tilde{\gamma}_{i,r}$ (see (1.8)) and its length $d(\tilde{\gamma}_{i,r})$ (for $\mathcal{Q}_m^{i,r}$) is shown in Figure 1.10.

Consequently, the estimation of α appearing in $d(\tilde{\gamma}_r) - d(\gamma) = O(\delta^\alpha)$ and computed by linear regression from the collection of non-reduced data $\{(\mathcal{Q}_{jr}, \{t_i\}_{i=0}^{jr})\}_{j=m_1}^{m_2}$ (see Section 1.6), where $min_r = m_1 * r \leq m \leq m_2 * r = max_r$ (with $m_1 \geq 1$) is encoded in the main program loop shown in Figure 1.11.

Note that to execute the last command one needs to invoke *Mathematica statistical package* with the instruction: $\ll$*Statistics'LinearRegression;*. A slight modification of the main program loop, yields the pseudocode to plot $\tilde{\gamma}_r$, for a fixed $\mathcal{Q}_{j_{fix}}$. It requires calling procedure **LagKnotsKnown**(List1,List2)[1] (with the second list returned) and subsequently appending to the so-far computed values of the track-sum of $\tilde{\gamma}_{0,r}$, $\tilde{\gamma}_{r,r}, \ldots, \tilde{\gamma}_{(i-1)r,r}$ for $\mathcal{Q}_{j_{fix}r}^{0,r}, \mathcal{Q}_{j_{fix}r}^{r,r}, \ldots, \mathcal{Q}_{j_{fix}r}^{(i-1)r,r}$, the list of values of $\tilde{\gamma}_r$ for $\mathcal{Q}_{j_{fix}r}^{ir,r}$ $(0 \leq ir \leq j_{fix}r)$. The pseudocode for estimating α in $\tilde{\gamma}_r - \gamma = O(\delta^\alpha)$ with $\{t_i\}_{i=0}^m$ assumed to be known (or in $\tilde{\gamma}_r \circ \psi - \gamma = O(\delta^\alpha)$ for uniform case with $\{t_i\}_{i=0}^m$ unknown) is similar but involves more computationally expensive optimization procedure, though only one-dimensional one (see also Subsection 1.3.1) to determine the following (for each $min_r \leq m \leq max_r$): $\sup_{t\in[0,T]} |\tilde{\gamma}_r(t) - \gamma(t)|$ (or the $\sup_{t\in[0,T]} |\tilde{\gamma}_r \circ \psi(t) - \gamma(t)|$, where $\psi : [0,T] \to [0,\hat{T}]$ is a piecewise-C^∞ reparameterization - see Section 1.4).

1.6.2 Uniform samplings

We first discuss convergence of length estimates for piecewise Lagrange polynomial approximation based on reduced *uniform samplings* with $\hat{t}_i = i$. This addresses the issue of

$\mathbf{LagKnotsKnown}[list1\_, list2\_]$

$\{$

$Lag\_Int[t\_] := Lag\_Formula[list1, list2, r, t];$ /∗ see (1.8) ∗/

$Der\_Lag\_Int := D[Expand[Lag\_Int[t]], t];$

$Lag\_Der\_List := CoefficientList[Der\_Lag\_Int, t];$

$Norm[t\_] := Derivatives\_Norm[Lag\_Der\_List, t];$

$Length := NIntegrate[Norm[t], \{t, list2[0], list2[r]\}];$

/∗ see (1.1) over $[list2[0], list2[r]]$ ∗/

$Lag\_Plot\_List := ParametricPlot[Lag\_Int[t], \{t, list2[0], list2[r]\}];$

$\mathbf{return}\{Length, Lag\_Plot\_List\}$

$\}$

Fig. 1.10. Pseudocode for procedure **LagKnotsKnown**, which for two input lists $\{q_i\}_i^{i+r}$ and $\{t_i\}_i^{i+r}$, returns the list of $\{d(\tilde{\gamma}_{i,r}), plot\}$, where $plot$ represents a discrete set of $\tilde{\gamma}_{i,r}(t)$, for $t \in [t_i, t_{i+r}]$ (according to the ParametricPlot format)

Rys. 1.10. Pseudokod dla procedury **LagKnotsKnown**, która dla danych na wejściu (dwie listy $\{q_i\}_i^{i+r}$ i $\{t_i\}_i^{i+r}$) zwraca listę $\{d(\tilde{\gamma}_{i,r}), plot\}$, gdzie $plot$ jest listą dyskretnego zbioru wartości $\tilde{\gamma}_{i,r}(t)$ dla $t \in [t_i, t_{i+r}]$ (zgodnie z formatem ParametricPlot)

sharpness of convergence orders claimed by Theorem 1.1. Recall however, that by Remark 1.6 (as already verified in Figure 1.1(a-b)) interpolating the non-reduced data $(\mathcal{Q}_m, \{t_i\}_{i=0}^m)$ with $t_i = iT/m$ yields the same results as if $\hat{t}_i = i$ were use. Thus we only verify here reduced uniform data. Experiments are conducted here for *a semicircle* γ_{sc} (1.9) and for *a cubic* γ_c (1.25) with $r = 1, 2, 3, 4$, for which Theorem 1.1 asserts errors to be $O(\delta^2)$, $O(\delta^4)$, $O(\delta^4)$, and $O(\delta^6)$, respectively. The lengths of both curves read $d(\gamma_{sc}) = \pi$ and $d(\gamma_c) = 3.3452$. Let $\tilde{\gamma}_{r,m_r}$ represent a piecewise-r-degree polynomial interpolating $m_r = rn + 1$ points and let $E_{min_r}^{max_r} = \max_{min_r \leq m_r \leq max_r} E_{m_r-1}$. As before, for each r, the estimate of convergence rate α is found by applying a linear regression to the pairs of points $(\log(m_r - 1), -\log(E_{m_r-1}))$, where $min_r \leq m_r \leq max_r$.

Both Tables 1.1 and 1.2 indicate that (in these cases at least) the statements in Theorem 1.1 for $d(\gamma)$ estimation are sharp. The last two rows of Table 1.2 are somewhat irrelevant in that Lagrange interpolation returns, for $r \geq 3$, the same curve γ_c, *up to machine precision*.

1.6.3 Non-uniform admissible samplings

We confirm now experimentally *the sharpness* of Theorems 1.2 and 1.3 and Remark 1.4 for $r = 2, 3, 4$ and $n = 2$, where tabular points $\{t_i\}_{i=0}^m$ are assumed to be *known*. In doing so, the $\{t_i\}_{i=0}^m$ are sampled according to (1.24)(i) and the respective tests are performed for different planar curves *i.e. a cubic* γ_c (1.25) (for $r = 2$), *a semicircle* γ_{sc} (1.9) (for

```
For  [j = m_1; err = { }, j ≤ m_2, j = j + 1,
     Data[j] := {q_0, q_1, q_2, ..., q_jr};
     Knots[j] := {t_0, t_1, t_2, ..., t_jr};
     m_3 = j * r;
     For  [i = 0; length = 0, i ≤ m_3 - r, i = i + r,
          List1 := {Data[j][i], Data[j][i + 1], ..., Data[j][i + r]};
          List2 := {Knots[j][i], Knots[j][i + 1], ..., Knots[j][i + r]};
          length+ = LagKnotsKnown[List1, List2][0];
          ];
     err := AppendTo[err, {Log[m_3], -Log[|length - d(γ)|]}];
     ];
α := Slope_Coeff[Regress[err]];                          / * see Section1.6 * /
```

Fig. 1.11. Pseudocode for the main program loop to compute the estimate of α appearing in $d(\tilde{\gamma}_r) - d(\gamma) = O(\delta^\alpha)$ based on collection of non-reduced data $\{(\mathcal{Q}_{jr}, \{t_i\}_{i=0}^{jr})\}_{j=m_1}^{m_2}$

Rys. 1.11. Pseudokod głównej pętli programu obliczającej oszacowanie α w $d(\tilde{\gamma}_r) - d(\gamma) = O(\delta^\alpha)$ na podstawie rodziny danych niezredukowanych $\{(\mathcal{Q}_{jr}, \{t_i\}_{i=0}^{jr})\}_{j=m_1}^{m_2}$

$r = 3$) and *a quintic curve* $\gamma_{q_5} : [0, 1] \to \mathbb{R}^2$ defined as

$$\gamma_{q_5}(t) = (\pi t, (\pi t + 1)^5(\pi + 1)^{-5}) , \tag{1.66}$$

(for $r = 4$). Different values of $\varepsilon = 2, 1, 3/4, 1/2, 1/3, 1/10, 1/100, 0$ are considered to test sharpness of the Theorem 1.3. The corresponding lengths of curves in question are $d(\gamma_c) = 3452$, $d(\gamma_{sc}) = \pi$ and $d(\gamma_{q_5}) = 3.4319$. Note, that cubic curve γ_c is replaced for $r = 3, 4$ by γ_{sc} and γ_{q_5} as otherwise a piecewise-cubic or piecewise-quartic Lagrange interpolant $\tilde{\gamma}$ used with t_i known coincide with γ_c thus yielding error equal zero. As previously, a linear regression yielding computed $\alpha^m_{\gamma,r}$ is applied here to the collection of points $(\log(m), -\log(E_m))$, with $m_r^{\min} \le m \le m_r^{\max}$ (see (1.65)). Here $m_2^{\max} = 2$ and $m_2^{\min} = 200$, $m_3^{\max} = 198$ and $m_3^{\min} = 3$ and $m_4^{\max} = 40$ and $m_4^{\min} = 4$. The results for $r = 2, 3, 4$ are shown in Table 1.3. Evidently, as illustrated in Table 1.3, the computed approximation orders $\alpha^m_{\gamma,r}$ for length estimation nearly coincide with those minimal rates α_r claimed by Theorem 1.3. The computed estimates slightly differ from (1.26) with $r = 4$. A smaller number of interpolation points was considered here (*i.e.* $m = 40$) before reaching machine precision while computing the errors in lengths. Of course, the asymptotic nature of theorems established herein requires m to be sufficiently large.

Note also that the last column of Table 1.3 confirm sharpness of Theorem 1.2 at least for $r = 2, 3, 4$ and $n = 2$. Upon noting that sampling (1.24)(i) for $\varepsilon = 0$ is more-or-less uniform with $K_u = \frac{5}{3}$ and $K_l = \frac{1}{3}$ (see Definition 1.5) again the last column of Table 1.3 verifies sharpness of Remark 1.4, at least for $r = 2, 3, 4$ and $n = 2$.

r	min_r	max_r	computed α	$\alpha_{Th.1.1}$	$E_{min_r}^{max_r}$	E_{max_r-1}
1	7	101	1.99	2.00	0.0357641	1.29191×10^{-4}
2	7	101	3.98	4.00	0.0036266	5.09874×10^{-8}
3	7	100	3.98	4.00	0.0026509	3.98087×10^{-8}
4	9	101	5.99	6.00	0.0001136	3.19167×10^{-11}

Tab. 1.1. Results for length estimation $d(\gamma_{sc})$ by $d(\tilde{\gamma}_r)$, with $\hat{t}_i = i$ (for $r = 1, 2, 3, 4$ and uniform sampling), of *a semicircle* γ_{sc} (1.9)

r	min_r	max_r	computed α	$\alpha_{Th.1.1}$	$E_{min_r}^{max_r}$	E_{max_r-1}
1	7	101	2.00	2.00	0.0357641	5.18348×10^{-6}
2	7	201	4.10	4.00	0.0036266	1.22657×10^{-12}
3	7	100	n/a[a]	4.00	5.90639×10^{-14}[a]	4.44089×10^{-16}[a]
4	9	101	n/a[a]	6.00	2.73115×10^{-13}[a]	4.44089×10^{-16}[a]

[a] not applicable (see above)

Tab. 1.2. Results for length estimation $d(\gamma_c)$ by $d(\tilde{\gamma}_r)$, with $\hat{t}_i = i$ (for $r = 1, 2, 3, 4$ and uniform sampling), of *a cubic curve* γ_c (1.25)

Remark 1.8. Recall that although condition (1.51) holds within $\mathcal{V}_\varepsilon^m$ and $\mathcal{V}_{mol}^m$ (see Remark 1.7) it may not be satisfied by the whole $\mathcal{V}_G^m$ (see Example 1.2). Hence (1.51) is assumed explicitly only in Theorem 1.2. A piecewise-quadratic Lagrange interpolation applied to γ_c (1.25) and non-reduced sampling (1.17), for which (1.51) does not hold, renders convergence order for length estimation $\alpha = 2.23$ which is less than minimal $\alpha = 3$ predicted by Theorem 1.2 for $r = 2$. Thus (1.51) is a necessary condition for Theorem 1.2 to hold. However, it remains an open problem whether regularity of γ is necessary as all experiments indicate that for a non-regular curve γ the claims of Theorem 1.2 still prevail. Of course, the assumption of the regularity of γ is vital for the presented herein proof of Theorem 1.2 and as such is not abandoned.

1.7 Discussion and motivation for Chapter 2

In this chapter we introduced uniform, ε-uniform and more-or-less uniform samplings all contained in the family of the general class of admissible samplings. Piecewise-r-degree Lagrange interpolation for trajectory and length estimation in $\mathbb{R}^n$ is discussed here including investigation of approximation orders for the above samplings $\mathcal{V}_U^m$, $\mathcal{V}_\varepsilon^m$ and $\mathcal{V}_G^m$ (see

	computed $\alpha^m_{\gamma,r}$ for $\gamma = \gamma_{sc}, \gamma_c, \gamma_{q_5}$								
ε	2	1	$3/4$	$1/2$	$1/3$	$1/10$	$5/100$	$1/100$	0
$\alpha^{200}_{\gamma_c,2}$	4.00	4.01	3.77	3.48	3.32	3.09	3.04	3.00	3.00
$\alpha_{2,Th.1.3}$	4.00	4.00	3.75	3.50	3.33	3.10	3.05	3.01	3.00
$\alpha^{198}_{\gamma_{sc},3}$	3.99	4.00	4.02	4.02	4.04	4.02	3.96	3.93	3.92
$\alpha_{3,Th.1.3}$	4.00	4.00	4.00	4.00	4.00	4.00	4.00	4.00	4.00
$\alpha^{40}_{\gamma_{q_5},4}$	6.09	5.72	5.76	5.52	5.23	5.07	5.06	5.02	5.01
$\alpha_{4,Th.1.3}$	6.00	6.00	5.75	5.50	5.33	5.10	5.05	5.01	5.00

Tab. 1.3. $d(\gamma)$ estimation by $d(\tilde{\gamma}_r)$ (for $r = 2, 3, 4$), with $\{t_i\}_{i=0}^m$ known as in $(1.24)(i)$ for $\gamma = \gamma_{sc}, \gamma_c, \gamma_{q_5}$, defined by (1.9), (1.25) and (1.66), respectively

Definitions 1.2 and 1.4) forming first *non-reduced set of data* $(\mathcal{Q}_m, \{t_i\}_{i=0}^m)$. All results established in this chapter appear to be sharp (at least for $n = 2$, $r = 1, 2, 3, 4$ and $d(\gamma)$ estimation). The assumption $m\delta = O(1)$ for Theorem 1.2 is shown experimentally to be important (see Remark 1.8). We also showed potential difficulties which may arise for *reduced data* $\mathcal{Q}_m$ if unknown knot parameters $\{t_i\}_{i=0}^m$ are approximated blindly by $\{\hat{t}_i\}_{i=0}^m$ (*i.e.* $\hat{t}_i = i$). Before dealing properly with the latter and related Question I from Subsection 1.3.2 (see Chapter 3 onward), the next chapter analyzes in more detail the interpolation of reduced data for $r = 2$, where the unknown knots $\{t_i\}_{i=0}^m$ are replaced by inappropriate equidistant approximation $\hat{t}_i = i$. The following problem is next tackled:

What are the convergence orders for γ and $d(\gamma)$ estimation, if piecewise-quadratic Lagrange interpolation is used with guessed uniform $\hat{t}_i = i$ of $\{t_i\}_{i=0}^m$ in fitting reduced data $\mathcal{Q}_m$?

The answer and theoretical discussion to the above question is given in the Chapter 2.

Chapter 2

Uniform piecewise-quadratics

Abstract

Orders of convergence for uniform piecewise-quadratic Lagrange interpolation $\tilde{\gamma}_2$ *to fit both* ε*-uniform and more-or-less uniform samplings are established for reduced data* $\mathcal{Q}_m$ *in* $\mathbb{R}^n$ *with the improper guess of knot parameters, chosen as uniform* $\hat{t}_i = i$ *- see Theorems 2.1 and 2.2. The asymptotics established herein show a substantial deceleration (up to the "divergence"*[1]*) in approximation orders derived to estimate* γ *and* $d(\gamma)$*. The sharpness (or nearly sharpness) of these results are confirmed experimentally (at least for estimating length of planar curves). Part of this work is published in [46] and [47].*

2.1 Example and main results

In this section we first show that the problem of incorrect choices of $\{\hat{t}_i\}_{i=0}^m \approx \{t_i\}_{i=0}^m$ (signaled already in Example 1.1) to estimate the unknown trajectory γ and $d(\gamma)$ extends to the case of m large *i.e. to the reduced dense data* (see Example 2.1). Evidently, this has an undesirable impact on the corresponding approximation orders, subsequently claimed in Theorem 2.1. As proved in the latter, for the uniform guesses $\hat{t}_i = i$ of $\{t_i\}_{i=0}^m$ the corresponding asymptotic results can still be derived, at least for ε-uniform samplings. Thus Theorem 2.1 clarifies the slowing effects in convergence orders (with $\varepsilon \to 0$) and the existing *convergence versus "divergence" duality* (for $\varepsilon = 0$) illustrated in Example 2.1.

Example 2.1. Consider now two curves: a semicircle γ_{sc} (see (1.9)) and a cubic curve γ_c (see (1.25)). Figure 2.1 shows a good performance of piecewise-quadratic Lagrange interpolant $\tilde{\gamma}_2$ to estimate γ, derived for $\{t_i\}_{i=0}^m$ unknown and $\hat{t}_i = i$. Here $m = 20$ and γ_{sc}, γ_c are sampled according to both (1.24) (with $\varepsilon = 0$). On the other hand, the estimation of $d(\gamma)$ is severely crippled. Indeed, for $20 \leq m \leq 200$ a linear regression applied to the pair of points $(\log(m), -\log(E_m))$ (see (1.65)) yields slower convergence rates (see Table 2.1) than $\alpha = 3$ claimed by Theorem 1.2 and established for $\{t_i\}_{i=0}^m$ known and with

[1] In fact, $\lim_{\delta \to 0} d(\tilde{\gamma}_2)$ exists but $d(\tilde{\gamma}_2) \not\to d(\gamma)$.

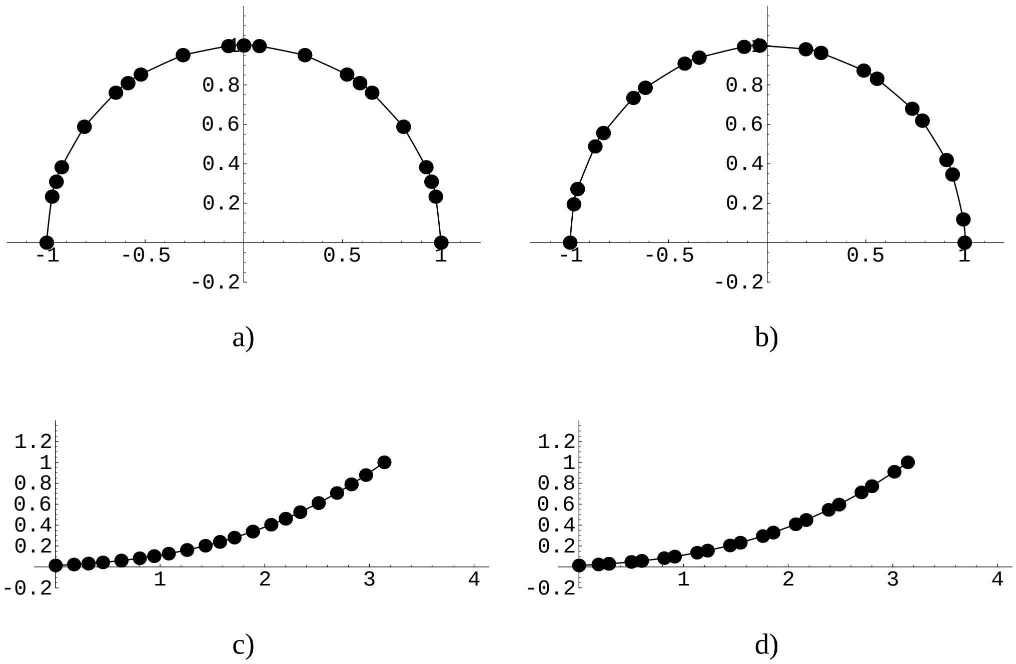


Fig. 2.1. Interpolation with *uniform piecewise-quadratics* $\tilde{\gamma}_2$ (solid) of reduced data $\mathcal{Q}_{20}$ (dotted) with $\hat{t}_i = i$, for: a) *a semicircle* γ_{sc} (1.9) and sampling (1.24)(ii), b) *a semicircle* γ_{sc} and sampling (1.24)(i), c) *a cubic* γ_c (1.25) and sampling (1.24)(ii), d) *a cubic* γ_c and sampling (1.24)(i). Here $\varepsilon = 0$

Rys. 2.1. Interpolacja *przedziałowo-kwadratowymi funkcjami sklejanymi* $\tilde{\gamma}_2$ (linia ciągła) danych zredukowanych $\mathcal{Q}_{20}$ (wytłuszczone punkty) z $\hat{t}_i = i$, dla: a) *półokręgu* γ_{sc} (1.9) oraz próbkowania (1.24)(ii), b) *półokręgu* γ_{sc} oraz próbkowania (1.24)(i), c) *krzywej kubicznej* γ_c (1.25) oraz próbkowania (1.24)(ii), d) *krzywej kubicznej* γ_c oraz próbkowania (1.24)(i). Przyjęto $\varepsilon = 0$

$r = 2$. Note that Table 2.1 shows an even more disturbing phenomenon. Namely, for both curves γ_{sc}, γ_c and sampling $(1.24)(i)$ there is a “divergence” in length estimation. By the latter (as already mentioned) we understand $\lim_{m\to\infty} d(\tilde{\gamma}_{2,m}) = g \neq d(\gamma)$. For comparison, Table 2.2 shows the corresponding computed convergence orders obtained for non-reduced data $(\mathcal{Q}_m, \{t_i\}_{i=0}^m)$ (*i.e.* with $\{t_i\}_{i=0}^m$ known). As expected they are faster then those in Table 2.1. Note also that the last column of Table 2.2 confirms experimentally the sharpness of Theorem 1.2 (for $r = 2$). ☐

As shown in Example 2.1, Lagrange estimates of length can behave badly at least for 0-uniform samplings and $r = 2$. To justify such ill-behavior (see Theorem 2.2) we first establish the following supplementary result (see [46]) covering ε-uniform samplings only for $\varepsilon > 0$:

curves:	semicircle γ_{sc} (1.9)		cubic curve γ_c (1.25)	
samplings	$(1.24)(ii)$	$(1.24)(i)$	$(1.24)(ii)$	$(1.24)(i)$
rates $\alpha_{d(\gamma)}$	1.44	$d(\tilde{\gamma}_{2,m}) \not\to d(\gamma_{sc})$	1.99	$d(\tilde{\gamma}_{2,m}) \not\to d(\gamma_c)$
errors: E_{200}	3.45x10^{-4}	0.1288	6.36x10^{-8}	0.1364

Tab. 2.1. $d(\gamma)$ estimation (for γ_{sc}, γ_c) by $d(\tilde{\gamma}_2)$ with $\{t_i\}_{i=0}^m$ unknown $\hat{t}_i = i$. For (1.24) $\varepsilon = 0$

curves:	semicircle γ_{sc} (1.9)		cubic curve γ_c (1.25)	
samplings	$(1.24)(ii)$	$(1.24)(i)$	$(1.24)(ii)$	$(1.24)(i)$
rates $\alpha_{d(\gamma)}$	3.99	4.02	3.99	2.99
$\alpha_{2,Th.1.2}$	3.00	3.00	3.00	3.00

Tab. 2.2. $d(\gamma)$ estimation (for γ_{sc}, γ_c) by $d(\tilde{\gamma}_2)$ with $\{t_i\}_{i=0}^m$ known. For (1.24) $\varepsilon = 0$

Theorem 2.1. *Let the unknown $\{t_i\}_{i=0}^m$ be sampled ε-uniformly, where $\varepsilon > 0$, and suppose that $\gamma \in C^4$. Then, there is a uniform piecewise-quadratic Lagrange interpolant*[2] *$\tilde{\gamma}_2 : [0, \hat{T}] :\to \mathbb{R}^n$ calculable in terms of $\mathcal{Q}_m$ and a piecewise-C^∞ reparameterization*[3] *$\psi : [0, T] \to [0, \hat{T}]$ such that*

$$\boxed{\tilde{\gamma}_2 \circ \psi \;=\; \gamma \;+\; O(\delta^{\min\{3,1+2\varepsilon\}}) \quad and \quad d(\tilde{\gamma}_2) \;=\; d(\gamma) \;+\; O(\delta^{\min\{4,4\varepsilon\}})\;.} \tag{2.1}$$

Whereas Theorem 1.1 permits length estimates of arbitrary accuracy (for r sufficiently large), Theorem 2.1 refers only to piecewise-quadratic estimates (*i.e.* $r = 2$), and accuracy is limited accordingly. More specifically, for the reduced data $\mathcal{Q}_m$ with $\hat{t}_i = i$, the smaller ε is the smaller convergence rates for γ and $d(\gamma)$ estimation result. Note also that, the case when $\varepsilon \geq 1$ coincides with the uniform one (see Theorem 1.1). Indeed, for $\{t_i\}_{i=0}^m \in \mathcal{V}_\varepsilon^m$ any perturbation of order $O(1/m^\beta)$, with $\beta \geq 1$ is “negligible” as it renders the same convergence orders as for $\mathcal{V}_U^m$. The sharpness (or nearly sharpness) of Theorem 2.1 is confirmed experimentally (at least for $n = 2$ and $d(\gamma)$ estimation) in Section 2.4 (see Tables 2.3 and 2.4).

As mentioned before, Theorem 2.1 has noting to say about the most interesting case *i.e.* when ε vanishes. This represents the biggest distortion of the uniform case (up to the

[2]See subsection 2.2 for details or Definition 1.1.
[3]See subsection 2.2 for details.

reparameterization ϕ) which may result in problems already highlighted in Example 2.1. Unfortunately, the proof of Theorem 2.1 (see Section 2.2) cannot be directly extended to $\varepsilon = 0$. We shall establish now the negative result for reduced data $\mathcal{Q}_m$ satisfying *more--or-less uniformity*. Evidently, the latter applies also to those 0-*uniform samplings* which satisfy (1.28).

Theorem 2.2. *Let the unknown $\{t_i\}_{i=0}^m$ be sampled more-or-less uniformly, and suppose that $\gamma \in C^4$. Then, there exists a uniform piecewise-quadratic Lagrange interpolant*[4] *$\tilde{\gamma}_2 : [0, \hat{T}] \to \mathbb{R}^n$ calculable in terms of $\mathcal{Q}_m$ and a piecewise-C^∞ reparameterization*[5] *$\psi : [0, T] \to [0, \hat{T}]$ such that*

$$\boxed{\tilde{\gamma}_2 \circ \psi \;=\; \gamma \;+\; O(\delta) \quad \textit{and} \quad d(\tilde{\gamma}_2) \;=\; d(\gamma) \;+\; O(1)\;.} \tag{2.2}$$

Clearly, the second formula in (2.2) justifies the *convergence versus "divergence" duality* illustrated in Example 2.1 - recall that both samplings from (1.24) are more-or-less and 0-uniform (for $\varepsilon = 0$). Example 2.1 shows also the sharpness of Theorem 2.2, at least for length estimation and $n = 2$. Note that as condition (1.51) is automatically satisfied by $\mathcal{V}_\varepsilon^m$ and $\mathcal{V}_{mol}^m$, both Theorems 2.1 and 2.2 do not invoke it explicitly.

Remark 2.1. Once can expect a similar effect of bad approximation property for all $r \geq 2$ when $\hat{t}_i = i$. The only exception stems from the case $r = 1$ (*i.e.* for *a piecewise-linear interpolation*). As explained in Remark 5.2, for $\hat{t}_i = i$ the corresponding approximation orders for trajectory and length estimation coincide with those derived for $\{t_i\}_{i=0}^m$ known which are of order $O(\delta^2)$. Piecewise-linear interpolation yields however a bad interpolant, with big discontinuities of derivatives at the consecutive knot points (in particular for sporadic data) and does not approximate the curvature of γ. In addition, in this monograph we introduce non-parametric interpolants with faster convergence rates than $O(\delta^2)$. For that reason the applicability of piecewise-linear Lagrange interpolation is limited.

In the next Sections we prove Theorems 2.1 and 2.2.

2.2 $\mathcal{E}$-uniform reduced samplings - asymptotics

In this section we prove the Theorem 2.1 (see also [46]).

Proof. Let $\gamma : [0, T] \to \mathbb{R}^n$ and its reparameterizations are at least C^4. Recall here, that as for proving Theorem 1.3, without loss we may assume that $[0, T] = [0, 1]$ with constant velocity $\|\dot{\gamma}\| = d(\gamma)$ (see Chapter 1; Proposition 1.1.5 of [26]). Fix $0 < \varepsilon \leq 1$, and let the $\{t_i\}_{i=0}^m$ be sampled ε-uniformly. We are going to prove Theorem 2.1. Without loss of generality m is even. For each triple $\mathcal{Q}_m^{i,2} = (q_i, q_{i+1}, q_{i+2})$, where $0 \leq i \leq m-2$, let

[4] See subsection 2.2 for details or Defintion 1.1.
[5] See subsection 2.2 for details.

$\tilde{\gamma}_{i,2} : [0,2] \to \mathbb{R}^n$ be the quadratic curve (expressed in local parameter $s \in [0,2]$; see also (1.8) with $r = 2$) satisfying

$$\boxed{\tilde{\gamma}_{i,2}(0) \;=\; q_i \;, \quad \tilde{\gamma}_{i,2}(1) \;=\; q_{i+1} \;, \quad \text{and} \quad \tilde{\gamma}_{i,2}(2) \;=\; q_{i+2} \;.} \tag{2.3}$$

Write

$$\boxed{\tilde{\gamma}_{i,2}(s) \;=\; a_0 \;+\; a_1 s \;+\; a_2 s^2 \;,}$$

where $s \in [0,2]$. Let $\tilde{\gamma}_2 : [0,m] \to \mathbb{R}^n$ be the track-sum of $\{\tilde{\gamma}_{2i,2}\}_{i=0}^{\frac{m}{2}-1}$ (see Remark 1.1). We shall examine now the asymptotics of the derivatives of $\tilde{\gamma}_{i,2}$. In doing so note that by (2.3) we obtain

$$a_0 \;=\; q_i \;, \quad a_1 \;=\; \frac{4q_{i+1} - 3q_i - q_{i+2}}{2} \;, \quad \text{and} \quad a_2 \;=\; \frac{q_{i+2} - 2q_{i+1} + q_i}{2} \;. \tag{2.4}$$

By Taylor's Theorem

$$\gamma(t_q) \;=\; \gamma(t_i) \;+\; \dot{\gamma}(t_i)(t_q - t_i) \;+\; \frac{\ddot{\gamma}(\xi_q)}{2}(t_q - t_i)^2 \;,$$

for either $q = i+1$ or $q = i+2$ and some $t_i < \xi_q < t_q$. Combining the latter with $\gamma(t_i) = q_i$, $\gamma(t_{i+1}) = q_{i+1}$, $\gamma(t_{i+2}) = q_{i+2}$ and substituting into (2.4) yields

$$a_2 \;=\; \frac{\dot{\gamma}(t_i)}{2}(t_{i+2} - 2t_{i+1} + t_i) \;+\; O((t_{i+1} - t_i)^2) \;+\; O((t_{i+2} - t_i)^2) \;. \tag{2.5}$$

Because sampling is ε-uniform the Mean Value Theorem gives

$$t_q - t_i \;=\; \phi'(\eta_q)\frac{q-i}{m} \;+\; O(\frac{1}{m^{1+\varepsilon}}) \;=\; O(\frac{1}{m}) \;, \tag{2.6}$$

for either $q = i+1$ or $q = i+2$ and some $t_i < \eta_q < t_q$. Thus by (2.5) and (2.6)

$$a_2 \;=\; \frac{t_{i+2} - 2t_{i+1} + t_i}{2}\dot{\gamma}(t_i) \;+\; O(\frac{1}{m^2}) \;. \tag{2.7}$$

Furthermore, up to a term $O(1/m^{1+\varepsilon})$,

$$t_{i+2} - 2t_{i+1} + t_i \;=\; \phi(\frac{i+2}{m}) - \phi(\frac{i+1}{m}) - (\phi(\frac{i+1}{m}) - \phi(\frac{i}{m})) \;, \tag{2.8}$$

because sampling is ε-uniform. By Taylor's Theorem the following holds

$$\phi(\frac{i+1}{m}) \;=\; \phi(\frac{i}{m}) \;+\; \dot{\phi}(\frac{i}{m})\frac{1}{m} \quad \text{and} \quad \phi(\frac{i+2}{m}) \;=\; \phi(\frac{i}{m}) \;+\; \dot{\phi}(\frac{i}{m})\frac{2}{m} \;, \tag{2.9}$$

up to a $O(1/m^2)$ term. Substituting (2.9) into (2.8) renders

$$t_{i+2} - 2t_{i+1} + t_i \;=\; O(\frac{1}{m^{\min\{2,1+\varepsilon\}}}) \; . \tag{2.10}$$

The latter combined with (2.7) yields

$$a_2 \;=\; O(\frac{1}{m^{\min\{2,1+\varepsilon\}}}) \;+\; O(\frac{1}{m^2}) \;=\; O(\frac{1}{m^{\min\{2,1+\varepsilon\}}}) \; . \tag{2.11}$$

A similar argument results in

$$a_1 \;=\; \frac{4t_{i+1} - 3t_i - t_{i+2}}{2}\dot{\gamma}(t_i) \;+\; O(\frac{1}{m^2}) \;=\; O(\frac{1}{m}) \; . \tag{2.12}$$

From (2.11) and (2.12) we get

$$\boxed{\frac{d\tilde{\gamma}_{i,2}}{ds} \;=\; a_1 \;+\; 2sa_2 \;=\; O(\frac{1}{m}) \quad \text{and} \quad \frac{d^2\tilde{\gamma}_{i,2}}{ds^2} \;=\; 2a_2 \;=\; O(\frac{1}{m^{\min\{2,1+\varepsilon\}}}) \; ,} \tag{2.13}$$

with $s \in [0,2]$ being of order $O(1)$.

For the need of comparison between curves γ and $\tilde{\gamma}_{i,2}$ we *reparameterize* now $\tilde{\gamma}_{i,2}$ so that both γ and $\tilde{\gamma}_{i,2}$ are defined over the same domain $[t_i, t_{i+2}]$. In doing so, let the function $\psi^i : [t_i, t_{i+2}] \to [0,2]$ be the quadratic (see (1.8) with $r = 2$)

$$\boxed{\psi^i(t) \;=\; b_0 \;+\; b_1 t \;+\; b_2 t^2}$$

satisfying

$$\boxed{\psi^i(t_i) \;=\; 0 \; , \quad \psi^i(t_{i+1}) \;=\; 1 \; , \quad \text{and} \quad \psi^i(t_{i+2}) \;=\; 2 \; .}$$

Of course, the latter yields, for $k = 0, 1, 2$

$$b_0 \;+\; b_1 t_{i+k} \;+\; b_2 t_{i+k}^2 \;=\; k \; . \tag{2.14}$$

Inspection reveals that the first two equations of (2.14) give

$$b_1 \;=\; (t_{i+1} - t_i)^{-1} \;-\; b_2(t_{i+1} + t_i) \; , \tag{2.15}$$

which combined with the last two equations of (2.14) renders

$$b_2 \;=\; \frac{(t_{i+1} - t_i) - (t_{i+2} - t_{i+1})}{(t_{i+1} - t_i)(t_{i+2} - t_{i+1})(t_{i+2} - t_i)} \; .$$

Furthermore, as before, by (2.9) and ε-uniformity

$$(t_{i+1} - t_i) - (t_{i+2} - t_{i+1}) = O(\frac{1}{m^{\min\{2,1+\varepsilon\}}}) ,$$

and $m^3(t_{i+1}-t_i)(t_{i+2}-t_{i+1})(t_{i+2}-t_i) = O(1)$, where the right-hand side of the latter is bounded away from 0 (as ϕ is a diffeomorphism defined over a compact set $[0,1]$). Note that we use here the assumption that $\varepsilon > 0$. Hence,

$$b_2 = O(m^{\max\{1,2-\varepsilon\}}) \quad \text{and} \quad \boxed{\ddot{\psi}^i(t) = 2b_2 = O(m^{\max\{1,2-\varepsilon\}}) .} \tag{2.16}$$

Similarly, as easily verified $(t_{i+1} - t_i)^{-1} = O(m)$. Hence, coupling (2.15) with (2.16) yields for $\dot{\psi}^i(t) = b_1 + 2b_2 t = O(m) + b_2(2t - (t_{i+1} + t_i))$ and thus

$$\boxed{\dot{\psi}^i(t) = O(m) + O(m^{\{0,1-\varepsilon\}}) = O(m) ,} \tag{2.17}$$

as sampling is ε-uniform and $2t - (t_{i+1} + t_i) = O(1/m)$, for $t \in [t_i, t_{i+2}]$. In particular, ψ is *a diffeomorphism* for m large. Similarly we define $\psi : [0,1] \to [0,m]$ as a track-sum of $\{\psi^{2i}\}_{i=0}^{\frac{m}{2}-1}$ (see Remark 1.1).

Define now a reparameterized curve $\tilde{\gamma}_{i,2}$ as

$$\boxed{\tilde{\gamma}_i = \tilde{\gamma}_{i,2} \circ \psi^i : [t_i, t_{i+2}] \to \mathbb{R}^n .} \tag{2.18}$$

Then $\tilde{\gamma}_i$ is polynomial of degree at most 4. Its derivatives of order $1 \leq p \leq 4$, are $O(m^{\max\{0,(p-1)(1-\varepsilon)\}})$. Indeed, (2.13), (2.16), (2.17), $\deg(\psi) \leq 2$ and $\deg(\tilde{\gamma}_{i,2}) \leq 2$ combined with the Chain Rule yield

$$\dot{\tilde{\gamma}}_i = \tilde{\gamma}_{i,2}' \dot{\psi}^i = O(\frac{1}{m})O(m) = O(1) ,$$

and

$$\begin{aligned} \ddot{\tilde{\gamma}}_i &= \tilde{\gamma}_{i,2}'' \dot{\psi}^{i\,2} + \tilde{\gamma}_{i,2}' \ddot{\psi}^i = O(\frac{1}{m^{\min\{2,1+\varepsilon\}}})O(m^2) + O(\frac{1}{m})O(m^{\max\{1,2-\varepsilon\}}) \\ &= O(m^{\max\{0,1-\varepsilon\}}) , \end{aligned}$$

and

$$\begin{aligned} \tilde{\gamma}_i^{(3)} &= 3\tilde{\gamma}_{i,2}'' \dot{\psi}^i \ddot{\psi}^i = O(\frac{1}{m^{\min\{2,1+\varepsilon\}}})O(m)O(m^{\max\{1,2-\varepsilon\}}) \\ &= O(m^{\max\{0,2-2\varepsilon\}}) , \end{aligned} \tag{2.19}$$

and

$$\begin{aligned}\tilde{\gamma}_i^{(4)} &= 3\tilde{\gamma}''_{i,2}\ddot{\psi}^{i\,2} = O(\frac{1}{m^{\min\{2,1+\varepsilon\}}})O(m^{\max\{2,4-2\varepsilon\}}) \\ &= O(m^{\max\{0,3-3\varepsilon\}}) \ . \end{aligned} \tag{2.20}$$

Then the *difference* between functions $\tilde{\gamma}_i$ and γ

$$\boxed{f = \tilde{\gamma}_i - \gamma : [t_i, t_{i+2}] \to \mathbb{R}^n}$$

is C^4 and satisfies

$$f(t_i) = f(t_{i+1}) = f(t_{i+2}) = \vec{0} \ . \tag{2.21}$$

By (2.19), (2.20), and ε-uniformity we have

$$\frac{d^3 f}{dt^3} = O(m^{\max\{0,2(1-\varepsilon)\}}) \quad \text{and} \quad \frac{d^4 f}{dt^4} = O(m^{\max\{0,3(1-\varepsilon)\}}) \ . \tag{2.22}$$

Use Lemma 1.1 to write

$$f(t) = (t-t_i)(t-t_{i+1})(t-t_{i+2})h(t) \ , \tag{2.23}$$

where $h : [t_i, t_{i+2}] \to \mathbb{R}^n$ is C^1, respectively. Then again by Lemma 1.1 and (2.22) we have

$$h = O(\frac{d^3 f}{dt^3}) = O(m^{\max\{0,2(1-\varepsilon)\}}) \ . \tag{2.24}$$

Hence by (2.23) and ε-uniformity we obtain

$$\boxed{f = O(\tfrac{1}{m^3})O(m^{\max\{0,2(1-\varepsilon)\}}) = O(\tfrac{1}{m^{\min\{3,1+2\varepsilon\}}}) \ .} \tag{2.25}$$

Clearly, the latter proves the first formula in (2.1) claimed by Theorem 2.1. Note that here, as in the end of the proof of Theorem 1.3 we have $1/m \leq \delta$ (recall $T = 1$) and thus (2.25) can be expressed in the similar δ-asymptotics.

To prove the first formula in (2.1) recall that (1.39) coupled with (2.22) renders

$$\dot{h} = O(m^{\max\{0,3(1-\varepsilon)\}}) \ . \tag{2.26}$$

The latter combined with the ε-uniformity yields

$$\begin{aligned} \dot{f} &= ((t-t_i)(t-t_{i+1}) + (t-t_i)(t-t_{i+2}) + (t-t_{i+1})(t-t_{i+2}))h(t) \\ &\quad + (t-t_i)(t-t_{i+1})(t-t_{i+2})\dot{h}(t) \\ &= O(\frac{1}{m^2})O(m^{\max\{0,2(1-\varepsilon)\}}) \;+\; O(\frac{1}{m^3})O(m^{\max\{0,3(1-\varepsilon)\}}) \\ &= O(\frac{1}{m^{\min\{2,2\varepsilon\}}}) \;. \end{aligned} \tag{2.27}$$

As in the proof of Theorem 1.1 define $V_{\dot{\gamma}}^{\perp}(t)$ to be the orthogonal complement to the space spanned by $\dot{\gamma}(t)$. Then expand $\dot{\tilde{\gamma}}_i(t)$ according to (1.41), where $v(t)$ is the orthogonal projection of $\dot{\tilde{\gamma}}_i(t)$ onto $V_{\dot{\gamma}}^{\perp}(t)$. Since $\|\dot{\gamma}\| = d(\gamma)$,

$$\dot{\tilde{\gamma}}_i(t) \;=\; \frac{\langle \dot{\tilde{\gamma}}_i(t), \dot{\gamma}(t)\rangle}{d(\gamma)^2}\dot{\gamma}(t) \;+\; v(t) \;,$$

which combined with $\dot{\tilde{\gamma}}_i = \dot{f} + \dot{\gamma}$ yields

$$\dot{\tilde{\gamma}}_i(t) \;=\; (1 + \frac{\langle \dot{f}(t), \dot{\gamma}(t)\rangle}{d(\gamma)^2})\dot{\gamma}(t) \;+\; v(t) \;. \tag{2.28}$$

Furthermore, (2.27) coupled with $\dot{\tilde{\gamma}}_i = \dot{f} + \dot{\gamma}$ render

$$v(t) \;=\; \dot{f}(t) \;-\; \frac{\langle \dot{f}(t), \dot{\gamma}(t)\rangle}{d(\gamma)^2}\dot{\gamma}(t) \;=\; O(\frac{1}{m^{\min\{2,2\varepsilon\}}}) \;.$$

Combining the latter with (2.27), (2.28), and $\langle \dot{\gamma}(t), v(t)\rangle = 0$ leads to

$$\begin{aligned} \|\dot{\tilde{\gamma}}_i(t)\| &= \|\dot{\gamma}(t)\|\sqrt{1 \;+\; 2\frac{\langle \dot{f}(t), \dot{\gamma}(t)\rangle}{d(\gamma)^2} \;+\; \frac{\langle \dot{f}(t), \dot{\gamma}(t)\rangle^2}{d(\gamma)^4} \;+\; O(\frac{1}{m^{\min\{4,4\varepsilon\}}})} \\ &= \|\dot{\gamma}(t)\|(1 + \frac{\langle \dot{f}(t), \dot{\gamma}(t)\rangle}{d(\gamma)^2}) \;+\; O(\frac{1}{m^{\min\{4,4\varepsilon\}}}) \;, \end{aligned} \tag{2.29}$$

for which we use $\varepsilon \in (0,1]$, so that for $g(x) = \sqrt{1+x} = 1 + 0.5x + 0.5g''(\xi)x^2$ (where $\xi \in [0,x]$ or $\xi \in [x,0]$) the second derivative $g''(\xi)$ is uniformly bounded as $x = O(1/m^{\min\{2,2\varepsilon\}})$ is asymptotically separated from -1. Consequently, by (2.21), (2.29), $\|\dot{\gamma}\| = d(\gamma)$, and integration by parts for

$$\mathcal{I}_0 = \int_{t_i}^{t_{i+2}} (\|\dot{\tilde{\gamma}}_i(t)\| \;-\; \|\dot{\gamma}(t)\|)\, dt$$

leads to

$$\begin{aligned}\mathcal{I}_0 &= \int_{t_i}^{t_{i+2}} \frac{\langle \dot{f}(t), \dot{\gamma}(t)\rangle}{d(\gamma)}\,dt \;+\; O(\frac{1}{m^{\min\{5,1+4\varepsilon\}}})\\ &= \frac{1}{d(\gamma)}\sum_{j=1}^{n}\left((f_j(t_{i+2})\dot{\gamma}_j(t_{i+2}) - f_j(t_i)\dot{\gamma}_j(t_i)) - \int_{t_i}^{t_{i+2}} f_j(t)\ddot{\gamma}_j(t)\,dt\right)\\ &\quad + O(\frac{1}{m^{\min\{5,1+4\varepsilon\}}})\,,\end{aligned}$$

where $f = (f_1, f_2, \ldots, f_n) \in \mathbb{R}^n$ and $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_n) \in \mathbb{R}^n$. Hence

$$\boxed{\int_{t_i}^{t_{i+2}}(\|\dot{\hat{\gamma}}_i(t)\| \;-\; \|\dot{\gamma}(t)\|)\,dt \;=\; -\int_{t_i}^{t_{i+2}} \frac{\langle f(t),\ddot{\gamma}(t)\rangle}{d(\gamma)}\,dt \;+\; O(\frac{1}{m^{\min\{5,1+4\varepsilon\}}})\;.} \tag{2.30}$$

Now by (2.23)

$$\langle f(t), \ddot{\gamma}(t)\rangle \;=\; (t-t_i)(t-t_{i+1})(t-t_{i+2})r(t)\,, \tag{2.31}$$

where $r(t) = \langle h(t), \ddot{\gamma}(t)\rangle \in C^1$. Taylor's Theorem applied to r at t_i yields the following $r(t) = r(t_i) + (t - t_i)\dot{r}(\xi)$, for some $t_i < \xi < t_{i+2}$. As previously, by (2.24)

$$r(t_i) \;=\; \langle h(t_i), \ddot{\gamma}(t_i)\rangle \;=\; O(h) \;=\; O(m^{\max\{0,2(1-\varepsilon)\}})\,, \tag{2.32}$$

and again by (2.24) and by (2.26)

$$\begin{aligned}\dot{r}(t) &= \langle \dot{h}(t), \ddot{\gamma}(t)\rangle + \langle h(t), \gamma^{(3)}(t)\rangle = O(\dot{h}) + O(h)\\ &= O(m^{\max\{0,3(1-\varepsilon)\}})\,.\end{aligned} \tag{2.33}$$

Note now that for $\mathcal{I}_1 = \int_{t_i}^{t_{i+2}}(t-t_i)(t-t_{i+1})(t-t_{i+2})\,dt$ integration by parts yields

$$\begin{aligned}\mathcal{I}_1 &= -\frac{1}{2}\int_{t_i}^{t_{i+2}}(t-t_i)^2(t-t_{i+1})\,dt \;-\; \frac{1}{2}\int_{t_i}^{t_{i+2}}(t-t_i)^2(t-t_{i+2})\,dt\\ &= \frac{(t_i-t_{i+2})^3(t_{i+2}-t_{i+1})}{6} \;+\; \frac{1}{3}\int_{t_i}^{t_{i+2}}(t-t_i)^3\,dt\\ &= \frac{(t_i-t_{i+2})^3}{12}(t_{i+2}-2t_{i+1}+t_i)\,.\end{aligned} \tag{2.34}$$

Consequently, by ε-uniformity and (2.10) we obtain

$$\int_{t_i}^{t_{i+2}}(t-t_i)(t-t_{i+1})(t-t_{i+2})\,dt \;=\; O(\frac{1}{m^{\min\{5,4+\varepsilon\}}}) \tag{2.35}$$

and hence by (2.31), (2.32), and (2.33) the integral $\mathcal{I}_2 = \int_{t_i}^{t_{i+2}} \frac{\langle f(t), \ddot{\gamma}(t)\rangle}{d(\gamma)}\, dt$ satisfies

$$\begin{aligned}\mathcal{I}_2 &= \frac{r(t_i)}{d(\gamma)} \int_{t_i}^{t_{i+2}} (t-t_i)(t-t_{i+1})(t-t_{i+2})\, dt \;+\; O(\frac{1}{m^5})O(m^{\max\{0,3(1-\varepsilon)\}}) \\ &= O(m^{\max\{0,2(1-\varepsilon)\}})O(\frac{1}{m^{\min\{5,4+\varepsilon\}}}) + O(\frac{1}{m^{\min\{5,2+3\varepsilon\}}})\ ,\end{aligned}$$

and therefore

$$\boxed{\int_{t_i}^{t_{i+2}} \frac{\langle f(t), \ddot{\gamma}(t)\rangle}{d(\gamma)}\, dt \;=\; O(\frac{1}{m^{\min\{5,2+3\varepsilon\}}})\ .} \tag{2.36}$$

So again by ε-uniformity, (2.30), and (2.36) we obtain

$$\boxed{\int_{t_i}^{t_{i+2}} (\|\dot{\tilde{\gamma}}_i(t)\| \;-\; \|\dot{\gamma}(t)\|)\, dt \;=\; O(\frac{1}{m^{\min\{5,1+4\varepsilon\}}})\ .}$$

Thus we finally arrive at

$$d(\tilde{\gamma}_2) \;=\; \Sigma_{j=0}^{\frac{m}{2}-1} d(\tilde{\gamma}_{2j,2}) \;=\; \Sigma_{j=0}^{\frac{m}{2}-1} d(\tilde{\gamma}_{2j}) \;=\; d(\gamma) \;+\; O(\frac{1}{m^{\min\{4,4\varepsilon\}}})\ . \tag{2.37}$$

Again, as at the end of the proof of Theorem 1.3 we have $1/m \leq \delta$ (recall $T = 1$). The latter substituted into (2.37) transforms it immediately into the corresponding δ-asymptotics. This completes the proof of Theorem 2.1. □

2.3 More-or-less and 0-uniform reduced data - asymptotics

We justify now the Theorem 2.2 .

Proof. We only outline the proof with the notation from Theorem 2.1. Note that as sampling is more-or-less uniform the differences $(t_j - t_k)^{-1} = O(m)$ and $t_j - t_k = O(1/m)$, where $k, j = 0, 1, 2$ for $k \neq j$. Thus upon inspection we have $a_1 = O(1/m)$ and $a_2 = O(1/m)$. Hence

$$\boxed{\frac{d\tilde{\gamma}_{i,2}}{ds} \;=\; O(\frac{1}{m}) \quad \text{and} \quad \frac{d^2\tilde{\gamma}_{i,2}}{ds^2} \;=\; O(\frac{1}{m})\ .} \tag{2.38}$$

Similarly, $b_2 = O(m^2)$ and $b_1 = O(m)$ and thus

$$\dot{\psi}^i \;=\; O(m) \quad \text{and} \quad \ddot{\psi}^i \;=\; O(m^2)\ . \tag{2.39}$$

Combining (2.38) and (2.39)

$$\tilde{\gamma}_i^{(1)} \;=\; O(1) \quad \text{and} \quad \tilde{\gamma}_i^{(3)} \;=\; O(m^2)$$

which in turn gives

$$\boxed{\dot{f} \;=\; O(1)\;, \quad h \;=\; O(\tfrac{d^3 f}{dt^3}) \;=\; O(m^2) \quad \text{and thus} \quad f \;=\; O(\tfrac{1}{m})\;.} \tag{2.40}$$

The last equation together with $1/m \leq \delta$ prove the first formula in (2.2). Furthermore, by (2.40)

$$\|\dot{\tilde{\gamma}}_i\| \;=\; \sqrt{\langle \dot{f} + \dot{\gamma}, \dot{f} + \dot{\gamma} \rangle} \;=\; \sqrt{O(1)} \;=\; O(1) \;=\; 1 \;+\; O(1)\;,$$

and thus

$$\boxed{\textstyle\int_{t_i}^{t_{i+2}} (\|\dot{\tilde{\gamma}}_i(t)\| \;-\; \|\dot{\gamma}(t)\|)\, dt \;=\; O(\tfrac{1}{m})\;.}$$

Again the latter combined with $1/m \leq \delta$ yield the second formula in (2.2). The proof is completed. ☐

2.4 Experiments

We test now sharpness of the theoretical results claimed by Theorems 2.1 and 2.2 for length estimation (see also [47]). Our test curves are as previously two planar curves, *i.e.* a semicircle and a cubic $\gamma_{sc}, \gamma_c : [0,1] \to \mathbb{R}^2$ (see (1.9) and (1.25)). Recall also that $d(\gamma_{sc}) = \pi$ and $d(\gamma_c) = 3.3452$. The *linear regression is applied* here to the data as indicated in Section 1.6 (see in particular (1.65)).

2.4.1 Pseudocode

We outline first the pseudocode (in *Mathematica*) for implementing a uniform piecewise-quadratic Lagrange interpolant $\tilde{\gamma}_2$ based on reduced data.

Let $list$ of triples (from reduced data $\mathcal{Q}_m^{i,2}$) be initialized to $list := \{q_i, q_{i+1}, q_{i+2}\}$ (we assume $i = 2l$). The pseudocode for the similar procedure as in Section 1.6.1 (with $r = 2$ and $\hat{t}_i = i$; we can set[6] for each triple $\mathcal{Q}_m^{2l,2}$: $\hat{t}_i = 0$, $\hat{t}_{i+1} = 1$ and $\hat{t}_{i+2} = 2$) reads as in Figure 2.2.

The estimation of α appearing in $d(\tilde{\gamma}_2) - d(\gamma) = O(\delta^\alpha)$ and computed from linear regression (see Section 1.6) applied to the collection of reduced data $\{\mathcal{Q}_{2j}\}_{j=m_1}^{m_2}$, where $m_{min} = 2m_1 \leq m \leq 2m_2 = m_{max}$ (where $m_1 \geq 1$), yields a similar pseudocode (see Section 1.6.1) for the main program loop shown in Figure 2.3.

Recall that we index here any list from the label set to zero. As before, a slight modification of the main program loop yields the pseudocode to plot $\tilde{\gamma}_2$ for a given $\mathcal{Q}_{j_{fix}}$. The

[6] Note that $\bar{\gamma}(\hat{t}) = \hat{\gamma}(\hat{t} - \bar{T})$ has the same trajectory and length over $[-\bar{T}, \hat{T} - \bar{T}]$ as $\hat{\gamma}$ over $[0, \hat{T}]$

$\mathbf{LagKnotsUniform}[list\_]$

$\{$

$Lag\_Int[s\_] := Lag\_Formula[list, \{0, 1, 2\}, 2, s];$

$/ *$ see (1.8) with $r = 2$ and $\{t_i = k - i\}_{k=i}^{i+2} * /$

$Der\_Lag\_Int := D[Expand[Lag\_Int[s]], s];$

$Lag\_Der\_List := CoefficientList[Der\_Lag\_Int, s];$

$Norm[s\_] := Derivatives\_Norm[Lag\_Der\_List, s];$

$Length := NIntegrate[Norm[s], \{s, 0, 2\}];$ $/ *$ see (1.1) over $[0, 2] * /$

$Lag\_Plot\_List := ParametricPlot[Lag\_Int[s], \{s, 0, 2\}];$

$\mathbf{return}\{Length, Lag\_Plot\_List\}$

$\}$

Fig. 2.2. Pseudocode for procedure **LagKnotsUniform**, which for one input list $\{q_i\}_i^{i+2}$, returns the list of $\{d(\tilde{\gamma}_{i,2}), plot\}$ (here $\{\hat{t}_i = k - i\}_{k=i}^{i+2}$), where $plot$ represents a discrete set of $\tilde{\gamma}_{i,2}(s)$, for $s \in [0, 2]$ (according to the ParametricPlot format)

Rys. 2.2. Pseudokod dla procedury **LagKnotsUniform**, która dla danych na wejściu (lista $\{q_i\}_i^{i+2}$) zwraca listę $\{d(\tilde{\gamma}_{i,2}), plot\}$ (dla $\{\hat{t}_i = k-i\}_{k=i}^{i+2}$), gdzie $plot$ jest listą dyskretnego zbioru wartości $\tilde{\gamma}_{i,2}(s)$ dla $s \in [0, 2]$ (zgodnie z formatem ParametricPlot)

pseudocode for estimating α in $\tilde{\gamma}_2 \circ \psi - \gamma = O(\delta^\alpha)$ is similar but as previously involves computationally expensive optimization procedure (see also Subsection 1.3.1) to determine (for each $m_{min} \leq m \leq m_{max}$) the $\sup_{t\in[0,T]} |\tilde{\gamma}_2 \circ \psi(t) - \gamma(t)|$, where $\psi : [0, T] \to [0, \hat{T}]$ defines a piecewise-C^∞ reparameterization - see Section 2.2. Therefore it is omitted.

2.4.2 $\mathcal{E}$-uniform random samplings

We consider first *random sampling* (1.23) with three types of ϕ_1, ϕ_2 and ϕ_3 introduced in Example 1.4. The expected convergence rates for $\varepsilon = 1$, $\varepsilon = 1/2$ and $\varepsilon = 1/3$ are not slower that $\alpha_1 = 4$, $\alpha_{1/2} = 2$, $\alpha_{1/3} = 4/3$, respectively. Table 2.3 indicates faster convergence rates than those claimed by Theorem 2.1. Note also that sampling (1.23) for $\varepsilon = 0$ may not be more-or-less uniform and therefore Theorem 2.2 is not tested in this subsection.

Remark 2.2. We remark here that if in definition of ε-uniformity (see (1.15)) $\phi \notin C^1$ then the Theorem 2.1 does not hold. Indeed, if for the sampling (1.23), $\phi(t) = \sqrt{t}$ (where $t \in [0, 1]$) and $\varepsilon = 1$, then a piecewise-quadratic Lagrange interpolation used with $\hat{t}_i = i$ yields $\alpha = 1.78 < 4$ (claimed by Theorem 2.1).

```
For  [j = m_1; err = { }, j ≤ m_2, j = j + 1,
     Data[j] := {q_0, q_1, q_2, ..., q_{2*j}};
     m_3 = 2 * j;
     For  [i = 0; length = 0, i ≤ m_3 − 2, i = i + 2,
          List1 := {Data[j][i], Data[j][i + 1], Data[j][i + 2]};
          length+ = LagKnotsUniform[List1][0];
          ];
     err := AppendTo[err, {Log[m_3], −Log[|length − d(γ)|]}];
     ];
α := Slope_Coeff[Regress[err]];                    / * see Section1.6 * /
```

Fig. 2.3. Pseudocode for the main program loop computing an α estimate in $d(\tilde{\gamma}_2)-d(\gamma) = O(\delta^\alpha)$ based on collection of reduced data $\{\mathcal{Q}_{2j}\}_{j=m_1}^{m_2}$

Rys. 2.3. Pseudokod głównej pętli programu obliczającej oszacowanie α w $d(\tilde{\gamma}_2)-d(\gamma) = O(\delta^\alpha)$ na podstawie rodziny danych zredukowanych $\{\mathcal{Q}_{2j}\}_{j=m_1}^{m_2}$

	computed α for γ_{sc} (1.9)			computed α for γ_c (1.25)		
	ε_1	$\varepsilon_{1/2}$	$\varepsilon_{1/3}$	ε_1	$\varepsilon_{1/2}$	$\varepsilon_{1/3}$
ϕ_1	3.99	2.79	2.49	4.09	2.82	2.45
ϕ_2	3.96	3.03	2.47	3.97	3.66	3.20
ϕ_3	3.97	2.94	2.36	3.90	3.89	3.79
$\alpha_{Th.2.1}$	4.00	2.00	1.33	4.00	2.00	1.33

Tab. 2.3. $d(\gamma)$ estimation (for γ_{sc}, γ_c) by $d(\tilde{\gamma}_2)$ for sampling (1.23) and $\hat{t}_i = i$

2.4.3 Skew-symmetric ε-uniform samplings

We experiment now with two other families of *skew-symmetric* ε-uniform sampling introduced by (1.24). Those samplings are also more-or-less uniform and hence testing is also performed for ε vanishing. Table 2.4 shows sharpness of the results claimed by Theorem 2.2. Still, experiments in Table 2.4 indicate faster convergence orders for $\varepsilon > 0$, as compared to those claimed by Theorem 2.1.

	computed α for γ_{sc} (1.9)				computed α for γ_c (1.25)			
	ε_1	$\varepsilon_{0.5}$	$\varepsilon_{0.33}$	ε_0	ε_1	$\varepsilon_{0.5}$	$\varepsilon_{0.33}$	ε_0
(1.24)(i)	4.01	2.69	2.53	$d(\tilde{\gamma}_2) \not\to d(\gamma)$	3.98	3.03	2.76	$d(\tilde{\gamma}_2) \not\to d(\gamma)$
(1.24)(ii)	4.00	2.68	2.40	1.44	3.97	2.92	2.65	1.99
$\alpha_{Th.2.1,2.2}$	4.00	2.00	1.43	"div."/conv.	4.00	2.00	1.33	"div."/conv.

Tab. 2.4. $d(\gamma)$ estimation (for γ_{sc}, γ_c) by $d(\tilde{\gamma}_2)$ for sampling (1.24) and $\hat{t}_i = i$

2.5 Discussion and motivation for Chapter 3

Note that both proofs of Theorems 2.1 and 2.2 adapted for *reduced data* $\mathcal{Q}_m$, involve also a non-trivial step of determining the asymptotics of the respective derivatives of the family of $\tilde{\gamma}_2 \circ \psi$, where for a given m, $\tilde{\gamma}_2 : [0, \hat{T}] \to \mathbb{R}^n$ is the piecewise-quadratic Lagrange interpolant and ψ defines a piecewise-C^∞ reparameterization $\psi : [0, T] \to [0, \hat{T}]$. This step will be present in the next chapters which deal with reduced data $\mathcal{Q}_m$.

This chapter examines a class of reduced ε-uniform and more-or-less uniform samplings for which piecewise-quadratic Lagrange interpolation is used with $\hat{t}_i = i$. The investigation of corresponding convergence rates for trajectory and length estimation is performed. Our results appear to be sharp (or almost sharp) for the class of samplings studied in this chapter (at least the latter was verified for length estimation and $n = 2$). The established asymptotics yield slower rates with $\varepsilon \to 0$. In particular, a piecewise-quadratic Lagrange interpolation $\tilde{\gamma}_2$ does not work well in general, for 0-uniform samplings differing the most from the uniform case. The assumption about $\phi \in C^\infty$ from (1.15) for Theorem 2.1 to hold is shown to be essential (see Remark 2.2). The obvious question arises now:

Is it possible to remove the weaknesses of Theorems 2.1 and 2.2 which yield slow convergence for $\varepsilon \to 0$ (or even "divergence" for $\varepsilon = 0$) in γ and $d(\gamma)$ estimation?

Such weaknesses of the above approach will be first corrected in the next chapter only for *convex planar curves* (*i.e* $n = 2$) and *more-or-less uniform samplings*. Ultimately, subsequent chapters will fix the weakness of Theorems 2.1 and 2.2 into a general class of reduced admissible samplings $\mathcal{V}_G^m$ (see Definition 1.2) and n arbitrary.

Chapter 3

Piecewise-4-point quadratics

Abstract

Fast, sharp, quartic orders of convergence to estimate trajectory and length of γ *are established, for reduced data* $\mathcal{Q}_m$ *representing more-or-less uniformly sampled convex planar curves. In doing so, first the piecewise-4-point quadratic non-Lagrange interpolant passing through four consecutive data points together with estimates* $\{\hat{t}_i\}_{i=0}^m$ *of the corresponding unknown tabular parameters* $\{t_i\}_{i=0}^m$ *are found via advanced algebraic computation. Subsequently, a non-trivial asymptotic analysis follows and the theory derived herein is illustrated and confirmed by examples (at least for length estimation). As shown, the piecewise-4-point quadratic interpolant not only outperforms (when applicable) the piecewise-quadratic Lagrange interpolation used either with* $\{t_i\}_{i=0}^m$ *known or uniformly guessed* $\hat{t}_i = i$ *(for* $\{t_i\}_{i=0}^m$ *unknown), but also matches the performance of the piecewise--cubic Lagrange interpolation used with* $\{t_i\}_{i=0}^m$ *known. Additionally, a good performance of piecewise-4-point quadratic is confirmed experimentally on sporadic data (not covered by the asymptotic analysis from this chapter). Part of this work is published in [43] and [44].*

3.1 Main result

As shown in the last chapter the uniform guess of $\hat{t}_i = i$ estimating the unknown parameters $\{t_i\}_{i=0}^m$ does not yield a good Lagrange piecewise-quadratic interpolation scheme, especially for 0-uniform samplings. We shall fix this problem under special assumption of considering *convex planar curves sampled more-or-less uniformly*.

Before stating the main result again recall the following example:

Example 3.1. Consider now a semicircle γ_{sc} (see (1.9)) sampled more-or-less uniformly according to (1.24)(i) (with $\varepsilon = 0$). When m is small the image of $\tilde{\gamma}_{sc_2}$ (see Chapter 2) does not much resemble a semicircle, as in Figure 3.1(a) with $m = 6$ we have $d(\tilde{\gamma}_{sc_2}) - d(\gamma) = 0.0601$. The error in length estimate with piecewise-linear interpolation

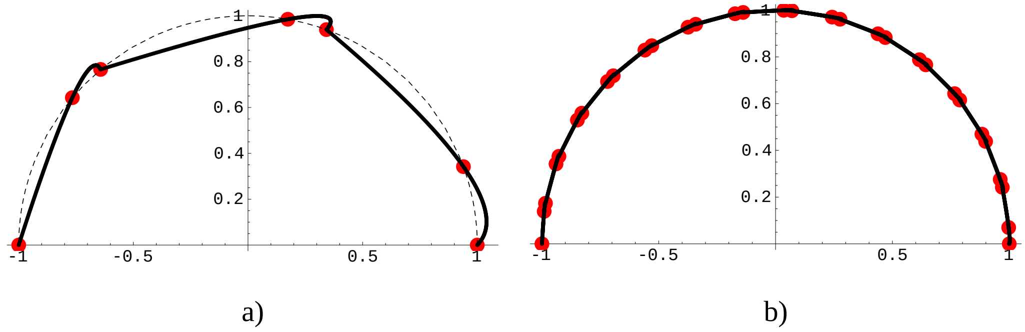


Fig. 3.1. Interpolating *a semicircle* γ_{sc} (1.9) (dashed) by *uniform piecewise-quadratics* $\tilde{\gamma}_{sc_2}$ (solid) with $\hat{t}_i = i$, for sampling (1.24)(i) ($\varepsilon = 0$) and: a) 3 successive triples of sampling points $\mathcal{Q}_6$ (dotted) with length estimate: $\pi + 0.0601$, b) 15 successive triples of sampling points $\mathcal{Q}_{30}$ (dotted) with length estimate: $\pi + 0.1194$

Rys. 3.1. Interpolacja *półokręgu* γ_{sc} (1.9) (linia przerywana) *przedziałowo-kwadratowymi funkcjami sklejanymi* $\tilde{\gamma}_{sc_2}$ (linia ciągła) z $\hat{t}_i = i$, dla próbkowania (1.24)(i) ($\varepsilon = 0$) oraz: a) 3 kolejnych trójek punktów ze zbioru próbkowań $\mathcal{Q}_6$ (wytłuszczone) z oszacowaniem długości: $\pi + 0.0601$, b) 15 kolejnych trójek punktów ze zbioru próbkowań $\mathcal{Q}_{30}$ (wytłuszczone) z oszacowaniem długości: $\pi + 0.1194$

is -0.0712. When m is large the image of $\tilde{\gamma}_{sc_2}$ looks semicircular, as in Figure 3.1 (b) where $m = 30$. In this case however $d(\tilde{\gamma}_{sc_2}) - d(\gamma) = 0.1194$, an error nearly twice as large as for $m = 6$. Even piecewise-linear interpolation with 31 points gives a better estimate, with error -0.0033. Indeed as m increases (at least for $m \leq 100$), then piecewise-quadratic interpolation tends to *accumalate* errors of length estimates. Linear interpolation is better, but not impressive. □

To state our main result, first take $n = 2$ and suppose that γ is C^4 and (without loss) parameterized by arc-length, namely $\|\dot{\gamma}\|$ is identically 1. The *curvature* of planar γ (see Chapter 1; Paragraphs 1-5, Remark 1 of [14]) is defined as

$$\boxed{k(t) \;=\; \det(M(t))\;,} \tag{3.1}$$

where $M(t)$ is the 2×2 matrix with columns $\dot{\gamma}(t), \ddot{\gamma}(t)$. When $k(t) \neq 0$ for all $t \in [0, T]$, γ is said to be *strictly convex*. The following holds (see [44]):

Theorem 3.1. *Let $\gamma \in C^4$ be strictly convex and be sampled more-or-less uniformly (i.e. $\{t_i\}_{i=0}^m \in \mathcal{V}_{mol}^m$). Then, there is a piecewise-quadratic*[1] $Q : [0, \hat{T}] \to \mathbb{R}^2$ *calculable in terms of $\mathcal{Q}_m$ and piecewise-C^∞ reparameterization*[2] $\psi : [0, T] \to [0, \hat{T}]$ *with*

$$\boxed{\boxed{Q \circ \psi \;=\; \gamma \;+\; O(\delta^4) \quad \textit{and} \quad d(Q) \;=\; d(\gamma) \;+\; O(\delta^4)\;.}} \tag{3.2}$$

[1] See Section 3.2 for more details.
[2] See Section 3.3 for more details.

In Sections 3.2, 3.3 and 3.4 we prove Theorem 3.1, constructing $d(Q)$ as a sum of lengths of quadratic arcs interpolating *quadruples* of sample points. In Section 3.5 some examples are given, showing that the quartic convergence of Theorem 3.1 is the best-possible for our construction.

3.2 Quadratics interpolating planar quadruples

In this Section we find the 4-point quadratic Q^i passing through the four consecutive planar points $\mathcal{Q}_m^{i,3}$ of $\mathcal{Q}_m$ together with the corresponding estimates $\hat{t}_{i+k}$ of the unknown interpolation knots t_{i+k}, where $k = 0,1,2,3$. This task involves a non-trivial algebraic step followed by the asymptotic analysis. The latter, as a precondition requires explicit formulas determining later the piecewise-4-point quadratic interpolant Q.

Let $\mathcal{Q}_m$ be sampled more-or-less uniformly (*i.e.* $\{t_i\}_{i=0}^m \in \mathcal{V}_{mol}^m$) from γ, and suppose (without loss) that m is a positive integer multiple of 3. For a given quadruple of sampling points $\mathcal{Q}_m^{i,3} = (q_i, q_{i+1}, q_{i+2}, q_{i+3})$, where $0 \leq i \leq m-3$, define $a_0, a_1, a_2 \in \mathbb{R}^2$ and $Q^i : [0, \beta_i] \to \mathbb{R}^2$

$$\boxed{Q^i(s) \;=\; a_0 + a_1 s + a_2 s^2} \tag{3.3}$$

by

$$\boxed{Q^i(0) \;=\; q_i\;, \quad Q^i(1) \;=\; q_{i+1}\;, \quad Q^i(\alpha_i) \;=\; q_{i+2} \quad \text{and} \quad Q^i(\beta_i) \;=\; q_{i+3}\;.} \tag{3.4}$$

For simplicity index i in α_i and β_i is from now on omitted. Then as $a_0 = q_i$ and $a_2 = q_{i+1} - a_0 - a_1$ we obtain two vector equations

$$\boxed{\boxed{a_1\alpha \;+\; (p_1 - a_1)\alpha^2 \;=\; p_\alpha \quad \text{and} \quad a_1\beta \;+\; (p_1 - a_1)\beta^2 \;=\; p_\beta\;,}} \tag{3.5}$$

where $(p_1, p_\alpha, p_\beta) \equiv (q_{i+1} - q_i, q_{i+2} - q_i, q_{i+3} - q_i)$. Then (3.5) amounts to *four quadratic scalar equations* in *four scalar unknowns* $a_1 = (a_{11}, a_{12})$, α, β. In the first step we find (α, β) which, for a given $\mathcal{Q}_m^{i,3}$ estimate the unknown parameters $\hat{t}_{i+3}$ and $\hat{t}_{i+4}$ subject to the *normalization condition* $\hat{t}_{i+1} - \hat{t}_i = 1$. In doing so, set

$$c \;=\; -\det(p_\alpha, p_\beta), \quad d \;=\; -\det(p_\beta, p_1)/c\;, \quad e \;=\; -\det(p_\alpha, p_1)/c\;, \tag{3.6}$$

where $c, d, e \neq 0$ by strict convexity, and define

$$\rho_1 \;=\; \sqrt{e(1+d-e)/d}, \quad \rho_2 \;=\; \sqrt{d(1+d-e)/e}\;. \tag{3.7}$$

Lemma 3.1. *Assume that a planar curve $\gamma \in C^3$ is strictly convex i.e. either $k(t) < 0$ or $k(t) > 0$ and is sampled more-or-less uniformly (i.e. $\{t_i\}_{i=0}^m \in \mathcal{V}_{mol}^m$). Then system* (3.5) *has two solutions in (α, β)*

$$\boxed{\boxed{(\alpha_+, \beta_+) \;=\; \tfrac{(1+\rho_1, 1+\rho_2)}{e-d} \quad \textit{and} \quad (\alpha_-, \beta_-) \;=\; \tfrac{(1-\rho_1, 1-\rho_2)}{e-d}\;,}} \tag{3.8}$$

provided ρ_1, ρ_2 *are real and* $e - d \neq 0$. *We show now that indeed these conditions hold and thus* (3.8) *follows. Moreover for* $(\alpha_\pm, \beta_\pm)$ *we have*

$$\boxed{\boxed{a_1 \;=\; \frac{p_{\alpha_\pm} - \alpha_\pm^2 p_1}{\alpha_\pm - \alpha_\pm^2} \;=\; \frac{p_{\beta_\pm} - \beta_\pm^2 p_1}{\beta_\pm - \beta_\pm^2} \;\; and \;\; a_2 \;=\; \frac{\alpha_\pm p_1 - p_{\alpha_\pm}}{\alpha_\pm - \alpha_\pm^2} \;=\; \frac{\beta_\pm p_1 - p_{\beta_\pm}}{\beta_\pm - \beta_\pm^2} \;.}} \tag{3.9}$$

In addition the pair (α_+, β_+) *satisfies the additional condition*

$$\boxed{\boxed{1 \;<\; \alpha_+ \;<\; \beta_+ \;.}} \tag{3.10}$$

It suffices[3] to deal with the case where $k(t) < 0$ for all $t \in [0, T]$.

Proof. First we solve (3.5). Note that α (and β) cannot vanish as otherwise, by (3.5), the vector $p_\alpha = q_2 - q_0 = \vec{0}$ ($p_\beta = q_3 - q_0 = \vec{0}$) - *a contradiction* as interpolation points $\mathcal{Q}$ are assumed to be different. Similarly, as $q_2 \neq q_1$ and $q_3 \neq q_1$ we have $\alpha \neq 1$ and $\beta \neq 1$. Thus elimination of a_1 from (3.5) and further simplification yields a vector equation in two scalar unknowns (α, β)

$$\boxed{\alpha\beta(\alpha - \beta)p_1 \;=\; (\beta - \beta^2)p_\alpha \;-\; (\alpha - \alpha^2)p_\beta \;.} \tag{3.11}$$

Consider now two vectors $p_\beta^\perp = (-p_{\beta 2}, p_{\beta 1})$ and $p_\alpha^\perp = (-p_{\alpha 2}, p_{\alpha 1})$, which are perpendicular to p_β and p_α, respectively. Taking the dot product $\langle \cdot, \cdot \rangle$ of (3.11) first with $p_\beta^\perp$ and then with $p_\alpha^\perp$ results in

$$\begin{aligned} \alpha\beta(\alpha - \beta)\langle p_1, p_\beta^\perp \rangle &= (\beta - \beta^2)\langle p_\alpha, p_\beta^\perp \rangle \;, \\ \alpha\beta(\alpha - \beta)\langle p_1, p_\alpha^\perp \rangle &= -(\alpha - \alpha^2)\langle p_\beta, p_\alpha^\perp \rangle \;. \end{aligned} \tag{3.12}$$

Note here that since $\{p_\beta^\perp, p_\alpha^\perp\} = \mathbb{R}^2$ holds asymptotically, both systems (3.11) and (3.12) are equivalent. Since α and β cannot vanish and $\langle p_\alpha, p_\beta^\perp \rangle \neq 0$ and $\langle p_\beta, p_\alpha^\perp \rangle \neq 0$ hold asymptotically (as γ is strictly convex) we obtain

$$\frac{\alpha(\alpha - \beta)\langle p_1, p_\beta^\perp \rangle}{\langle p_\alpha, p_\beta^\perp \rangle} \;=\; 1 - \beta \quad \text{and} \quad \frac{\beta(\alpha - \beta)\langle p_1, p_\alpha^\perp \rangle}{\langle p_\beta, p_\alpha^\perp \rangle} \;=\; \alpha - 1 \;. \tag{3.13}$$

Note that by (3.6) and convexity of γ, $c \neq 0$ asymptotically. A simple verification shows:

$$c \;=\; -\langle p_\beta, p_\alpha^\perp \rangle \;=\; \langle p_\alpha, p_\beta^\perp \rangle \;. \tag{3.14}$$

Similarly

$$d \;=\; \frac{-\langle p_1, p_\beta^\perp \rangle}{c} \qquad \text{and} \qquad e \;=\; \frac{-\langle p_1, p_\alpha^\perp \rangle}{c} \;.$$

[3] The other case $k(t) > 0$, is dealt with by considering the reversed curve $\gamma_r(t) = (\gamma_1(T - t), \gamma_2(T - t))$.

The latter coupled with (3.14) yields

$$d \;=\; \frac{-\langle p_1, p_\beta^\perp\rangle}{\langle p_\alpha, p_\beta^\perp\rangle} \qquad \text{and} \qquad e \;=\; \frac{\langle p_1, p_\alpha^\perp\rangle}{\langle p_\beta, p_\alpha^\perp\rangle}\;,$$

which combined with (3.13) renders

$$\boxed{\alpha(\alpha-\beta)d \;=\; \beta - 1 \qquad \text{and} \qquad \beta(\alpha-\beta)e \;=\; \alpha - 1\;.} \tag{3.15}$$

The first equation of (3.15) yields

$$\alpha^2 d + 1 \;=\; \beta(1+d\alpha)\;. \tag{3.16}$$

Note that $1+d\alpha \neq 0$ as otherwise since $\alpha \neq 0$ we would have $d = -\alpha^{-1}$ and by (3.16) $\alpha^2 d + 1$ would vanish which combined with $d = -\alpha^{-1}$ would lead to $\alpha = 1$, which yields *a contradiction*. Thus by (3.16)

$$\beta \;=\; \frac{\alpha^2 d + 1}{1 + d\alpha}\;. \tag{3.17}$$

Substituting (3.17) into the second equation of (3.15) yields

$$\frac{\alpha^2 d + 1}{(1+d\alpha)^2}(\alpha-1)e \;=\; \alpha - 1$$

and taking into account that $\alpha \neq 1$ results in

$$(d^2 - de)\alpha^2 + 2d\alpha + 1 - e \;=\; 0\;. \tag{3.18}$$

Assuming temporarily that

$$d \;<\; 0\;, \quad e \;<\; 0\;, \quad d - e \;<\; 0\;, \quad \Delta = 4de(1+d-e) \;>\; 0 \tag{3.19}$$

(the proof for (3.19) commence from (3.20)) we obtain $\alpha_\pm = (1 \pm \rho_1)/(e-d)$. It is rather not straightforward for each $\alpha_\pm$ to compute a unique β from (3.17), so that the expression (3.8) are matched. Instead, the second equation of (3.15) yields

$$\alpha \;=\; \frac{\beta^2 e - 1}{e\beta - 1}\;,$$

which when substituted to the first equation of (3.15) renders

$$\boxed{(e^2 - de)\beta^2 - 2e\beta + 1 + d \;=\; 0\;.}$$

Thus $\beta_\pm \;=\; (1 \pm \rho_2)/(e-d)$. As $d\rho_1 \;=\; e\rho_2$, it can be verified that pairs (α_+, β_+) (α_-, β_-) satisfy (3.15) and thus (3.11).

Having found $(\alpha_\pm, \beta_\pm)$ the corresponding formulas (3.9) follow immediately.

To show (3.19) recall that

$$d \;=\; \frac{\det(p_\beta, p_1)}{\det(p_\alpha, p_\beta)} \qquad \text{and} \qquad e \;=\; \frac{\det(p_\alpha, p_1)}{\det(p_\alpha, p_\beta)}\,. \tag{3.20}$$

As $\det(v,w) \;=\; \|v\|\|w\| sin(\sigma)$ (where σ is the oriented angle between v and w) for convex γ both $e \;<\; 0$ and $d \;<\; 0$ hold. Similarly, since γ is convex and $e - d$ equals $\det(p_\beta - p_\alpha, p_1)c^{-1}$, then for $k \;<\; 0$ both c and $\det(p_\beta - p_\alpha, p_1)$ are positive, and for $k > 0$ both c and $\det(p_\beta - p_\alpha, p_1)$ are negative, asymptotically. Thus to justify (3.19) it is therefore enough to show $1 + d - e \;>\; 0$. In fact, as γ is strictly convex and sampled more-or-less uniformly we also show analytically that the above inequalities are separated from zero. The second-order Taylor's expansion of γ at $t = t_i$ yields

$$\boxed{\gamma(t) \;=\; \gamma(t_i) \;+\; \dot\gamma(t_i)(t-t_i) \;+\; \tfrac{\ddot\gamma(t_i)}{2}(t-t_i)^2 \;+\; O(\tfrac{1}{m^3})\;,}$$

as $0 < T < \infty$ and $\gamma \in C^4$ (in fact to prove the lemma in question $\gamma \in C^3$ suffices). Thus taking into account that $\gamma(t_i) = q_i$, $\gamma(t_{i+1}) = q_{i+1}$, $\gamma(t_{i+2}) = q_{i+2}$, and $\gamma(t_{i+3}) = q_{i+3}$ we have

$$\begin{aligned} p_1 &= \dot\gamma(t_i)(t_{i+1}-t_i) \;+\; (1/2)\ddot\gamma(t_i)(t_{i+1}-t_i)^2 \;+\; O(\frac{1}{m^3})\;, \\ p_\alpha &= \dot\gamma(t_i)(t_{i+2}-t_i) \;+\; (1/2)\ddot\gamma(t_i)(t_{i+2}-t_i)^2 \;+\; O(\frac{1}{m^3})\;, \\ p_\beta &= \dot\gamma(t_i)(t_{i+3}-t_i) \;+\; (1/2)\ddot\gamma(t_i)(t_{i+3}-t_i)^2 \;+\; O(\frac{1}{m^3})\;. \end{aligned} \tag{3.21}$$

Introducing $\gamma_2(t) \;=\; \dot\gamma(t_i)(t-t_i) + (1/2)\ddot\gamma(t_i)(t-t_i)^2$ and coupling it with (3.21) and more-or-less uniformity results in:

$$\begin{aligned} \det(p_\beta, p_\alpha) &= \det(\gamma_2(t_{i+3}), \gamma_2(t_{i+2})) \;+\; O(\frac{1}{m^4})\;, \\ \det(p_1, p_\beta) &= \det(\gamma_2(t_{i+1}), \gamma_2(t_{i+3})) \;+\; O(\frac{1}{m^4})\;, \\ \det(p_\alpha, p_1) &= \det(\gamma_2(t_{i+2}), \gamma_2(t_{i+1})) \;+\; O(\frac{1}{m^4})\;. \end{aligned} \tag{3.22}$$

Set now $\mathcal{P}(c,d,e) = c(1+d-e)$. Thus by (3.6) and (3.20) we have

$$\mathcal{P}(c,d,e) \;=\; \det(p_\beta, p_\alpha) \;+\; \det(p_1, p_\beta) \;+\; \det(p_\alpha, p_1)\,. \tag{3.23}$$

Furthermore, by (3.1) and (3.22), up to the term $O(\frac{1}{m^4})$, we have

$$\begin{aligned}\det(p_\beta, p_\alpha) &= (1/2)k(t_i)\left((t_{i+3}-t_i)(t_{i+2}-t_i)^2 - (t_{i+3}-t_i)^2(t_{i+2}-t_i)\right)\\ &= (1/2)k(t_i)(t_{i+3}-t_i)(t_{i+2}-t_i)(t_{i+2}-t_{i+3}) \ ,\end{aligned}$$

$$\begin{aligned}\det(p_1, p_\beta) &= (1/2)k(t_i)\left((t_{i+1}-t_i)(t_{i+3}-t_i)^2 - (t_{i+1}-t_i)^2(t_{i+3}-t_i)\right)\\ &= (1/2)k(t_i)(t_{i+1}-t_i)(t_{i+3}-t_i)(t_{i+3}-t_{i+1}) \ ,\end{aligned}$$

$$\begin{aligned}\det(p_\alpha, p_1) &= (1/2)k(t_i)\left((t_{i+1}-t_i)^2(t_{i+2}-t_i) - (t_{i+1}-t_i)(t_{i+2}-t_i)^2\right)\\ &= (1/2)k(t_i)(t_{i+1}-t_i)(t_{i+2}-t_i)(t_{i+1}-t_{i+2}) \ .\end{aligned}$$

Denote $K = (t_{i+1}-t_i)(t_{i+2}-t_i)(t_{i+1}-t_{i+2}) + I$, where

$$I \ = \ (t_{i+3}-t_i)(t_{i+2}-t_i)(t_{i+2}-t_{i+3}) \ + \ (t_{i+1}-t_i)(t_{i+3}-t_i)(t_{i+3}-t_{i+1}) \ .$$

Then for $J = I/(t_i - t_{i+3})$ we have

$$\begin{aligned}J &= -t_{i+2}^2 + t_{i+2}t_{i+3} + t_i t_{i+2} - t_{i+1}t_{i+3} + t_{i+1}^2 - t_i t_{i+1} + t_{i+1}t_{i+2} - t_{i+1}t_{i+2}\\ &= t_{i+1}(t_{i+1}-t_i) + t_{i+1}(t_{i+2}-t_{i+3}) - t_{i+2}(t_{i+2}-t_{i+3}) - t_{i+2}(t_{i+1}-t_i)\\ &= (t_{i+1}-t_i)(t_{i+1}-t_{i+2}) + (t_{i+2}-t_{i+3})(t_{i+1}-t_{i+2}) \ .\end{aligned}$$

Hence $K = (t_{i+1}-t_{i+2})L$, where

$$\begin{aligned}L &= (t_{i+1}-t_i)(t_{i+2}-t_i) + (t_i - t_{i+3})(t_{i+1}-t_i) + (t_i - t_{i+3})(t_{i+2}-t_{i+3})\\ &= t_{i+3}t_i - t_{i+3}t_{i+1} + t_{i+1}t_{i+2} - t_i t_{i+2} + (t_i - t_{i+3})(t_{i+2}-t_{i+3})\\ &= t_{i+2}(t_{i+1}-t_i) - t_{i+3}(t_{i+1}-t_i) + (t_i - t_{i+3})(t_{i+2}-t_{i+3})\\ &= (t_{i+1}-t_i)(t_{i+2}-t_{i+3}) + (t_i - t_{i+3})(t_{i+2}-t_{i+3})\\ &= (t_{i+2}-t_{i+3})(t_{i+1}-t_{i+3}) \ .\end{aligned}$$

Thus $K = (t_{i+1}-t_{i+2})(t_{i+2}-t_{i+3})(t_{i+1}-t_{i+3})$. The latter combined with (3.23) yields

$$\mathcal{P}(c,d,e) \ = \ (1/2)k(t_i)(t_{i+1}-t_{i+2})(t_{i+1}-t_{i+3})(t_{i+2}-t_{i+3}) \ + \ O(\frac{1}{m^4}) \ . \quad (3.24)$$

Of course, as $c = \det(p_\beta, p_\alpha)$, we have

$$c \ = \ (1/2)k(t_i)(t_{i+3}-t_i)(t_{i+2}-t_i)(t_{i+2}-t_{i+3}) \ + \ O(\frac{1}{m^4}) \ . \quad (3.25)$$

Upon coupling (3.24), (3.25) with more-or-less uniformity (1.28) and using geometric series expansion we obtain:

$$\begin{aligned}
1+d-e &= \frac{\frac{(t_{i+1}-t_{i+2})(t_{i+1}-t_{i+3})}{(t_{i+3}-t_{i+1})(t_{i+2}-t_i)}+O(\frac{1}{m})}{1+O(\frac{1}{m})} \\
&= \left(\frac{(t_{i+1}-t_{i+2})(t_{i+1}-t_{i+3})}{(t_{i+3}-t_{i+1})(t_{i+2}-t_i)} + O(\frac{1}{m})\right) \\
&\quad \cdot \left(1 + O(\frac{1}{m}) + O(\frac{1}{m^2}) + \ldots + O(\frac{1}{m^s}) + \ldots\right) \\
&= \frac{(t_{i+2}-t_{i+1})(t_{i+3}-t_{i+1})}{(t_{i+3}-t_i)(t_{i+2}-t_i)} + O(\frac{1}{m}) , \qquad (3.26)
\end{aligned}$$

where $s \geq 0$ integer. Note that curvature $k(t)$ is here bounded and separated from zero. Again as sampling is more-or-less uniform (1.28) the latter amounts to

$$0 < \frac{K_l^2}{3K_u^2} \leq \frac{(t_{i+2}-t_{i+1})(t_{i+3}-t_{i+1})}{(t_{i+3}-t_i)(t_{i+2}-t_i)} + O(\frac{1}{m}) .$$

Hence (3.19) follows. In fact, $0 < \delta \leq 1+d-e$ (for some $\delta > 0$), asymptotically, by (1.28).

We show now that one of the pairs $(\alpha_\pm, \beta_\pm)$ satisfies (3.10). More precisely, if curvature of curve γ satisfies $k(t) < 0$ then the pair (α_+, β_+) fulfills

$$\boxed{1 < \alpha_+ < \beta_+ .} \qquad (3.27)$$

The opposite case of $k(t) > 0$ involves also the pair (α_+, β_+). It is sufficient (due to the analogous argument to be used) to justify the first case only. To prove (3.27) we combine more-or-less uniformity, convexity of γ, with (3.6) and (3.21) to obtain for $\mathcal{R}(c,d) = cd$ which coincides with

$$\begin{aligned}
\mathcal{R}(c,d) &= \det(\gamma_2(t_{i+1}), \gamma_2(t_{i+3})) + O(\frac{1}{m^4}) \\
&= (1/2)k(t_i)(t_{i+1}-t_i)(t_{i+3}-t_i)(t_{i+3}-t_{i+1}) + O(\frac{1}{m^4}) .
\end{aligned}$$

Hence, repeating the argument as for (3.26) we get

$$d = \frac{-(t_{i+1}-t_i)(t_{i+3}-t_{i+1})}{(t_{i+2}-t_i)(t_{i+3}-t_{i+2})} + O(\frac{1}{m}) , \qquad (3.28)$$

which also, by (1.28) yields $d \leq -\delta < 0$, asymptotically, for some $\delta > 0$. Similarly, by (3.6), (3.25) (taking also into account (3.28)) we arrive at

$$\begin{aligned}
e &= \frac{-(t_{i+1}-t_i)(t_{i+2}-t_{i+1})}{(t_{i+3}-t_i)(t_{i+3}-t_{i+2})} + O(\frac{1}{m}) , \\
\frac{e}{d} &= \frac{(t_{i+2}-t_{i+1})(t_{i+2}-t_i)}{(t_{i+3}-t_{i+1})(t_{i+3}-t_i)} + O(\frac{1}{m}) ,
\end{aligned}$$

which also render both $d < 0$, $e/d > 0$ separated from zero by (1.28). Thus the latter combined with (3.26) yields

$$\begin{aligned}\frac{e}{d}(1+d-e) &= \frac{(t_{i+2}-t_{i+1})^2}{(t_{i+3}-t_i)^2} + O(\frac{1}{m}) ,\\ e-d &= 1-\frac{(t_{i+2}-t_{i+1})(t_{i+3}-t_{i+1})}{(t_{i+3}-t_i)(t_{i+2}-t_i)} + O(\frac{1}{m}) .\end{aligned} \tag{3.29}$$

To see that $e-d > 0$ and is separated from zero (not straightforward from (3.29)), we expand the numerator η in

$$\chi = \frac{(t_{i+3}-t_i)(t_{i+2}-t_i)-(t_{i+2}-t_i)(t_{i+3}-t_{i+1})}{(t_{i+3}-t_i)(t_{i+2}-t_i)} ,$$

upon adding and subtracting $t_i t_{i+1}$, to

$$\begin{aligned}\eta &= -t_{i+3}t_i - t_i t_{i+2} + t_i^2 + t_{i+1}t_{i+2} + t_{i+1}t_{i+3} - t_{i+1}^2 + t_i t_{i+1} - t_i t_{i+1}\\ &= t_i(t_i+t_{i+1}-t_{i+2}-t_{i+3}) - t_{i+1}(t_i+t_{i+1}-t_{i+2}-t_{i+3})\\ &= (t_{i+1}-t_i)(t_{i+3}-t_i+t_{i+2}-t_{i+1}) .\end{aligned}$$

The latter combined with $e-d=\chi+O(1/m)$ and (1.28) yields $e-d \geq \delta > 0$, holding asymptotically, for some $\delta > 0$.

Furthermore, Taylor's Theorem applied to $f(x) = \sqrt{1+x}$ at $x = 0$, coupled with (1.28), (3.7) and first equation from (3.29) yields asymptotically

$$\rho_1 = \frac{t_{i+2}-t_{i+1}}{t_{i+3}-t_i}\sqrt{1 + O(\frac{1}{m})} = \frac{t_{i+2}-t_{i+1}}{t_{i+3}-t_i} + O(\frac{1}{m}) .$$

Combining the latter with (1.28), (3.8) and (3.29) yields

$$\begin{aligned}\alpha_+ &= \frac{\frac{(t_{i+3}-t_i)(t_{i+2}-t_i)+(t_{i+2}-t_{i+1})(t_{i+2}-t_i)}{(t_{i+3}-t_i)(t_{i+2}-t_i)-(t_{i+2}-t_{i+1})(t_{i+3}-t_{i+1})}+O(\frac{1}{m})}{1+O(\frac{1}{m})}\\ &= \frac{(t_{i+2}-t_i)\left((t_{i+3}-t_i)+(t_{i+2}-t_{i+1})\right)}{(t_{i+3}-t_i)(t_{i+2}-t_i)-(t_{i+2}-t_{i+1})(t_{i+3}-t_{i+1})} + O(\frac{1}{m}) .\end{aligned} \tag{3.30}$$

Expanding the denominator θ of (3.30) as follows:

$$\begin{aligned}\theta &= -t_i t_{i+3} - t_i t_{i+2} + t_i^2 + t_{i+1}t_{i+2} + t_{i+1}t_{i+3} - t_{i+1}^2\\ &= -t_i t_{i+3} - t_i t_{i+2} + t_i^2 + t_{i+1}t_{i+2} + t_{i+1}t_{i+3} - t_{i+1}^2 + t_i t_{i+1} - t_i t_{i+1}\\ &= t_{i+1}(t_{i+2}-t_{i+1}) - t_i(t_{i+2}-t_{i+1}) - t_i(t_{i+3}-t_i) + t_{i+1}(t_{i+3}-t_i)\\ &= (t_{i+1}-t_i)(t_{i+2}-t_{i+1}) + (t_{i+1}-t_i)(t_{i+3}-t_i)\\ &= (t_{i+1}-t_i)\left((t_{i+2}-t_{i+1})+(t_{i+3}-t_i)\right) .\end{aligned}$$

Finally, combining the latter with (3.30) results in

$$\boxed{\alpha_+ \;=\; \tfrac{t_{i+2}-t_i}{t_{i+1}-t_i} \;+\; O(\tfrac{1}{m}) \;=\; 1 \;+\; \tfrac{t_{i+2}-t_{i+1}}{t_{i+1}-t_i} \;+\; O(\tfrac{1}{m})\;.} \tag{3.31}$$

A similar analysis used to prove (3.31) shows that

$$\boxed{\beta_+ \;=\; 1 \;+\; \tfrac{t_{i+3}-t_{i+1}}{t_{i+1}-t_i} \;+\; O(\tfrac{1}{m})\;.} \tag{3.32}$$

Because sampling is more-or-less uniform the formulas (3.31) and (3.32) guarantee that $1 < \alpha_+ < \beta_+$ hold asymptotically. Even more, by (1.28), (3.31), and (3.32) both α_+ and β_+ are separated from 1 and from each other, asymptotically. ☐

3.3 Auxiliary results

From now on the third-order Taylor's expansion of γ is needed to justify the fast approximation results claimed by Theorem 3.1. Even more advanced computational effort is here involved.

We assume now $\gamma \in C^4$ and hence

$$\boxed{\gamma(t) \;=\; \gamma(t_i) + \dot{\gamma}(t_i)(t-t_i) + \tfrac{\ddot{\gamma}}{2}(t_i)(t-t_i)^2 + \frac{\frac{d^3\gamma}{dt^3}(t_i)}{6}(t-t_i)^3 + O(\tfrac{1}{m^4})\;.} \tag{3.33}$$

The Lemma 3.2 below gives sharper estimates for (α_+, β_+). In doing so, define

$$l(t) \;=\; \frac{\det(\frac{d\gamma}{dt}, \frac{d^3\gamma}{dt^3})}{k(t)}\;. \tag{3.34}$$

Then, for $t, u \in [t_i, t_{i+3}]$, and up to a $O(1/m^5)$ term we have

$$\det(\gamma(t) - q_i, \gamma(u) - q_i) \;=\; k\frac{(t-t_i)(u-t_i)(u-t)}{2}(1 + (u - 2t_i + t)\frac{l}{3})\;, \tag{3.35}$$

where k, l are evaluated at t_i. Indeed, by (3.1), (3.33), and (3.34), upon some simplifications, we have for $\xi = \det(\gamma(t) - q_i, \gamma(u) - q_i)$

$$\begin{aligned}
\xi \;&=\; \frac{\det(\dot{\gamma}, \ddot{\gamma})}{2}(t-t_i)(u-t_i)^2 \;+\; \frac{\det(\frac{d\gamma}{dt}, \frac{d^3\gamma}{dt^3})}{6}(t-t_i)(u-t_i)^3 \\
&\quad - \frac{\det(\dot{\gamma}, \ddot{\gamma})}{2}(t-t_i)^2(u-t_i) \;-\; \frac{\det(\frac{d\gamma}{dt}, \frac{d^3\gamma}{dt^3})}{6}(t-t_i)^3(u-t_i) \\
&=\; \frac{k}{2}(t-t_i)(u-t_i)((u-t) + \frac{l}{3}((u-t_i)^2 - (t-t_i)^2)) \;+\; O(\frac{1}{m^5}) \\
&=\; \frac{k}{2}(t-t_i)(u-t_i)((u-t) + \frac{l}{3}(u-t)(u-2t_i+t)) \;+\; O(\frac{1}{m^5}) \\
&=\; k\frac{(t-t_i)(u-t_i)(u-t)}{2}(1 + (u-2t_i+t)\frac{l}{3}) \;+\; O(\frac{1}{m^5})\;.
\end{aligned}$$

The following holds:

Lemma 3.2. *For $\gamma \in C^4$ and sampled more-or-less uniformly (i.e. $\{t_i\}_{i=0}^m \in \mathcal{V}_{mol}^m$)*

$$\boxed{(\alpha_+, \beta_+) \;=\; \frac{((t_{i+2}-t_i)(1+\frac{l(t_{i+2}-t_{i+1})}{6}),(t_{i+3}-t_i)(1+\frac{l(t_{i+3}-t_{i+1})}{6}))}{t_{i+1}-t_i}\,,} \tag{3.36}$$

up to a $O(1/m^2)$ term.

Proof. By (3.35)

$$\begin{aligned} c &= -k\frac{(t_{i+2}-t_i)(t_{i+3}-t_i)(t_{i+3}-t_{i+2})}{2}(1+(t_{i+3}-2t_i+t_{i+2})\frac{l}{3})\,, \\ cd &= -k\frac{(t_{i+3}-t_i)(t_{i+1}-t_i)(t_{i+1}-t_{i+3})}{2}(1+(t_{i+1}-2t_i+t_{i+3})\frac{l}{3})\,, \\ ce &= -k\frac{(t_{i+2}-t_i)(t_{i+1}-t_i)(t_{i+1}-t_{i+2})}{2}(1+(t_{i+1}-2t_i+t_{i+2})\frac{l}{3})\,, \end{aligned} \tag{3.37}$$

up to a $O(1/m^5)$ term. Consequently

$$\begin{aligned} -d &= \frac{(t_{i+1}-t_i)(t_{i+3}-t_{i+1})}{(t_{i+3}-t_{i+2})(t_{i+2}-t_i)}(1-(t_{i+2}-t_{i+1})\frac{l}{3}) \;+\; O(\frac{1}{m^2})\,, \\ -e &= \frac{(t_{i+1}-t_i)(t_{i+2}-t_{i+1})}{(t_{i+3}-t_{i+2})(t_{i+3}-t_i)}(1-(t_{i+3}-t_{i+1})\frac{l}{3}) \;+\; O(\frac{1}{m^2})\,. \end{aligned} \tag{3.38}$$

We justify only the first formula from (3.38) as the second one follows analogously. Indeed, let $-(d/\tau) = \mu$, where $\tau = ((t_{i+1}-t_i)(t_{i+3}-t_{i+1}))/((t_{i+3}-t_{i+2})(t_{i+2}-t_i))$. By (3.37), $|k| \geq \delta > 0$, and (1.28) we arrive at

$$\begin{aligned} \mu &= \frac{1+(t_{i+1}-2t_i+t_{i+3})\frac{l}{3}+O(\frac{1}{m^2})}{1+(t_{i+3}-2t_i+t_{i+2})\frac{l}{3}+O(\frac{1}{m^2})} \\ &= \frac{1}{1-(2t_i-t_{i+3}-t_{i+2})\frac{l}{3}+O(\frac{1}{m^2})} \;+\; \frac{(t_{i+1}-2t_i+t_{i+3})\frac{l}{3}+O(\frac{1}{m^2})}{1-(2t_i-t_{i+3}-t_{i+2})\frac{l}{3}+O(\frac{1}{m^2})}\,. \end{aligned}$$

By (1.28) $2t_i - t_{i+3} - t_{i+2} = -(t_{i+3}-t_i) - (t_{i+2}-t_i) = O(1/m)$ and since $l = O(1)$

(as γ is C^4; in fact C^3 suffices here) the formula for geometric series yields:

$$\begin{aligned}
\mu &= \frac{1}{1-(2t_i-t_{i+3}-t_{i+2})\frac{l}{3}+O(\frac{1}{m^2})}+\frac{(t_{i+1}-2t_i+t_{i+3})\frac{l}{3}+O(\frac{1}{m^2})}{1-O(\frac{1}{m})}\\
&= 1+(2t_i-t_{i+3}-t_{i+2})\frac{l}{3}+O(\frac{1}{m^2})+\left((2t_i-t_{i+3}-t_{i+2})\frac{l}{3}+O(\frac{1}{m^2})\right)^2\\
&\quad +\cdots\\
&\quad +\left((t_{i+1}-2t_i+t_{i+3})\frac{l}{3}+O(\frac{1}{m^2})\right)\left(1+O(\frac{1}{m})+O(\frac{1}{m^2})+\cdots\right)\\
&= 1-(t_{i+2}-t_{i+1})\frac{l}{3}+O(\frac{1}{m^2})\ .
\end{aligned}$$

Hence, as $-d=\tau\mu$ and $\tau=O(1)$ (by (1.28)) we arrive at the first formula in (3.38). To substantiate formulas (3.36), the *Mathematica* symbolic calculations (see the URL address `http://www.cs.uwa.edu.au/~ryszard/4points/`) can be used as originally performed in [44]. Alternatively, a full analytical proof justifying (3.36) can be derived. Indeed, by (3.38) and more-or-less uniformity we have $e-d$

$$\begin{aligned}
&= \frac{t_{i+1}-t_i}{t_{i+3}-t_{i+2}}\left(\frac{t_{i+3}-t_{i+1}}{t_{i+2}-t_i}-\frac{t_{i+2}-t_{i+1}}{t_{i+3}-t_i}\right.\\
&\quad \left.-\frac{l}{3}\left(\frac{(t_{i+2}-t_{i+1})(t_{i+3}-t_{i+1})}{t_{i+2}-t_i}-\frac{(t_{i+3}-t_{i+1})(t_{i+2}-t_{i+1})}{t_{i+3}-t_i}\right)\right)+O(\frac{1}{m^2})\\
&= \frac{t_{i+1}-t_i}{t_{i+3}-t_{i+2}}\left(\frac{(t_i+t_{i+1}-t_{i+2}-t_{i+3})(t_{i+2}-t_{i+3})}{(t_{i+2}-t_i)(t_{i+3}-t_i)}\right.\\
&\quad \left.-\frac{l}{3}\frac{(t_{i+3}-t_{i+1})(t_{i+2}-t_{i+1})(t_{i+3}-t_{i+2})}{(t_{i+2}-t_i)(t_{i+3}-t_i)}\right)+O(\frac{1}{m^2})\ .
\end{aligned}$$

Hence, up to $O(1/m^2)$ term, we obtain

$$e-d=\frac{(t_{i+1}-t_i)(t_i+t_{i+1}-t_{i+2}-t_{i+3})}{(t_{i+2}-t_i)(t_i-t_{i+3})}\left(1+\frac{l}{3}\frac{(t_{i+2}-t_{i+1})(t_{i+3}-t_{i+1})}{t_i+t_{i+1}-t_{i+2}-t_{i+3}}\right)\ . \tag{3.39}$$

Thus, (1.28) and formula for geometric series yield

$$\begin{aligned}
\frac{1}{e-d} &= \frac{(t_{i+2}-t_i)(t_i-t_{i+3})}{(t_{i+1}-t_i)(t_i+t_{i+1}-t_{i+2}-t_{i+3})}\\
&\quad \left(1-\frac{l}{3}\frac{(t_{i+2}-t_{i+1})(t_{i+3}-t_{i+1})}{t_i+t_{i+1}-t_{i+2}-t_{i+3}}+O(\frac{1}{m^2})\right.\\
&\quad \left.+\left(\frac{l}{3}\frac{(t_{i+2}-t_{i+1})(t_{i+3}-t_{i+1})}{t_i+t_{i+1}-t_{i+2}-t_{i+3}}+O(\frac{1}{m^2})\right)^2+\cdots\right)\ .
\end{aligned}$$

Therefore, up to $O(1/m^2)$ term, $(e-d)^{-1}=$

$$\frac{(t_{i+2}-t_i)(t_i-t_{i+3})}{(t_{i+1}-t_i)(t_i+t_{i+1}-t_{i+2}-t_{i+3})}\left(1-\frac{l}{3}\frac{(t_{i+2}-t_{i+1})(t_{i+3}-t_{i+1})}{t_i+t_{i+1}-t_{i+2}-t_{i+3}}\right)\ . \tag{3.40}$$

Furthermore, by (3.39), up to $O(1/m^2)$ term, $1+d-e$

$$
\begin{aligned}
&= 1-\frac{(t_{i+1}-t_i)(t_i+t_{i+1}-t_{i+2}-t_{i+3})}{(t_{i+2}-t_i)(t_i-t_{i+3})}\left(1+\frac{l}{3}\frac{(t_{i+2}-t_{i+1})(t_{i+3}-t_{i+1})}{t_i+t_{i+1}-t_{i+2}-t_{i+3}}\right)\\
&= \frac{(t_{i+2}-t_i)(t_i-t_{i+3})-(t_{i+1}-t_i)(t_i+t_{i+1}-t_{i+2}-t_{i+3})}{(t_{i+2}-t_i)(t_i-t_{i+3})}\\
&\quad -\frac{l}{3}\frac{(t_{i+2}-t_{i+1})(t_{i+3}-t_{i+1})(t_{i+1}-t_i)}{(t_i-t_{i+3})(t_{i+2}-t_i)}\\
&= \frac{t_{i+2}(t_{i+1}-t_{i+3})-t_{i+1}(t_{i+1}-t_{i+3})}{(t_{i+2}-t_i)(t_i-t_{i+3})}\\
&\quad -\frac{l}{3}\frac{(t_{i+2}-t_{i+1})(t_{i+3}-t_{i+1})(t_{i+1}-t_i)}{(t_i-t_{i+3})(t_{i+2}-t_i)}\ ,
\end{aligned}
$$

and hence

$$
1+d-e = \frac{(t_{i+1}-t_{i+2})(t_{i+1}-t_{i+3})}{(t_i-t_{i+2})(t_i-t_{i+3})}\big(1-\frac{l}{3}(t_i-t_{i+1})\big) + O(\frac{1}{m^2})\ . \tag{3.41}
$$

Again by (1.28), (3.38) and geometric series formula, we have

$$
\begin{aligned}
\frac{e}{d} &= \frac{(t_{i+2}-t_{i+1})(t_{i+2}-t_i)}{(t_{i+3}-t_i)(t_{i+3}-t_{i+1})}\frac{1-(t_{i+3}-t_{i+1})\frac{l}{3}+O(\frac{1}{m^2})}{1-(t_{i+2}-t_{i+1})\frac{l}{3}+O(\frac{1}{m^2})}\\
&= \frac{(t_{i+2}-t_{i+1})(t_{i+2}-t_i)}{(t_{i+3}-t_i)(t_{i+3}-t_{i+1})}(1-(t_{i+3}-t_{i+1})\frac{l}{3}+O(\frac{1}{m^2}))\\
&\quad \cdot\left((t_{i+2}-t_{i+1})\frac{l}{3}+O(\frac{1}{m^2})+\big((t_{i+2}-t_{i+1})\frac{l}{3}+O(\frac{1}{m^2})\big)^2+\dots\right)\\
&= \frac{(t_{i+2}-t_{i+1})(t_{i+2}-t_i)}{(t_{i+3}-t_i)(t_{i+3}-t_{i+1})}\big(1+\frac{l}{3}(t_{i+2}-t_{i+3})\big) + O(\frac{1}{m^2})\ .
\end{aligned} \tag{3.42}
$$

And similarly

$$
\frac{d}{e} = \frac{(t_{i+3}-t_{i+1})(t_{i+3}-t_i)}{(t_{i+2}-t_i)(t_{i+2}-t_{i+1})}\big(1+\frac{l}{3}(t_{i+3}-t_{i+2})\big) + O(\frac{1}{m^2})\ . \tag{3.43}
$$

Combining now (1.28), (3.41), and (3.42) yields, for $(e/d)(1+d-e)$

$$
\begin{aligned}
&= \left(\frac{t_{i+2}-t_{i+1}}{t_{i+3}-t_i}\right)^2\big(1+\frac{l}{3}(t_{i+2}-t_{i+3})\big)+O(\frac{1}{m^2})\big(1-\frac{l}{3}(t_i-t_{i+1})\big)\\
&\quad +O(\frac{1}{m^2})\\
&= \left(\frac{t_{i+2}-t_{i+1}}{t_{i+3}-t_i}\right)^2\big(1+\frac{l}{3}(t_{i+2}+t_{i+1}-t_i-t_{i+3})\big) + O(\frac{1}{m^2})\ .
\end{aligned}
$$

Similarly, by (1.28), (3.41), and (3.43) we arrive at

$$\frac{d}{e}(1+d-e) \;=\; \left(\frac{t_{i+3}-t_{i+1}}{t_{i+2}-t_i}\right)^2 \big(1+\frac{l}{3}(t_{i+3}+t_{i+1}-t_i-t_{i+2})\big) \;+\; O(\frac{1}{m^2}) \; .$$

Thus Taylor's Theorem applied to $f(x)=\sqrt{1+x}$ at $x=0$ renders

$$\sqrt{\frac{e}{d}(1+d-e)} \;=\; \frac{t_{i+2}-t_{i+1}}{t_{i+3}-t_i}\big(1+\frac{l}{6}(t_{i+2}+t_{i+1}-t_i-t_{i+3})\big) \;+\; O(\frac{1}{m^2}) \; , \quad (3.44)$$

and

$$\sqrt{\frac{d}{e}(1+d-e)} \;=\; \frac{t_{i+3}-t_{i+1}}{t_{i+2}-t_i}\big(1+\frac{l}{6}(t_{i+3}+t_{i+1}-t_i-t_{i+2})\big) \;+\; O(\frac{1}{m^2}) \; . \quad (3.45)$$

Combining now (3.8), (3.40), and (3.44) yield $\alpha_+ = ((t_{i+2}-t_i)/(t_{i+1}-t_i))\sigma_1\sigma_2$, where

$$\begin{aligned}
\sigma_1 &= 1+\frac{t_{i+2}-t_{i+1}}{t_{i+3}-t_i}\big(1+\frac{l}{6}(t_{i+2}+t_{i+1}-t_i-t_{i+3})\big) \;+\; O(\frac{1}{m^2}) \\
&= \frac{t_{i+3}-t_i+t_{i+2}-t_{i+1}}{t_{i+3}-t_i} \;+\; \frac{l}{6}\frac{(t_{i+2}+t_{i+1}-t_i-t_{i+3})(t_{i+2}-t_{i+1})}{t_{i+3}-t_i} \\
&\quad + O(\frac{1}{m^2}) \; , \\
\sigma_2 &= \frac{t_i-t_{i+3}}{t_i+t_{i+1}-t_{i+2}-t_{i+3}} \;-\; \frac{l}{3}\frac{(t_{i+2}-t_{i+1})(t_{i+3}-t_{i+1})(t_i-t_{i+3})}{(t_i+t_{i+1}-t_{i+2}-t_{i+3})^2} \\
&\quad + O(\frac{1}{m^2}) \; ,
\end{aligned}$$

and consequently, up to $O(1/m^2)$ term, we have

$$\begin{aligned}
\sigma_1\sigma_2 &= 1 \;+\; \frac{l}{3}\frac{(t_{i+2}-t_{i+1})(t_{i+3}-t_{i+1})}{t_{i+3}+t_{i+2}-t_{i+1}-t_i} \\
&\quad + \frac{l}{6}\frac{(t_{i+2}+t_{i+1}-t_i-t_{i+3})(t_{i+2}-t_{i+1})}{t_{i+3}+t_{i+2}-t_{i+1}-t_i} \\
&= 1 \;+\; \frac{l}{6}(t_{i+2}-t_{i+1})\frac{2(t_{i+3}-t_{i+1})+t_{i+2}+t_{i+1}-t_i-t_{i+3}}{t_{i+3}+t_{i+2}-t_{i+1}-t_i} \\
&= 1 \;+\; \frac{l}{6}(t_{i+2}-t_{i+1}) \; .
\end{aligned}$$

Thus (1.28) and $\alpha_+ = ((t_{i+2}-t_i)/(t_{i+1}-t_i))\sigma_1\sigma_2$ yield the first formula in (3.36). Clearly, a similar proof for β_+ from (3.36) follows from (3.8), (3.40), and (3.45). Indeed as now $\beta_+ = ((t_{i+3}-t_i)/(t_{i+2}-t_i))\eta_1\eta_2$, where up to a term of order $O(1/m^2)$ we have

$$\begin{aligned}
\eta_1 &= \frac{t_{i+2}-t_i+t_{i+3}-t_{i+1}}{t_{i+2}-t_i} \;+\; \frac{l}{6}\frac{(t_{i+3}+t_{i+1}-t_i-t_{i+2})(t_{i+3}-t_{i+1})}{t_{i+2}-t_i} \; , \\
\eta_2 &= \frac{t_i-t_{i+2}}{t_i+t_{i+1}-t_{i+2}-t_{i+3}} \;-\; \frac{l}{3}\frac{(t_{i+2}-t_{i+1})(t_{i+3}-t_{i+1})(t_i-t_{i+2})}{(t_i+t_{i+1}-t_{i+2}-t_{i+3})^2} \; .
\end{aligned}$$

And furthermore as previously, up to a $O(1/m^2)$ term, we obtain $\eta_1\eta_2$

$$\begin{aligned} &= 1 + \frac{l}{3}\frac{(t_{i+2}-t_{i+1})(t_{i+3}-t_{i+1})}{t_{i+3}+t_{i+2}-t_{i+1}-t_i} + \frac{l}{6}\frac{(t_{i+3}+t_{i+1}-t_i-t_{i+2})(t_{i+3}-t_{i+1})}{t_{i+3}+t_{i+2}-t_{i+1}-t_i} \\ &= 1 + \frac{l}{6}(t_{i+3}-t_{i+1}) \ . \end{aligned}$$

The proof is complete. ☐

We continue with the assumption that either $k(t) < 0$ or $k(t) > 0$ for all t. Then, for $0 \le s \le \beta$, and $Q^i(s) = q_i + a_1 s + a_2 s^2$ (see also (3.3)), where $a_0 = q_0$, and a_i, for $i = 1, 2$ are defined by (3.9), the following holds:

Lemma 3.3.

$$\boxed{a_1 \ = \ (t_{i+1}-t_i)\dot\gamma(t_i) \ + \ O(\tfrac{1}{m^2}) \quad \textit{and} \quad a_2 \ = \ O(\tfrac{1}{m^2}) \ .} \tag{3.46}$$

In particular, for $s \in [0, \beta_+]$,

$$\boxed{\tfrac{dQ^i}{ds} \ = \ O(\tfrac{1}{m}) \quad \textit{and} \quad \tfrac{d^2Q^i}{ds^2} \ = \ O(\tfrac{1}{m^2}) \ .} \tag{3.47}$$

Proof. From (3.31), the following holds

$$(\alpha^2, \alpha - \alpha^2) = \frac{(t_{i+2}-t_i)(t_{i+2}-t_i, t_{i+1}-t_{i+2})}{(t_{i+1}-t_i)^2} \ + \ O(\frac{1}{m}) \ . \tag{3.48}$$

By Taylor's Theorem $(p_1, p_\alpha) = \dot\gamma((t_{i+1}-t_i), (t_{i+2}-t_i)) + O(1/m^2)$, where $\dot\gamma$ is evaluated at t_i. The latter combined with (1.28), (3.9), and (3.48) yields

$$\begin{aligned} a_1 \ &= \ \frac{(t_{i+1}-t_i)^2}{(t_{i+2}-t_i)(t_{i+1}-t_{i+2})}(p_{\alpha_+} - \alpha_+^2 p_1)\left(1 \ + \ O(\frac{1}{m})\right) \\ &= \ \frac{(t_{i+1}-t_i)^2}{(t_{i+2}-t_i)(t_{i+1}-t_{i+2})}\left(\dot\gamma(t_{i+2}-t_i) \ - \ \dot\gamma\frac{(t_{i+2}-t_i)^2}{t_{i+1}-t_i} \ + \ O(\frac{1}{m^2})\right) \\ &\qquad \cdot\left(1 \ + \ O(\frac{1}{m})\right) \\ &= \ \left(\dot\gamma\frac{(t_{i+1}-t_i)^2}{t_{i+1}-t_{i+2}} \ - \ \dot\gamma\frac{(t_{i+2}-t_i)(t_{i+1}-t_i)}{t_{i+1}-t_{i+2}} \ + \ O(\frac{1}{m^2})\right)\left(1 \ + \ O(\frac{1}{m})\right) \\ &= \ \left((t_{i+1}-t_i)\dot\gamma \ + \ O(\frac{1}{m^2})\right)\left(1 \ + \ O(\frac{1}{m})\right) \\ &= \ (t_{i+1}-t_i)\dot\gamma \ + \ O(\frac{1}{m^2}) \ . \end{aligned}$$

In a similar fashion,

$$
\begin{aligned}
a_2 &= \frac{(t_{i+1}-t_i)^2}{(t_{i+2}-t_i)(t_{i+1}-t_{i+2})}(\alpha_+ p_1 - p_{\alpha_+})\left(1 \;+\; O(\frac{1}{m})\right) \\
&= \frac{(t_{i+1}-t_i)^2}{(t_{i+2}-t_i)(t_{i+1}-t_{i+2})}\left(\dot\gamma(t_{i+2}-t_i) \;-\; \dot\gamma(t_{i+2}-t_i) \;+\; O(\frac{1}{m^2})\right) \\
&\quad \cdot\left(1 \;+\; O(\frac{1}{m})\right) \\
&= O(\frac{1}{m^2}) \;.
\end{aligned}
$$

As $Q^i : [0,\beta_+] \to \mathbb{R}^2$, and as by (3.32) $\beta_+ = O(1)$ then $Q^{i'} = 2a_2 s + a_1 = O(1/m)$ and $Q^{i''} = 2a_2 = O(1/m^2)$. The proof is complete. □

The Lemma 3.3 yields the asymptotics of the derivatives of Q^i used later to prove Theorem 3.1.

Remark 3.1. Note that in proving Lemma 3.3 one can use a better approximation of α as specified by (3.36). Indeed repeating the above proof yields

$$
\begin{aligned}
a_1 &= (t_{i+1}-t_i)\dot\gamma(t_i)(1 \;+\; O(\frac{1}{m})) \;+\; O(\frac{1}{m^3}) \;, \\
a_2 &= \frac{(t_{i+1}-t_i)^2}{2}\ddot\gamma(t_i)(1 \;+\; O(\frac{1}{m})) \;+\; O(\frac{1}{m^2})\dot\gamma(t_i) \;+\; O(\frac{1}{m^3}) \;,
\end{aligned} \tag{3.49}
$$

which evidently gives the same asymptotics for $a_1 = O(1/m)$ and $a_2 = O(1/m)$. Thus (3.31) suffices to justify Lemma 3.3. Formulas (3.36) will be, however needed in proving the next Lemma.

The quadratics Q^i, determined by $\mathcal{Q}_m^{i,3}$, need to be *reparameterized* for comparison with the original curve γ and $Q^i \circ \psi^i$, where ψ^i is some reparameterization. In doing so let $\psi^i : [t_i, t_{i+3}] \to [0,\beta_+]$ be *the Lagrange cubic* (we suppress index i in β_+)

$$
\boxed{\psi^i(t) \;=\; b_0^i \;+\; b_1^i t \;+\; b_2^i t^2 \;+\; b_3^i t^3}
$$

given by

$$
\boxed{\psi^i(t_i) \;=\; 0, \quad \psi^i(t_{i+1}) \;=\; 1 \;, \quad \psi^i(t_{i+2}) \;=\; \alpha_+ \;, \quad \psi^i(t_{i+3}) \;=\; \beta_+ \;.} \tag{3.50}
$$

The existence and uniqueness of ψ^i follows immediately from classical Interpolation Theory (see *e.g.* Chapter 1 of [33] - see also (1.8) with $r = 3$). The following holds:

Lemma 3.4. *For a large m the cubic ψ^i satisfying* (3.50) *defines a diffeomorphism and*

$$
\boxed{\boxed{\frac{d^k\psi^i}{dt^k} \;=\; O(m) \;, \qquad \textit{for} \quad k = 1,2,3 \;.}} \tag{3.51}
$$

Proof. In order to prove (3.51) it suffices to find the asymptotics for the coefficients of ψ^i, *i.e.* b_1^i, b_2^i, and b_3^i. As shown in [44], using Lemma 3.2 and *Mathematica* symbolic calculations (see URL address `http://www.cs.uwa.edu.au/~ryszard/4points/`, especially for treatment of $O(1/m^2)$ errors in (3.36)) yield (3.51). Alternatively, as in Lemma 3.2, a full analytical proof justifying (3.51) is derived herein. Taking into account (3.36), for

$$\hat{\alpha}_+ = \frac{t_{i+2}-t_i}{t_{i+1}-t_i}\big(1 + \frac{l}{6}(t_{i+2}-t_{i+1})\big) , \qquad \tilde{\alpha}_+ = \alpha_+ - \hat{\alpha}_+ = O(\frac{1}{m^2}) , \quad (3.52)$$

$$\hat{\beta}_+ = \frac{t_{i+3}-t_i}{t_{i+1}-t_i}\big(1 + \frac{l}{6}(t_{i+3}-t_{i+1})\big) , \qquad \tilde{\beta}_+ = \beta_+ - \hat{\beta}_+ = O(\frac{1}{m^2}) \quad (3.53)$$

define now *two cubics*

$$\hat{\psi}^i(t) = \hat{b}_0^i + \hat{b}_1^i t + \hat{b}_2^i t^2 + \hat{b}_3^i t^3 \quad \text{and} \quad \tilde{\psi}^i(t) = \tilde{b}_0^i + \tilde{b}_1^i t + \tilde{b}_2^i t^2 + \tilde{b}_3^i t^3 ,$$

$\hat{\psi}^i, \tilde{\psi}^i : [t_i, t_{i+3}] \to [0,\beta]$, satisfying the following interpolation conditions

$$\hat{\psi}^i(t_i) = 0, \quad \hat{\psi}^i(t_{i+1}) = 1 , \quad \hat{\psi}^i(t_{i+2}) = \hat{\alpha}_+ , \quad \hat{\psi}^i(t_{i+3}) = \hat{\beta}_+ , \quad (3.54)$$

$$\tilde{\psi}^i(t_i) = 0, \quad \tilde{\psi}^i(t_{i+1}) = 1 , \quad \tilde{\psi}^i(t_{i+2}) = \tilde{\alpha}_+ , \quad \tilde{\psi}^i(t_{i+3}) = \tilde{\beta}_+ . \quad (3.55)$$

Again the existence and uniqueness of $\hat{\psi}^i, \tilde{\psi}^i$ follows from the classical Interpolation Theory (see *e.g.* Chapter 1 of [33]). Since both domains of ψ^i and $\hat{\psi}^i + \tilde{\psi}^i$ coincide and $\deg(\psi^i) \leq 3$, $\deg(\hat{\psi}^i + \tilde{\psi}^i) \leq 3$, formulas (3.50), (3.52), (3.53), (3.54), (3.55) and uniqueness of Lagrange Interpolation result in $\psi^i = \hat{\psi}^i + \tilde{\psi}^i$. Hence

$$\frac{d^k\psi^i}{dt^k} = \frac{d^k\hat{\psi}^i}{dt^k} + \frac{d^k\tilde{\psi}^i}{dt^k} , \qquad \text{for} \quad k = 1,2,3 . \quad (3.56)$$

Lagrange interpolation formula (see [33]; Paragraph 1.2) renders $\hat{\psi}^i(t)$

$$\begin{aligned} &= \frac{(t-t_i)(t-t_{i+2})(t-t_{i+3})}{(t_{i+1}-t_i)(t_{i+1}-t_{i+2})(t_{i+1}-t_{i+3})} \\ &\quad + \frac{t_{i+2}-t_i}{t_{i+1}-t_i}\big(1+\frac{l}{6}(t_{i+2}-t_{i+1})\big)\frac{(t-t_i)(t-t_{i+1})(t-t_{i+3})}{(t_{i+2}-t_i)(t_{i+2}-t_{i+1})(t_{i+2}-t_{i+3})} \\ &\quad + \frac{t_{i+3}-t_i}{t_{i+1}-t_i}\big(1+\frac{l}{6}(t_{i+3}-t_{i+1})\big)\frac{(t-t_i)(t-t_{i+1})(t-t_{i+2})}{(t_{i+3}-t_i)(t_{i+3}-t_{i+1})(t_{i+3}-t_{i+2})} \\ &= \frac{(t-t_i)(t-t_{i+2})(t-t_{i+3})}{(t_{i+1}-t_i)(t_{i+1}-t_{i+2})(t_{i+1}-t_{i+3})} + \frac{(t-t_i)(t-t_{i+1})(t-t_{i+3})}{(t_{i+1}-t_i)(t_{i+2}-t_{i+1})(t_{i+2}-t_{i+3})} \\ &\quad + \frac{(t-t_i)(t-t_{i+1})(t-t_{i+2})}{(t_{i+1}-t_i)(t_{i+3}-t_{i+1})(t_{i+3}-t_{i+2})} + \frac{l}{6}\frac{(t-t_i)(t-t_{i+1})}{t_{i+1}-t_i} . \end{aligned} \quad (3.57)$$

Let $\eta = \hat{\psi}^i - (l/6)(t-t_i)(t-t_{i+1})(t_{i+1}-t_i)^{-1}$. By (3.57) we have

$$\begin{aligned} \eta &= \frac{t-t_i}{t_{i+1}-t_i}\Bigg(\frac{(t-t_{i+2})(t-t_{i+3})}{(t_{i+1}-t_{i+2})(t_{i+1}-t_{i+3})} + \frac{(t-t_{i+1})(t-t_{i+3})}{(t_{i+2}-t_{i+1})(t_{i+2}-t_{i+3})} \\ &\qquad + \frac{(t-t_{i+1})(t-t_{i+2})}{(t_{i+3}-t_{i+1})(t_{i+3}-t_{i+2})}\Bigg) . \end{aligned} \quad (3.58)$$

Thus, for $\kappa = \eta(t_{i+1} - t_i)(t - t_i)^{-1}$, we obtain

$$\begin{aligned}\kappa &= \frac{t - t_{i+3}}{t_{i+2} - t_{i+1}}\left(\frac{t - t_{i+2}}{t_{i+3} - t_{i+1}} + \frac{t - t_{i+1}}{t_{i+2} - t_{i+3}}\right) + \frac{(t - t_{i+1})(t - t_{i+2})}{(t_{i+3} - t_{i+1})(t_{i+3} - t_{i+2})} \\ &= \frac{(t - t_{i+3})(t + t_{i+3} - t_{i+2} - t_{i+1})}{(t_{i+3} - t_{i+1})(t_{i+2} - t_{i+3})} + \frac{(t - t_{i+1})(t - t_{i+2})}{(t_{i+3} - t_{i+1})(t_{i+3} - t_{i+2})} \\ &= \frac{t_{i+3}(t_{i+3} - t_{i+2}) - t_{i+1}(t_{i+3} - t_{i+2})}{(t_{i+3} - t_{i+1})(t_{i+3} - t_{i+2})} \\ &= 1\,. \end{aligned} \tag{3.59}$$

Note that (3.59) results independently from (3.58), where κ is expressed exactly by Lagrange interpolating formula as a quadratic satisfying $\kappa(t_{i+k}) = 1$, for $k = 1, 2, 3$. Since the function $f(x) \equiv 1$ mets these interpolation conditions and $\deg(f) \leq 2$, the uniqueness of such interpolant guarantees that $\kappa \equiv 1$. Thus, by (3.57) and (3.59) the cubic $\hat{\psi}^i$ collapses to a quadratic

$$\hat{\psi}^i(t) = \frac{l}{6}\frac{(t - t_i)(t - t_{i+1})}{t_{i+1} - t_i} + \frac{t - t_i}{t_{i+1} - t_i}$$

and therefore

$$(b_0^i, b_1^i, b_2^i, b_3^i) = \left(\frac{t_i(t_{i+1}l - 6)}{6(t_{i+1} - t_i)}, \frac{6 - l(t_i + t_{i+1})}{6(t_{i+1} - t_i)}, \frac{l}{6(t_{i+1} - t_i)}, 0\right).$$

Thus as $t_i \leq t \leq t_{i+3}$ by (1.28)

$$\boxed{\frac{d^k\hat{\psi}^i}{dt^k} = O(m)\,, \quad \text{for} \quad k = 1, 2; \quad \text{and} \quad \frac{d^3\hat{\psi}^i}{dt^3} = 0\,.} \tag{3.60}$$

Similarly, Lagrange interpolation formula coupled with (3.55) yields $\tilde{\psi}^i(t)$

$$= \frac{\tilde{\alpha}_+(t - t_i)(t - t_{i+1})(t - t_{i+3})}{(t_{i+2} - t_i)(t_{i+2} - t_{i+1})(t_{i+2} - t_{i+3})} + \frac{\tilde{\beta}_+(t - t_i)(t - t_{i+1})(t - t_{i+2})}{(t_{i+3} - t_i)(t_{i+3} - t_{i+1})(t_{i+3} - t_{i+2})}\,.$$

Thus by (1.28), $\tilde{\alpha}_+ = O(1/m^2)$, and $\beta_+ = O(1/m^2)$ we obtain

$$\frac{d^3\tilde{\psi}^i}{dt^3} = \frac{1}{t_{i+3} - t_{i+2}}\left(\frac{-6\tilde{\alpha}_+}{(t_{i+2} - t_i)(t_{i+2} - t_{i+1})} + \frac{6\tilde{\beta}_+}{(t_{i+3} - t_i)(t_{i+3} - t_{i+1})}\right)$$

and thus

$$\boxed{\frac{d^3\tilde{\psi}^i}{dt^3} = O(m)\,.}$$

Similarly, as $t_i \leq t \leq t_{i+3}$, we obtain

$$\boxed{\dot{\tilde{\psi}}^i = O(\tfrac{1}{m}) \quad \text{and} \quad \ddot{\tilde{\psi}}^i = O(1)\,.}$$

Combining the latter with (3.60) and (3.56) yields (3.51) and in particular

$$\dot{\psi}^i(t) = \frac{l}{6}\frac{t-t_i}{t_{i+1}-t_i} + \frac{l}{6}(t-t_i) + \frac{1}{t_{i+1}-t_i} + O(\frac{1}{m}) = \frac{1}{t_{i+1}-t_i} + O(1) > 0\,, \tag{3.61}$$

asymptotically. Thus ψ^i defines a diffeomorphism for large m. The proof is complete. □

The Lemma 3.4 yields the asymptotics of the derivatives of ψ^i used to prove Theorem 3.1.

Remark 3.2. Note that, in order to prove Lemma 3.4 we used for (α_+, β_+) the formulas from (3.36), instead of those claimed in (3.31) and (3.32). This stands in contrast with proving the Lemma 3.3, where (3.31) and (3.32) suffice to establish (3.46). If only (3.31) and (3.32) are used to prove Lemma 3.4 then upon repeating the above proof $\dot{\tilde{\psi}}^i = O(1)$, $\ddot{\tilde{\psi}}^i = O(m)$, $(d^3\tilde{\psi}^i/dt^3) = O(m^2)$ and $\dot{\hat{\psi}}^i = O(m)$, $\ddot{\hat{\psi}}^i = O(m)$, $(d^3\hat{\psi}^i/dt^3) = O(m^2)$ resulting in

$$\dot{\psi}^i \;=\; O(m)\,, \quad \ddot{\psi}^i \;=\; O(m)\,, \quad and \quad (d^3\psi^i/dt^3) \;=\; O(m^2)\,.$$

The latter would force (3.62) to be of order $O(m)$, which in turn would only establish cubic convergence orders in Theorem 3.1.

3.4 Proof of main result

In this section we prove the main result *i.e.* the Theorem 3.1.

Proof. Let $\psi : [0,T] \to [0,\hat{T}]$, where $\hat{T} = \sum_{i=0}^{\frac{m}{3}-1}\beta_{+3i}$ be the track-sum of $\{\psi^{3i}\}_{i=0}^{\frac{m}{3}-1}$ (see Remark 1.1). Similarly, let $Q : [0,\hat{T}] \to \mathbb{R}^2$ be the track-sum of $\{Q^{3i}\}_{i=0}^{\frac{m}{3}-1}$. For comparison reasons, define reparameterized

$$\boxed{\tilde{\gamma}_i \;=\; Q^i \circ \psi^i : [t_i, t_{i+3}] \to \mathbb{R}^2\;.}$$

Then $\tilde{\gamma}_i$ is polynomial of degree at most 6 and, using (3.47), (3.51) and Chain Rule

$$\frac{d^4\tilde{\gamma}_i}{dt^4} \;=\; 3Q^{i''}\ddot{\psi}^{i^2} + 4Q^{i''}\dot{\psi}^i\frac{d^3\psi^i}{dt^3} \;=\; O(\frac{1}{m^2})O(m^2) + O(\frac{1}{m^2})O(m^2) \;=\; O(1)\,. \tag{3.62}$$

In fact as it turns out that its derivatives of all orders are $O(1)$. The C^4 function $f = \tilde{\gamma}_i - \gamma$ is $\vec{0}$ at $t_i, t_{i+1}, t_{i+2}, t_{i+3}$, and consequently by Lemma 1.1

$$f(t) \;=\; (t-t_i)(t-t_{i+1})(t-t_{i+2})g(t)\,, \quad \text{where} \quad g(t) \;=\; (t-t_{i+3})h(t)\,,$$

and $g, h : [t_i, t_{i+3}] \to \mathbb{R}^2$ are C^1, C^0 respectively. As before we use Lemma 2.1 of Part I of [38] to estimate errors (see also Lemma 1.1 and Remark 1.5). Because $\frac{d^4 f}{dt^4} = O(1)$, $h = O(1)$. Therefore $g = O(\frac{1}{m})$. Also $\dot{g} = O(\frac{d^4 f}{dt^4}) = O(1)$. Therefore

$$\boxed{\dot{f} \; = \; O(\tfrac{1}{m^3}) \quad \text{and} \quad f \; = \; O(\tfrac{1}{m^4}) \; .} \tag{3.63}$$

Write $\dot{\tilde{\gamma}}_i(t)$ in the form $(1 + \langle \dot{f}(t), \dot{\gamma}(t) \rangle)\dot{\gamma}(t) + v(t)$, where $v(t)$ is the projection of $\dot{f}(t)$ onto the line orthogonal to $\dot{\gamma}(t)$. By (3.63), $v = O(\frac{1}{m^3})$ and, because $\|\dot{\gamma}\| = 1$,

$$\|\dot{\tilde{\gamma}}_i(t)\| \; = \; (1 + \langle \dot{f}(t), \dot{\gamma}(t) \rangle)\|\dot{\gamma}(t)\| \; + \; O(\frac{1}{m^6}) \; .$$

Then

$$\begin{aligned} \int_{t_i}^{t_{i+3}} (\|\dot{\tilde{\gamma}}_i(t)\| - \|\dot{\gamma}(t)\|) \, dt \;\; &= \;\; \int_{t_i}^{t_{i+3}} \langle \dot{f}(t), \dot{\gamma}(t) \rangle \, dt \; + \; O(\frac{1}{m^7}) \\ &= \;\; -\int_{t_i}^{t_{i+3}} \langle f(t), \ddot{\gamma}(t) \rangle \, dt \; + \; O(\frac{1}{m^7}) \; , \end{aligned} \tag{3.64}$$

after integration by parts. By (3.63) the right-hand side is $O(\frac{1}{m^5})$, namely

$$\boxed{\textstyle\int_{t_i}^{t_{i+3}} \|\dot{\gamma}(t)\| \, dt \; - \; d(Q^i) \; = \; O(\frac{1}{m^5}) \; ,}$$

and so

$$\boxed{d(Q) \; \equiv \; \Sigma_{j=0}^{\frac{m}{3}-1} d(Q^{3j}) \; = \; d(\gamma) \; + \; O(\tfrac{1}{m^4}) \; .} \tag{3.65}$$

As clearly $m\delta \geq T$, we have $\delta^4 \geq (T/m)^4$. This combined with (3.63) and (3.65) yields both claims of Theorem 3.1 expressed in δ-asymptotics. The proof is complete. □

3.5 Experiments

We test now sharpness of the theoretical results claimed by Theorem 3.1 (see also [43]). Again the algorithm was implemented and tested in *Mathematica*.

3.5.1 Pseudocode

We give first the pseudocode (in *Mathematica*) for implementing piecewise-4-point quadratic Q based on reduced data.

Initialize $list$ of quadruples (from reduced data $\mathcal{Q}_m^{i,3}$) to $list := \{q_i, q_{i+1}, q_{i+2}, q_{i+3}\}$ (*i.e.* $list[k] = q_{i+k}$, where $0 \leq k \leq 3$; we assume here $i = 3l$). The pseudocode for the

$\mathbf{4PointQuadratic}[list\_]$

$\{$

$Knots := 4PFormula\_1[list];$ $/*$ see (3.8) $*/$

$4P\_Int[s\_] := 4PFormula\_2[list, Knots, s];$ $/*$ see (3.9) $*/$

$Der\_4P\_Int := D[4P\_Int[s], s];$

$4P\_Der\_List := CoefficientList[Der\_4P\_Int, s];$

$Norm[s\_] := Derivatives\_Norm[4P\_Der\_List, s];$

$Length := NIntegrate[Norm[s], \{s, 0, Knots[3]\}];$

$/*$ see (1.1) over $[0, Knots[3]]$ $*/$

$4P\_Plot\_List := ParametricPlot[4P\_Int[s], \{s, 0, Knots[3]\}];$

$\mathbf{return}\{Length, 4P\_Plot\_List\}$

$\}$

Fig. 3.2. Pseudocode for procedure **4PointQuadratic**, which for one input list $\{q_i\}_i^{i+3}$, returns the list of $\{d(Q^i), plot\}$, where $plot$ represents a discrete set of $Q^i(s)$, for $s \in [0, \beta_{+_i}]$ (according to the ParametricPlot format)

Rys. 3.2. Pseudokod dla procedury **4PointQuadratic**, która dla danych na wejściu (lista $\{q_i\}_i^{i+3}$) zwraca listę $\{d(Q^i), plot\}$, gdzie $plot$ jest listą dyskretnego zbioru wartości $Q^i(s)$ dla $s \in [0, \beta_{+_i}]$ (zgodnie z formatem ParametricPlot)

similar procedure as in Section 1.6.1 (with $r = 3$, and for each quadruple $\mathcal{Q}_m^{3l,3}$ we can set[4] $\hat{t}_i = 0$, $\hat{t}_{i+1} = 1$, $\hat{t}_{i+2} = \alpha_{+_i}$, and $\hat{t}_{i+3} = \beta_{+_i}$ - see (3.8) with suppressed index i in α_{+_i} and β_{+_i}) reads as in Figure 3.2.

The estimation of α (for $d(Q) - d(\gamma) = O(\delta^\alpha)$) by linear regression (see Section 1.6) from the collection of reduced data $\{\mathcal{Q}_{3j}\}_{j=m_1}^{m_2}$, for $m_{min} = 3m_1 \leq m \leq 3m_2 = m_{max}$ (where $m_1 \geq 1$), yields a similar pseudocode (see Section 1.6.1) for the main program loop shown in Figure 3.3.

Recall that we index here any list from the label set to zero. As before, a slight modification of the main program loop yields the pseudocode to plot Q for a given $\mathcal{Q}_{j_{fix}}$. Again, the pseudocode for estimating α in $Q \circ \psi - \gamma = O(\delta^\alpha)$ is similar but as previously involves computationally expensive optimization procedure (see also Subsection 1.3.1) to determine (for each $m_{min} \leq m \leq m_{max}$) the $\sup_{t \in [0,T]} |Q \circ \psi(t) - \gamma(t)|$, where $\psi : [0, T] \to [0, \hat{T}]$ defines a piecewise-C^∞ reparameterization - see Section 3.4). Thus it is omitted.

3.5.2 Testing

We start with the following example:

[4]Recall that $\bar{\gamma}(\hat{t}) = \hat{\gamma}(\hat{t} - \bar{T})$ has the same trajectory and length over $[-\bar{T}, \hat{T} - \bar{T}]$ as $\hat{\gamma}$ over $[0, \hat{T}]$

```
For  [j = m_1; err = { }, j ≤ m_2, j = j + 1,
     Data[j] := {q_0, q_1, q_2, ..., q_{3*j}};
     m_3 = 3 * j;
     For  [i = 0; length = 0, i ≤ m_3 − 3, i = i + 3,
          List := {Data[j][i], Data[j][i + 1], Data[j][i + 2], Data[j][i + 3]};
          length+ = 4PointQuadratic[List][0];
          ];
     err := AppendTo[err, {Log[m_3], −Log[|length − d(γ)|]}];
     ];
α := Slope_Coeff[Regress[err]];                         / * see Section1.6 * /
```

Fig. 3.3. Pseudocode for the main program loop computing an α estimate in $d(Q)-d(\gamma) = O(\delta^\alpha)$ based on collection of reduced data $\{\mathcal{Q}_{3j}\}_{j=m_1}^{m_2}$

Rys. 3.3. Pseudokod głównej pętli programu obliczającej oszacowanie α w $d(Q) - d(\gamma) = O(\delta^\alpha)$ na podstawie rodziny danych zredukowanych $\{\mathcal{Q}_{3j}\}_{j=m_1}^{m_2}$

Example 3.2. Consider *a convex spiral* $\gamma_{spl} : [0, 5\pi] \to \mathbb{R}^2$ defined as

$$\gamma_{spl}(t) \; = \; ((6\pi - t)\cos(t), (6\pi - t)\sin(t)) \; . \tag{3.66}$$

and sampled as Example 3.1 (rescaled by factor 5π). Figure 3.4 shows performance of piecewise-4-point quadratic interpolant Q_{spl}. The linear regression applied to the pair of points $(\log(m), -\log(E_m))$ (see (1.65)) yields the estimate of convergence order for length approximation equal to 4.03. This supports the claim of Theorem 3.1. □

In Example 3.1 uniform piecewise-3-point quadratic interpolation with $\hat{t}_i = i$ (see Chapter 2) gives a poor estimate of the semicircle, in particular of its length (see Figure 3.5(a)). Now we check the performance of the alternative piecewise-4-point quadratic interpolation for both sporadic and dense data $\mathcal{Q}_m$.

Example 3.3. As in Example 3.1, take $m = 6$ and use the same more-or-less uniform sampling of parameters $\{t_i\}_{i=0}^m$ (assumed to be unknown). The piecewise-4-point quadratic interpolant Q_{sc} in Figure 3.5(b) is more semicircular than the uniform piecewise-quadratic Lagrange interpolant $\tilde{\gamma}_{sc_2}$ (see Chapter 2) used with $\hat{t}_i = i$ shown in Figure 3.5(a) and the error in the length estimate is reduced from 0.0601 to -0.0072.

Let E_m be the absolute value of the error in the length estimate (see (1.65)) using piecewise-4-point quadratic interpolation (with sampling (1.24)(i)), where values of m not divisible by 3 are accounted for by a simple modification. The plot in Figure 3.6 of $-\log(E_m)$ against $\log(m)$ for $3 \leq m \leq 100$ appears linear, and the least-squares estimate of slope is approximately 3.83. According to Theorem 3.1 the limiting slope is at least 4 as $m \to \infty$. So the evidence points to exactly quartic convergence in this example. □

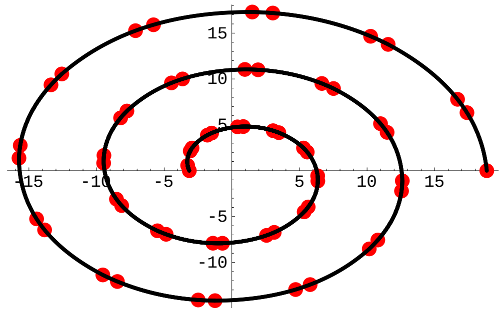


Fig. 3.4. *A piecewise-4-point quadratic* Q_{spl} (solid) for *a spiral* γ_{spl} (3.66), using more-or-less uniform sampling (1.24)(i) ($\varepsilon = 0$; rescaled by factor 5π) with $\mathcal{Q}_{60}$ (dotted). True length: $d(\gamma_{spl}) = 173.608$, $d(Q_{spl}) = 173.539$; *piecewise-3-point quadratic* length estimate: $d(\tilde{\gamma}_{spl_2}) = 181.311$

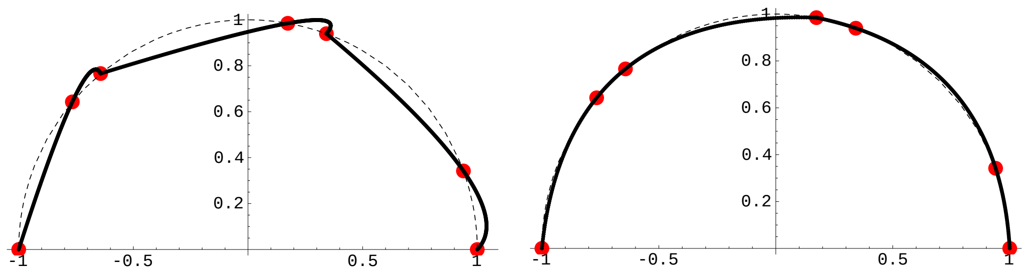
Rys. 3.4. *4-punktowa przedziałowo-kwadratowa interpolacja* Q_{spl} (linia ciągła) *spirali* γ_{spl} (3.66) próbkowanej mniej lub bardziej równomiernie (1.24)(i) ($\varepsilon = 0$; przeskalowane przez czynnik 5π) dla $\mathcal{Q}_{60}$ (wytłuszczone punkty). Rzeczywista długość: $d(\gamma_{spl}) = 173.608$, $d(Q_{spl}) = 173.539$; oszacowanie długości *3-punktową przedziałowo-kwadratową funkcją sklejaną* $\tilde{\gamma}_{spl_2}$: $d(\tilde{\gamma}_{spl_2}) = 181.311$

Notice that a track-sum Q (see Remark 1.1) of the arcs swept out by the $\{Q^{3j}\}_{j=0}^{\frac{m}{3}-1}$ gives a $O(\frac{1}{m^4})$ uniformly accurate approximation of the image of γ. Although $\tilde{\gamma}$ is not C^1 at junction parameters $t_3, t_6, \ldots, t_{m-3}$, the differences in left and right derivatives are $O(\frac{1}{m^3})$, and hardly discernible when m is large (see Figure 3.4). As it turns out, the experiments show that this important property is also preserved for *sporadic data i.e.* when m is small (see Figure 3.5(b)).

Our experiences with other curves and other more-or-less uniform samplings are similar to Example 3.3. We have also mentioned the spiral in connection with the sampling of Example 3.1. We give one further example, of a cubic. Not a lot changes.

Example 3.4. Consider *the cubic curve* given by $\gamma_{c_0}(t) = (t, -t^3)$, for $t \in [0.1, 0.5]$ and sampled in the random fashion (accordingly rescaled) of Example 1.7, for $3 \leq m \leq 100$. The plot of $-\log(E_m)$ against $\log(m)$ is shown in Figure 3.7. The least-squares estimate of slope is 4.00. So the evidence suggests only quartic convergence for Theorem 3.1. $\square$

Remark 3.3. The assumption about *strict convexity* of γ is important for claims of Theorem 3.1 to prevail. Indeed, for the *quartic curve* $\gamma_q(t) = (t, t^4)$, with $t \in [-1, 1]$, the curvature (see (3.1)) $k(t) \geq 0$ and vanishes at $t = 0$ (*i.e.* no inflection points). By geometric arguments used in Section 3.2, a piecewise-4-point-quadratic interpolant Q is still defined. However, for more-or-less uniform sampling (1.24)(i) ($\varepsilon = 0$; rescaled accordingly), computed length estimate is slower and equals $3.09 < 4$. The matter can get worse if the curve γ has *an inflection point*. Indeed, for *the cubic curve* $\gamma_{c_1}(t) = (t, t^3)$ with $t \in [-1, 1]$, we have $k(t) < 0$ for $t \in [-1, 0)$, $k(t) > 0$ for $t \in (0, 1]$ and $k(0) = 0$. The attempt to find the piecewise-4-point quadratic Q for sampling (1.24)(i) and parameters (α, β) fails for algebraic reason (in the neighborhood of inflection point). In general, for



a) b)

Fig. 3.5. Interpolating *a semicircle* γ_{sp} (1.9) (dashed) sampled as (1.24)(i) (with $\varepsilon = 0$) for $\mathcal{Q}_6$ (dotted) by: a) *a uniform piecewise-quadratic* $\tilde{\gamma}_{sc_2}$ (solid) with $\hat{t}_i = i$; length estimate: $\pi + 0.0601$, b) *a piecewise-4-point quadratic* Q_{sc} (solid); length estimate: $\pi - 0.0072$

Rys. 3.5. Interpolacja *półokręgu* γ_{sp} (1.9) (linia przerywana) próbkowanego w takt (1.24)(i) (dla $\varepsilon = 0$) dla $\mathcal{Q}_6$ (wytłuszczone punkty) *funkcją sklejaną*: a) *przedziałowo-kwadratową* $\tilde{\gamma}_{sc_2}$ (linia ciągła) z $\hat{t}_i = i$; oszacowanie długości: $\pi + 0.0601$, b) *4-punktową przedziałowo-kwadratową* Q_{sc} (linia ciągła); oszacowanie długości: $\pi - 0.0072$

curves with inflection points, in the neighborhood of the latter either $1 < \alpha < \beta$ does not hold or expressions (3.7) contain negative arguments passed to $f(x) = \sqrt{x}$, or zero may appear in both denominators of (3.7) and of (3.8). In each of such case the interpolation scheme based on piecewise-4-point quadratic Q collapses.

Remark 3.4. The testing was made also to verify the necessity of the assumption of *more--or-less uniformity* for Theorem 3.1. Taking the sampling (1.17), for *the strictly convex cubic curve* $\gamma_{c_2}(t) = (t, ((t+1)/2)^3)$, with $t \in [0,1]$, yields the length estimate equal to $2.61 < 4$ claimed by Theorem 3.1. Note, that sampling (1.17) does not preserve the right-hand side of the condition for more-or-less uniformity (1.28) (and also $m\delta \neq O(1)$). The experiments were also made to verify whether violation of the left-hand side of (1.28) (see *e.g.* sampling (1.31)) contradicts the claims of Theorem 3.1. For such samplings, all tested strictly convex curves the experiments confirm the validity of Theorem 3.1. Thus necessity of the left-hand side inequality of (1.28) for Theorem 3.1 to hold, remain an open problem. Also the tests indicate that for non-regular convex planar curves sampled more--or-less uniformly the claims of Theorem 3.1 still hold. On the other hand, the regularity of γ is a vital component for the proof of Theorem 3.1 and as such is not dropped.

3.6 Discussion and motivation for Chapter 4 (and 6)

Fast quartic orders of approximation for γ and $d(\gamma)$ estimation are proved for the interpolation constructed from the piecewise-4-point quadratics. The scheme derived herein, applies to C^4 regular planar curves, sampled more-or-less uniformly and forming the set of reduced data $\mathcal{Q}_m$. The presented experiments confirm sharpness of the main result (al least for $d(\gamma)$ estimation). Both Remarks 3.4 and 3.4 illustrate the importance of the assumptions appearing in the formulation of Theorem 3.1 (*i.e. strict convexity, more-or-less*

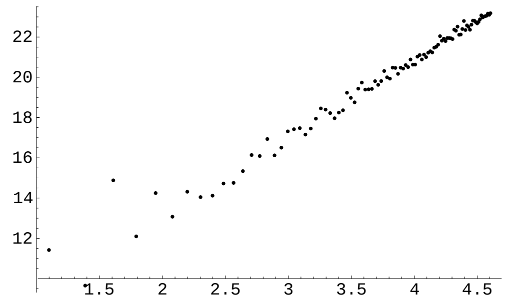


Fig. 3.6. Plot of $\mathcal{D} = (\log(m), -\log(E_m))$ (see (1.65)) for *a semicircle* γ_{sc} (1.9) sampled as in (1.24)(i) ($\varepsilon = 0$, $3 \leq m \leq 100$) and interpolated by *piecewise-4-point quadratics* Q_{sc}. The slope of linear regression line applied to $\mathcal{D}$: $\alpha = 3.83 \approx 4$ (see Theorem 3.1)

Rys. 3.6. Wykres zbioru punktów $\mathcal{D} = (\log(m), -\log(E_m))$ (patrz (1.65)) dla interpolacji *półokręgu* γ_{sc} (1.9) próbkowanego w takt (1.24)(i) ($\varepsilon = 0$, $3 \leq m \leq 100$) *4-punktowymi przedziałowo-kwadratowymi funkcjami sklejanymi* Q_{sc}. Współczynnik nachylenia prostej otrzymanej z liniowej regresji dla $\mathcal{D}$: $\alpha = 3.83 \approx 4$ (patrz Twierdzenie 3.1)

uniformity and $n = 2$). Good performance of the piecewise-4-point quadratic interpolant is also experimentally confirmed for sporadic data, resulting not only in good trajectory and length estimation but also yielding small jumps of derivatives at junction points for two consecutive 4-point quadratic interpolants Q^i and Q^{i+3} *i.e.* the jump between derivatives $Q^{i'}(\beta_i)$ and $Q^{i+3'}(0)$. Evidently, the asymptotic analysis does cover the case when m is small.

The piecewise-4-point quadratic interpolant outperforms the piecewise-quadratic Lagrange interpolant (applied to unparameterized samplings) used with $\hat{t}_i = i$ for both dense (see Theorems 2.1, 2.2 and 3.1) and sporadic (see *e.g.* Example 3.3) data. Clearly, the latter happens as an effort is made here to find Q^i and (α, β) by incorporating the geometry of the available data $\mathcal{Q}_m^{i,3}$. This permits estimating the distribution of the unknown $(t_i, t_{i+1}, t_{i+2}, t_{i+3})$ subject to the *normalization condition* $\hat{t}_i = 0$ and $\hat{t}_{i+1} = 1$, and with $\hat{t}_{i+2} = \alpha_i$ and $\hat{t}_{i+3} = \beta_i$.

Remark 3.5. Note finally that the piecewise-4-point quadratic interpolant (when applicable) outperforms (or match) the piecewise-quadratic and piecewise-cubic Lagrange interpolation based on non-reduced data $(\mathcal{Q}_m, \{t_i\}_{i=0}^m)$ (*i.e.* when t_i is known) for which the corresponding convergence rates are 3 and 4, respectively (see Theorems 1.1 1.2, 1.3 and 3.1). Thus we have a positive answer to Question I from Subsection 1.3.2, at least for $r = 3$, $n = 2$, and γ strictly convex, sampled more-or-less uniformly. In other words in the special case from above, we can *compensate for the stripped information contained in reduced data* $\mathcal{Q}_m$ as compared with the corresponding non-reduced counterpart $(\mathcal{Q}_m, \{t_i\}_{i=0}^m)$.

In a pursuit to extend the above interpolation procedure to $\mathbb{R}^3$ one may try to pass through *five space points* $\mathcal{Q}_m^{i,4} = (q_i, q_{i+1}, q_{i+2}, q_{i+3}, q_{i+4})$ *a cubic curve* $C^i : [0, \gamma_i] \to \mathbb{R}^3$

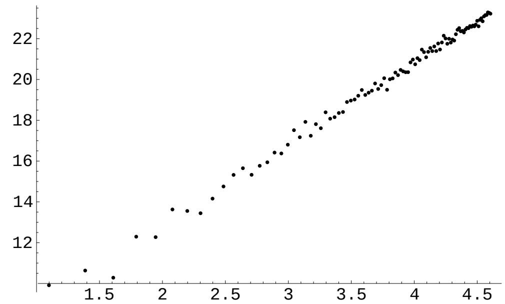


Fig. 3.7. Plot of $\mathcal{D} = (\log(m), -\log(E_m))$ (see (1.65)) for *a cubic* γ_{c_0} defined and sampled as in Example 3.4 ($3 \leq m \leq 100$), and interpolated by *piecewise-4-point quadratics* Q_{c_0} . The slope of linear regression line applied to $\mathcal{D}$: $\alpha = 4.00 \approx 4.00$ (see Theorem 3.1)

Rys. 3.7. Wykres $\mathcal{D} = (\log(m), -\log(E_m))$ (patrz (1.65)) dla interpolacji *krzywej kubicznej* γ_{c_0} zdefiniowanej i próbkowanej jak w Przykładzie 3.4 ($3 \leq m \leq 100$) *4-punktowymi przedziałowo-kwadratowymi funkcjami sklejanymi* Q_{c_0} . Współczynnik nachylenia prostej otrzymanej z liniowej regresji dla $\mathcal{D}$: $\alpha = 4.00 \approx 4.00$ (patrz Twierdzenie 3.1)

satisfying

$$C^i(0) = q_i \ , \ \ C^i(1) = q_{i+1} \ , \ \ C^i(\alpha_i) = q_{i+2} \ , \ \ C^i(\beta_i) = q_{i+3} \ , \ \ C^i(\gamma_i) = q_{i+4} \tag{3.67}$$

with the estimates $\{\hat{t}_{i+k}\}_{k=0}^{k=4}$ of the unknown interpolation parameters $\{t_{i+k}\}_{k=0}^{k=4}$ equal to $(0, 1, \alpha_i, \beta_i, \gamma_i)$ and satisfying $1 < \alpha_i < \beta_i < \gamma_i$. Problem (3.67) reduces itself into solving of 15 *scalar cubic equations* in 15 *unknowns*, where 12 of them represent the unknown coefficients of C^i expressed in linear form. The remaining three unknowns correspond to $(\alpha_i, \beta_i, \gamma_i)$ and appear in linear, quadratic and cubic form. Even if one resolves this much more complicated task (one certainly expects even more complicated asymptotic analysis as compared to that presented in this chapter *i.e.* for $n = 2$), a limited application of this approach to *specific samplings and special curves* in $\mathbb{R}^3$ (or $\mathbb{R}^2$) suggests rather to turn the attention to other possible schemes for *fitting given reduced data*. Chapter 4 and the following after, discuss one such scheme based on *cumulative chord parameterization* working fast for the general class of admissible samplings $\mathcal{V}_G^m$ (see Definition 1.2) and for an arbitrary regular smooth curve γ in $\mathbb{R}^n$.

Chapter 4

Cumulative chord piecewise-quadratics-cubics

Abstract

Cumulative chord piecewise-quadratics and piecewise-cubics are examined in detail and compared with other low degree interpolants for reduced data $\mathcal{Q}_m$ from regular curves in $\mathbb{R}^n$, especially piecewise-4-point quadratics. Orders of approximation (i.e. cubic and quartic, respectively) for arbitrary admissible or special samplings (i.e. $\min\{4, 3+\varepsilon\}$ for ε-uniform and $r = 2$) are calculated and compared with numerical experiments. Their sharpness is verified for $n = 2, 3$ and for length estimation. As shown, cumulative chord piecewise-quadratics and piecewise-cubics approximate to the same order as the piecewise-quadratic and piecewise-cubic Lagrange interpolants used with non-reduced data $(\mathcal{Q}_m, \{t_i\}_{i=0}^m)$ (for $\{t_i\}_{i=0}^m$ known). Cumulative chord piecewise-quadratics and piecewise-cubics are also experimentally confirmed to perform well on sporadic data (at least for $n = 2, 3$) which is not covered by the asymptotic analysis presented herein. Part of this work is published in [42] and [45].

4.1 Preliminaries and main result

In the last chapter a partial solution to solve the problem of *fitting the reduced data* $\mathcal{Q}_m$ is proposed and analyzed (see Theorem 3.1). More specifically, for $n = 2$ and curve $\gamma : [0, T] \to \mathbb{R}^2$, C^4 and strictly convex, sampled more-or-less uniformly, Theorem 3.1 guarantees order four of approximations, for both curve γ and its length $d(\gamma)$ estimation, by the piecewise-quadratics $Q : [0, \hat{T}] \to \mathbb{R}^2$ (see Section 3.2), called *piecewise-4-point quadratics* because the quadratic arcs interpolate *quadruples* of points in $\mathcal{Q}_m$ rather than triples as in Example 4.1. Again we start with the example which not only emphasizes already known phenomenon of bad choice of $\hat{t}_i = i$, but also is later used for comparison purposes.

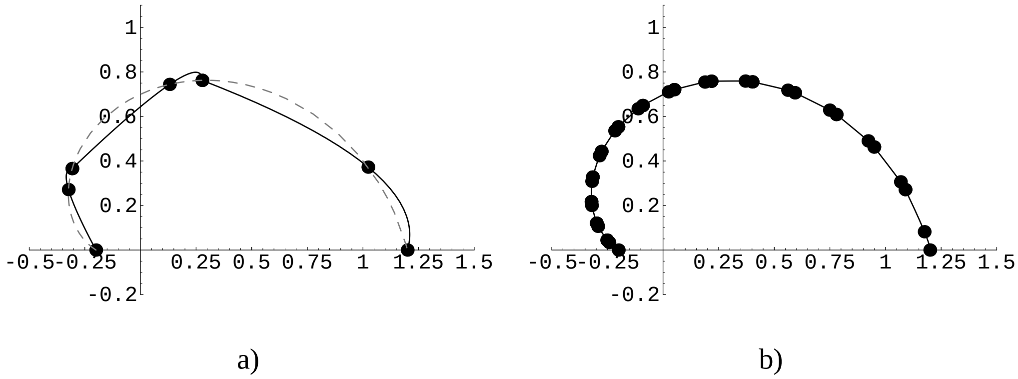


Fig. 4.1. Interpolating *a spiral* γ_{sp} (1.18) (dashed) sampled as in Example 4.1 by *uniform piecewise-quadratics* $\tilde{\gamma}_{sp_2}$ (solid) with knots $\hat{t}_i = i$, for samplings: a) $\mathcal{Q}_6$ (dotted): $d(\tilde{\gamma}_{sp_2}) = d(\gamma_{sp}) + 2.422 \times 10^{-3}$, b) $\mathcal{Q}_{30}$ (dotted): $d(\tilde{\gamma}_{sp_2}) = d(\gamma_{sp}) + 8.163 \times 10^{-2}$

Rys. 4.1. Interpolacja *spirali* γ_{sp} (1.18) (linia przerywana) próbkowanej jak w Przykładzie 4.1 *przedziałowo-kwadratowymi funkcjami sklejanymi* $\tilde{\gamma}_{sp_2}$ (linia ciągła) z węzłami $\hat{t}_i = i$, dla: a) $\mathcal{Q}_6$ (wytłuszczone punkty): $d(\tilde{\gamma}_{sp_2}) = d(\gamma_{sp}) + 2.422 \times 10^{-3}$, b) $\mathcal{Q}_{30}$ (wytłuszczone punkty): $d(\tilde{\gamma}_{sp_2}) = d(\gamma_{sp}) + 8.163 \times 10^{-2}$

Example 4.1. If we guess $t_i \approx \hat{t}_i = i$, then as we know the resulting *uniform piecewise-quadratic* Lagrange interpolant $\tilde{\gamma}_2 : [0, m] \to \mathbb{R}^n$ (see Chapter 2) is sometimes uninformative. For instance take a sampling (1.24)(i) with $\varepsilon = 0$, $n = 2$, and a spiral γ_{sp} defined by (1.18). Then $d(\gamma_{sp}) = 2.452$. When m is small $\tilde{\gamma}_{sp_2}$ does not much resemble γ_{sp}: in Figure 4.1(a), $m = 6$ and $d(\tilde{\gamma}_{sp_2}) = d(\gamma_{sp}) + 2.422 \times 10^{-3}$ Errors in length tend to cancel and the curve is a worse approximation than these numbers suggest. When m is large $\tilde{\gamma}_{sp_2}$ looks more like γ_{sp}: in Figure 4.1(b), $m = 30$. In this case however $d(\tilde{\gamma}_{sp_2}) = d(\gamma_{sp}) + 8.163 \times 10^{-2}$, with nearly 33 times the error for $m = 6$. Even *piecewise-linear* interpolation with 31 points is better, with error -3.622×10^{-3}. □

The requirement that γ be planar and strictly convex seems very restrictive. An alternative, noted in Chapter 11 of [33], is Lagrange interpolation based on *cumulative chord length parameterizations* [16] and [36]. More precisely (see also (1.10)), set for $j = 1, 2, \ldots, m$:

$$\boxed{\hat{t}_0 = 0 \quad \text{and} \quad \hat{t}_j = \hat{t}_{j-1} + \|q_j - q_{j-1}\| ,} \tag{4.1}$$

and $\hat{T} = \sum_{j=1}^{m} \|q_j - q_{j-1}\|$. For k dividing m, $i = 0, k, 2k, \ldots, m-k$, let

$$\hat{\gamma}_k : [0, \hat{T}] \to \mathbb{R}^n \tag{4.2}$$

be the curve (see (1.8)) satisfying

$$\hat{\gamma}_k(\hat{t}_j) = q_j , \tag{4.3}$$

for all $j = 0, 1, 2, \ldots, m$, and whose restriction $\hat{\gamma}_k^i$ to each $[\hat{t}_i, \hat{t}_{i+k}]$ is polynomial of degree at most k (interpolating $\mathcal{Q}_m^{i,k}$). Call $\hat{\gamma}_k$ the *cumulative chord piecewise-degree-* k approximation to γ defined by $\mathcal{Q}_m = (q_0, q_1, \ldots, q_m)$ (*i.e.* a track-sum of $\{\hat{\gamma}_k^{ik}\}_{i=0}^{\frac{m}{k}-1}$ - see also Remark 1.1). Note that as for 4-point quadratic Q^i (see Section 3.2) the $\hat{\gamma}_k^i$ defined by cumulative chord parameterization (1.10) and $\mathcal{Q}_m^{i,k}$, takes into account the *geometry of the sampling points*. Our main result (see [45]) stands as follows (holding for general admissible samplings $\{t_i\}_{i=0}^m \in \mathcal{V}_G^m$):

Theorem 4.1. *Suppose γ is a regular C^r curve in $\mathbb{R}^n$, where $r \geq k+1$ and k is 2 or 3. Let $\hat{\gamma}_k : [0, \hat{T}] \to \mathbb{R}^n$ be the cumulative chord piecewise-degree-k approximation defined by $\mathcal{Q}_m$. Then there is a piecewise-C^r reparameterization*[1] $\psi : [0, T] \to [0, \hat{T}]$, *with*

$$\boxed{\hat{\gamma}_k \circ \psi \;=\; \gamma \;+\; O(\delta^{k+1}) \quad \textit{and} \quad d(\hat{\gamma}_k) \;=\; d(\gamma) \;+\; O(\delta^{k+1})\;.} \tag{4.4}$$

Also, if $r \geq 4$, $k = 2$, and if, for some $\varepsilon \geq 0$, we have the general uniformity condition

$$t_{i+2} - 2t_{i+1} + t_i \;=\; O(\delta^{\min\{2,1+\varepsilon\}}) \quad \textit{for} \quad i = 0, 3, 5, \ldots, m-2\;, \tag{4.5}$$

then

$$\boxed{d(\hat{\gamma}_k) \;=\; d(\gamma) \;+\; O(\delta^{\min\{4,3+\varepsilon\}})\;.} \tag{4.6}$$

Note that for both results to estimate the length of γ the additional assumption is needed $m\delta = O(1)$.

Remark 4.1. Note that if $\{t_i\}_{i=0}^m$ satisfy ε-uniformity (1.15) then by (2.10) and $m\delta \geq T$ we have

$$t_{i+2} - 2t_{i+1} + t_i = O(\frac{1}{m^{\min\{2,1+\varepsilon\}}}) = O(\delta^{\min\{2,1+\varepsilon\}})\;,$$

and thus condition (4.5) is implicitly satisfied.

After some preliminaries in Section 4.2, Theorem 4.1, is proved in Section 4.3. In Section 4.4, Theorem 4.1 is illustrated by examples and compared with other results.

4.2 Divided differences and cumulative chords

First recall some facts about divided differences [52]: setting $\gamma[t_i] = \gamma(t_i)$, the *first divided difference* of γ at t_i is

$$\boxed{\gamma[t_i, t_{i+1}] \;\equiv\; \frac{\gamma(t_{i+1})-\gamma(t_i)}{t_{i+1}-t_i}\;,} \tag{4.7}$$

[1] See Section 4.2 for more details.

and, for $k = 2, 3, \ldots, m-i$, the k th *divided difference* is defined inductively as

$$\boxed{\gamma[t_i, t_{i+1}, \ldots, t_{i+k}] \equiv \frac{\gamma[t_{i+1}, t_{i+2}, \ldots, t_{i+k}] - \gamma[t_i, t_{i+1}, \ldots, t_{i+k-1}]}{t_{i+k} - t_i}.} \tag{4.8}$$

Newton's Interpolation Formula [52] is $\gamma(t) = L + R$, where

$$\begin{aligned} L \equiv\ & \gamma[t_i] + (t-t_i)\gamma[t_i, t_{i+1}] + (t-t_i)(t-t_{i+1})\gamma[t_i, t_{i+1}, t_{i+2}] + \\ & \ldots + (t-t_i)(t-t_{i+1})\ldots(t-t_{i+k-1})\gamma[t_i, t_{i+1}, \ldots, t_{i+k}] \end{aligned} \tag{4.9}$$

is the polynomial of degree at most k interpolating γ at $t_i, t_{i+1}, \ldots, t_{i+k}$, and

$$R \equiv (t-t_i)(t-t_{i+1})\ldots(t-t_{i+k})\gamma[t, t_i, t_{i+1}, \ldots, t_{i+k}].$$

When γ is C^{k+1} and $t_i, t_{i+1}, t_{i+2}, \ldots, t_{i+k+1} \in (t-\tilde{\delta}, t+\tilde{\delta})$ where $\tilde{\delta} > 0$ then, for $j = 1, 2, \ldots, n$, the j th component of the $k+1$ th divided difference is given by

$$\gamma[t, t_i, t_{i+1}, \ldots, t_{i+k}]_j = \frac{\gamma_j^{(k+1)}(\tilde{t}_j)}{(k+1)!}, \tag{4.10}$$

for some $\tilde{t}_j \in (t-\tilde{\delta}, t+\tilde{\delta})$.

We now work with the hypotheses of Theorem 4.1. In particular γ is C^r and regular, where $r \geq k+1$ and k is 2 or 3. After a C^r reparameterization, as in Chapter 1; Proposition 1.1.5 of [26], we can assume γ is parameterized by arc-length, namely $\|\dot{\gamma}\|$ is identically 1. Then

$$\langle\dot{\gamma}, \dot{\gamma}\rangle \equiv 1, \quad \langle\ddot{\gamma}, \dot{\gamma}\rangle \equiv 0, \quad \langle\frac{d^3\gamma}{dt^3}, \dot{\gamma}\rangle \equiv -\langle\ddot{\gamma}(t), \ddot{\gamma}(t)\rangle = -\kappa(t)^2, \tag{4.11}$$

where $\kappa(t)$ is the *curvature* of γ (see Chapter 1; Paragraphs 1-5 of [14]) equal to $\|\ddot{\gamma}(t)\|$ at $t \in [0, T]$. For any $i = 0, k, 2k, \ldots, m-k$, let

$$\boxed{\psi^i : [t_i, t_{i+k}] \to [\hat{t}_i, \hat{t}_{i+k}]} \tag{4.12}$$

be the polynomial function (see (1.8)) of degree at most k satisfying

$$\boxed{\psi^i(t_{i+j}) = \hat{t}_{i+j},}$$

for $0 \leq j \leq k$ (for convenience we suppressed here the subscript k in ψ^i). Substituting for the $\hat{t}_{i+j}$ in the first divided difference, we find

$$\psi^i[t_{i+j}, t_{i+j+1}] = \|\gamma[t_{i+j}, t_{i+j+1}]\|, \tag{4.13}$$

for $j = 0, 1, \ldots, k-1$. Using (4.13) to substitute in the second divided difference,

$$\psi^i[t_i, t_{i+1}, t_{i+2}] \;=\; \frac{\|\gamma[t_{i+1}, t_{i+2}]\| - \|\gamma[t_i, t_{i+1}]\|}{t_{i+2} - t_i} \;,$$

and so

$$|\psi^i[t_i, t_{i+1}, t_{i+2}]| \;\leq\; \frac{\|\gamma[t_{i+1}, t_{i+2}] - \gamma[t_i, t_{i+1}]\|}{t_{i+2} - t_i} \;=\; \|\gamma[t_i, t_{i+1}, t_{i+2}]\| \;.$$

By (4.10) and because γ is C^2, the right hand side is bounded, and therefore

$$\psi^i[t_i, t_{i+1}, t_{i+2}] \;=\; O(1) \;.$$

The same can be said for the right hand side of (4.13). The assumption that γ is C^3 permits us to say more:

Lemma 4.1. $\psi^i[t_{i+j}, t_{i+j+1}] = 1 + O(\delta^2)$, *for* $j = 0, 1$. *Also* $\psi^i[t_i, t_{i+1}, t_{i+2}] = O(\delta)$.

Proof. By (4.13) and Taylor's Theorem,

$$\begin{aligned}\psi^i[t_{i+j}, t_{i+j+1}] &= \|\gamma[t_{i+j}, t_{i+j+1}]\| \\ &= \|\dot\gamma(t_{i+j}) + \frac{1}{2}(t_{i+j+1} - t_{i+j})\ddot\gamma(t_{i+j}) + O((t_{i+j+1} - t_{i+j})^2)\| \;.\end{aligned}$$

Because $\|\dot\gamma\| \equiv 1$ and $\langle\dot\gamma, \ddot\gamma\rangle \equiv 0$, the Taylor's Theorem applied to $f(x) = \sqrt{1+x}$ at $x = 0$ gives

$$\psi^i[t_{i+j}, t_{i+j+1}] \;=\; \sqrt{1 + O((t_{i+j+1} - t_{i+j})^2)} \;=\; 1 + O((t_{i+j+1} - t_{i+j})^2) \;, \quad (4.14)$$

proving the first part. The second assertion follows by comparing (4.14) with the definition of the second divided difference, because t_i increases with i. The proof is complete. □

The next lemma uses the sampling condition (4.5).

Lemma 4.2. *For* $\gamma \in C^4$ $\psi^i[t_{i+j}, t_{i+j+1}, t_{i+j+2}] \;=\; O(\delta^{\min\{2, 1+\varepsilon\}})$,*with* $0 \leq j \leq k-1$. *Also if* $k = 3$, *then* $\psi^i[t_i, t_{i+1}, t_{i+2}, t_{i+3}] = O(1)$.

Proof. By Taylor's Theorem, $\|\gamma[t_{i+j}, t_{i+j+1}]\|$ is

$$\|\dot\gamma \;+\; \frac{\ddot\gamma}{2}(t_{i+j+1} - t_{i+j}) \;+\; \frac{d^3\gamma}{dt^3}\frac{(t_{i+j+1} - t_{i+j})^2}{6} \;+\; O((t_{i+j+1} - t_{i+j})^3)\| \;,$$

where $\dot{\gamma}, \ddot{\gamma}, \frac{d^3\gamma}{dt^3}$ are evaluated at t_{i+j}, and $0 \le j \le k$. Therefore, and by (4.11),

$$\begin{aligned} \|\gamma[t_{i+j}, t_{i+j+1}]\| &= \sqrt{1 - \frac{\kappa^2}{12}(t_{i+j+1} - t_{i+j})^2 \; + \; O((t_{i+j+1} - t_{i+j})^3)} \\ &= 1 \; - \; \frac{\kappa^2}{24}(t_{i+j+1} - t_{i+j})^2 \; + \; O((t_{i+j+1} - t_{i+j})^3) \; , \end{aligned}$$

by the Taylor's Theorem (applied to $f(x) = \sqrt{1+x}$ at $x = 0$), with κ evaluated at t_{i+j}. Therefore, for $0 \le j \le k-1$, by (4.5)

$$\begin{aligned} \psi^i[t_{i+j}, t_{i+j+1}, t_{i+j+2}] &= -\kappa^2 \frac{(t_{i+j+2} - t_{i+j+1})^2 - (t_{i+j+1} - t_{i+j})^2}{24(t_{i+j+2} - t_{i+j})} \\ &\quad + O((t_{i+3} - t_i)^2) \\ &= -\frac{\kappa^2}{24}(t_{i+j+2} - 2t_{i+j+1} + t_{i+j}) \; + \; O((t_{i+3} - t_i)^2) \; , \end{aligned} \tag{4.15}$$

where κ is evaluated at t_{i+j}. Thus combining (4.5) with (4.15) yields

$$\psi^i[t_{i+j}, t_{i+j+1}, t_{i+j+2}] \; = \; O(\delta^{\min\{2, 1+\varepsilon\}}) \; . \tag{4.16}$$

This proves the first part. For $k = 3$, by (4.15) we obtain

$$\psi^i[t_i, t_{i+1}, t_{i+2}, t_{i+3}] \; = \; -\frac{\kappa^2}{24} \frac{(t_{i+3} - 3t_{i+2} + 3t_{i+1} - t_i)}{t_{i+3} - t_i} + O(t_{i+3} - t_i) \; = \; O(1) \; , \tag{4.17}$$

where κ is evaluated at t_i. The proof is complete. □

By Newton's Interpolation Formula [52], and since ψ^i is polynomial of degree at most 3,

$$\begin{aligned} \psi^i(t) &= \psi^i(t_i) \; + \; (t - t_i)\psi^i[t_i, t_{i+1}] \; + \; (t - t_i)(t - t_{i+1})\psi^i[t_i, t_{i+1}, t_{i+2}] \\ &\quad + (k-2)(t - t_i)(t - t_{i+1})(t - t_{i+2})\psi^i[t_i, t_{i+1}, t_{i+2}, t_{i+3}] \; , \end{aligned}$$

for $k = 2, 3$. Differentiating twice, we first obtain

$$\begin{aligned} \dot{\psi}^i(t) &= \psi^i[t_i, t_{i+1}] \; + \; (2t - t_i - t_{i+1})\psi^i[t_i, t_{i+1}, t_{i+2}] \\ &\quad + (k-2)\zeta(t, t_i, t_{i+1}, t_{i+2})\psi^i[t_i, t_{i+1}, t_{i+2}, t_{i+3}] \; , \end{aligned}$$

where $\zeta(t, t_i, t_{i+1}, t_{i+2}) = (t - t_{i+1})(t - t_{i+2}) + (t - t_i)(t - t_{i+2}) + (t - t_i)(t - t_{i+1})$, and

$$\begin{aligned} \ddot{\psi}^i(t) &= 2\psi^i[t_i, t_{i+1}, t_{i+2}] \\ &\quad + 2(k-2)(3t - t_i - t_{i+1} - t_{i+2})\psi^i[t_i, t_{i+1}, t_{i+2}, t_{i+3}] \; . \end{aligned}$$

So, by Lemmas 4.1, 4.2, we arrive at

Lemma 4.3. *$\dot\psi^i = 1 + O(\delta^2)$ and $\ddot\psi^i = O(\delta)$. When γ is C^4 and $k = 3$, $\frac{d^3\psi^i}{dt^3} = O(1)$.*

In particular, ψ^i is a C^∞ diffeomorphism for δ small, which we assume from now on.

Lemma 4.4. *For $k = 2, 3$, and $s \in [\hat t_i, \hat t_{i+k}]$, $\hat\gamma^i_k(s), \frac{d\hat\gamma^i_k}{ds}, \frac{d^2\hat\gamma^i_k}{ds^2}$ are $O(1)$. For $k = 3$ and if γ is C^4, then $\frac{d^3\hat\gamma^i_k}{ds^3}$ is $O(1)$.*

Proof. $\hat\gamma^i_k$ is the polynomial of degree at most k interpolating function $\gamma \circ (\psi^i)^{-1}$ at $t_i, t_{i+1}, \ldots, t_{i+k}$, and the derivatives to order k of $\gamma \circ (\psi^i)^{-1}$ are $O(1)$ by Lemma 4.3. By (4.10) the divided differences to order k of $\gamma \circ (\psi^i)^{-1}$ are also $O(1)$. By Newton's Interpolation Formula [52], these are nonzero constant multiples of the derivatives of $\hat\gamma_k$ to order k. The proof is complete. □

Let $\psi : [0, T] \to [0, \hat T]$ be the track-sum of the $\{\psi^{ik}\}_{i=0}^{\frac{m}{k}-1}$ (see Remark 1.1). This function is used to reparameterize $\hat\gamma_k$. We shall pass now to the proof of Theorem 4.1.

4.3 Proof of the main result

Proof. The proof now follows the previous pattern and is performed for completion. By Lemmas 4.3, 4.4, all coefficients of the polynomials $\hat\gamma^i_k$ and ψ^i are $O(1)$, So the restriction $\hat\gamma^i_k \circ \psi^i$ of $\hat\gamma \circ \psi$ to $[t_i, t_{i+k}]$ is a polynomial of degree at most k^2 and with all coefficients $O(1)$. So all derivatives of the C^{k+1} function

$$\boxed{f^i \ \equiv\ \hat\gamma^i_k \circ \psi^i \ -\ \gamma : [t_i, t_{i+k}] \to \mathbb{R}^n}$$

are $O(1)$. By Remark 1.5, because $f^i(t_{i+j}) = \vec 0$ for $0 \le j \le k$,

$$f^i(t) \ =\ (t - t_i)\ldots(t - t_{i+k-1})g^i(t)\ , \quad \text{where} \quad g^i(t) \ =\ (t - t_{i+k})h^i(t)\ ,$$

with $g^i, h^i : [t_i, t_{i+k}] \to \mathbb{R}^n$ C^1 and C^0 respectively. Also $h^i = O(\frac{d^{k+1}f^i}{dt^{k+1}}) = O(1)$, so that $g^i = O(\delta)$, and $\dot g^i = O(\frac{d^{k+1}f^i}{dt^{k+1}}) = O(1)$. So

$$\boxed{f^i \ =\ O(\delta^{k+1}) \quad \text{and} \quad \dot f^i \ =\ O(\delta^k)\ .} \tag{4.18}$$

In particular $\hat\gamma_k \circ \psi$ approximates γ uniformly with $O(\delta^{k+1})$ errors. Thus the formula in (4.4) is proved.

To compare lengths, write $\hat\gamma^i_k \circ \psi^i \equiv \tilde\gamma^i$. Then

$$\dot{\tilde\gamma}^i(t) \ =\ (1 + \langle \dot f^i(t), \dot\gamma(t)\rangle)\dot\gamma(t) \ +\ v(t)\ ,$$

where $v(t)$ is the projection of $\dot{f}^i(t)$ onto the line orthogonal to $\dot{\gamma}(t)$. By (4.18), function $v = O(\delta^k)$ and, because $\|\dot{\gamma}\| \equiv 1$,

$$\|\dot{\tilde{\gamma}}^i(t)\| \;=\; (1 + \langle \dot{f}^i(t), \dot{\gamma}(t)\rangle)\|\dot{\gamma}(t)\| \;+\; O(\delta^{2k}) \; .$$

Then

$$\int_{t_i}^{t_{i+k}} (\|\dot{\tilde{\gamma}}^i(t)\| - \|\dot{\gamma}(t)\|)\, dt = \int_{t_i}^{t_{i+k}} \langle \dot{f}^i(t), \dot{\gamma}(t)\rangle \, dt \;+\; O(\delta^{2k+1})$$

which, on integration by parts, becomes $-\int_{t_i}^{t_{i+k}} \langle f^i(t), \ddot{\gamma}(t)\rangle \, dt + O(\delta^{2k+1})$. By (4.18) the right hand side is $O(\delta^{k+2})$, namely

$$\int_{t_i}^{t_{i+k}} \|\dot{\gamma}(t)\| \, dt \;-\; d(\tilde{\gamma}^i) \;=\; O(\delta^{k+2}) \; .$$

So as $m\delta = O(1)$

$$\boxed{d(\hat{\gamma}_k) \;=\; \Sigma_{j=0}^{\frac{m}{k}-1} d(\tilde{\gamma}^{jk}) \;=\; d(\gamma) \;+\; O(\delta^{k+1}) \; .}$$

This proves the second formula in (4.4).

When $r \geq 4$, $k = 2$, and (4.5) holds, we can say more. As before,

$$\int_{t_i}^{t_{i+2}} (\|\dot{\tilde{\gamma}}^i(t)\| - \|\dot{\gamma}(t)\|)\, dt \;=\; -\int_{t_i}^{t_{i+2}} \langle f^i(t), \ddot{\gamma}(t)\rangle \, dt \;+\; O(\delta^5) \; . \tag{4.19}$$

Now $h^i = O(1)$ and, since $r \geq 4$, h^i is C^1 with $\dot{h}^i = O(1)$ by Remark 1.5 again. Therefore

$$\langle h^i(t), \ddot{\gamma}(t)\rangle \;=\; a_i \;+\; O(\delta) \; ,$$

where $a_i \equiv \langle h^i(t_i), \ddot{\gamma}(t_i)\rangle = O(1)$, and

$$\langle f^i(t), \ddot{\gamma}(t)\rangle \;=\; a_i(t - t_i)(t - t_{i+1})(t - t_{i+2}) \;+\; O(\delta^4) \; .$$

Now

$$\begin{aligned}\int_{t_i}^{t_{i+2}} (t - t_i)(t - t_{i+1})(t - t_{i+2})\, dt \;&=\; \frac{1}{12}(t_i - t_{i+2})^3(t_{i+2} - 2t_{i+1} + t_i) \\ &=\; O(\delta^{\min\{5, 4+\varepsilon\}}) \; ,\end{aligned}$$

by (4.5). So (4.19) gives

$$\boxed{\int_{t_i}^{t_{i+2}} (\|\dot{\tilde{\gamma}}^i(t)\| \;-\; \|\dot{\gamma}(t)\|)\, dt \;=\; O(\delta^{\min\{5,4+\varepsilon\}})\;.}$$

Thus as $m\delta = O(1)$ we have

$$\boxed{d(\hat{\gamma}_k) \;=\; \Sigma_{j=0}^{\frac{m}{2}-1} d(\tilde{\gamma}^{2j}) \;=\; d(\gamma) \;+\; O(\delta^{\min\{4,3+\varepsilon\}})\;.}$$

This supplementary argument does not apply when $k=3$. Indeed, in the next Section the orders of approximation in Theorem 4.1 for cumulative chord piecewise-cubics are seen to be best-possible, even when sampling is ε-uniform. The proof is complete. □

4.4 Experiments

Here are some experiments, using *Mathematica*, and admissible samplings of smooth regular curves in $\mathbb{R}^2$ and $\mathbb{R}^3$. First we verify sharpness of Theorems 3.1 and 4.1 in the situation of Example 4.1 (see also [42] and [45]).

4.4.1 Pseudocode

We cover only the case of pseudocode (in *Mathematica*) for implementation of cumulative chord piecewise-quadratics $\hat{\gamma}_2$ (for $k=2$) based on reduced data. The case for cumulative chord piecewise-cubics (*i.e.* when $k=2$) is analogous and as such omitted.

Initialize $list$ of triple of points (from reduced data $\mathcal{Q}_m^{i,2}$) to $list := \{q_i, q_{i+1}, q_{i+2}\}$ (*i.e.* $list[s] = q_{i+s}$, for $0 \leq s \leq 2$; we assume here $i = 2l$). The pseudocode for the similar procedure as in Section 1.6.1 (with $r = 2$ and, for each triple $\mathcal{Q}_m^{2l,2}$ we can set[2] $\hat{t}_i = 0$, $\hat{t}_{i+1} = \|q_{i+1} - q_i\|$ and $\hat{t}_{i+2} = \hat{t}_{i+1} + \|q_{i+2} - q_{i+1}\|$ - see (4.1)) reads as in Figure 4.2.

The estimation of α (for $d(\hat{\gamma}_2) - d(\gamma) = O(\delta^{\alpha})$) which is computed by linear regression (see Section 1.6) from the collection of reduced data $\{\mathcal{Q}_{2j}\}_{j=m_1}^{m_2}$, where as previously $m_{min} = 2m_1 \leq m \leq 2m_2 = m_{max}$ (with $m \geq 1$), yields a similar pseudocode (see Section 1.6.1) for the main program loop shown in Figure 4.3.

Recall that we index here any list from the label set to zero. As before, a slight modification of the main program loop yields the pseudocode to plot $\hat{\gamma}_2$ for a given $\mathcal{Q}_{j_{fix}}$. Again, the pseudocode for estimating α in $\hat{\gamma}_2 \circ \psi - \gamma = O(m^{-\alpha})$ is similar but as previously involves computationally expensive optimization procedure (see also Subsection 1.3.1) to determine (for each $m_{min} \leq m \leq m_{max}$) the $\sup_{t\in[0,T]} |\hat{\gamma}_2 \circ \psi(t) - \gamma(t)|$, where $\psi : [0,T] \to [0,\hat{T}]$ is a piecewise-C^{∞} reparameterization - see Section 4.2).

[2] Note that $\bar{\gamma}(\hat{t}) = \hat{\gamma}(\hat{t} - \bar{T})$ has the same trajectory and length over $[-\bar{T}, \hat{T} - \bar{T}]$ as $\hat{\gamma}$ over $[0, \hat{T}]$

$\mathbf{QuadraticChord}[list\_]$

$\{$

$Knots := CC\_Q\_Formula[list];$

$/*$ see (4.1) with $k=2$ and $\{\hat{t}_{i+k} := \hat{t}_{i+k} - \hat{t}_i\}_{k=0}^{2} */$

$CC\_Q[s\_] := Lag\_Formula[list, Knots, 2, s];$ $/*$ see (1.8) with $r=2*/$

$Der\_CC\_Q\_Int := D[Expand[CC\_Q[s]], s];$

$CC\_Q\_Der\_List := CoefficientList[Der\_CC\_Q\_Int, s];$

$Norm[s\_] := Derivatives\_Norm[CC\_Q\_Der\_List, s];$

$Length := Integral[Norm[s], \{s, 0, Knots[2]\}];$

$/*$ see (1.1) over $[0, Knots[2]] */$

$CC\_Q\_Plot\_List := ParametricPlot[CC\_Q[s], \{s, 0, Knots[2]\}];$

$\mathbf{return}\{Length, CC\_Q\_Plot\_List\}$

$\}$

Fig. 4.2. Pseudocode for procedure **QuadraticChord**, which for one input list $\{q_i\}_i^{i+2}$, returns the list of $\{d(\hat{\gamma}_2^i), plot\}$, where $plot$ represents a discrete set of $\hat{\gamma}_2^i(s)$, for $s \in [0, \hat{t}_{i+2} - \hat{t}_i]$ (according to the ParametricPlot format)

Rys. 4.2. Pseudokod dla procedury **QuadraticChord**, która dla danych na wejściu (lista $\{q_i\}_i^{i+2}$) zwraca listę $\{d(\hat{\gamma}_2^i), plot\}$, gdzie $plot$ jest listą dyskretnego zbioru wartości $\hat{\gamma}_2^i(s)$ dla $s \in [0, \hat{t}_{i+2} - \hat{t}_i]$ (zgodnie z formatem ParametricPlot)

4.4.2 Testing

Example 4.2. Uniform piecewise-quadratics, piecewise-4-point quadratics together with cumulative chord piecewise-quadratics and cumulative chord piecewise-cubics based on the $\mathcal{Q}_6$ of Example 4.1 are shown as solid curves in Figures 4.4(a), 4.4(b), 4.4(c), and 4.4(d), respectively.

Figures 4.4(b) and 4.4(c) show markedly better approximations to γ_{sp} than Figure 4.4(a), and in Figure 4.4(d) the cumulative chord piecewise-cubic is nearly indistinguishable from the dashed spiral. The respective errors in length estimates are -1.155×10^{-2}, -2.459×10^{-2}, -1.048×10^{-3} and 2.422×10^{-3}. For larger values of m differences in performance are more marked:

- Uniform piecewise-quadratics $\tilde{\gamma}_{sp_2}$ (see Chapter 2) behave badly with respect to length estimates, as noted in Example 4.1.

- Piecewise-4-point quadratics Q_{sp} with $m = 30, 99, 198$ yield the following errors 5.499×10^{-5}, 5.127×10^{-7}, 3.200×10^{-8} respectively in length estimates. The numerical estimate of order of convergence for length estimates, based on samples of up to 199 points is 3.93.

For $[j = m_1;\ err = \{\ \},\ j \leq m_2,\ j = j + 1,$
$Data[j] := \{q_0, q_1, q_2, \ldots, q_{2*j}\};$
$m_3 = 2 * j;$
For $[i = 0;\ length = 0,\ i \leq m_3 - 2,\ i = i + 2,$
$List := \{Data[j][i], Data[j][i+1], Data[j][i+2]\};$
$length+ = \textbf{QuadraticChord}[List][0];$
$];$
$err := AppendTo[err, \{Log[m_3], -Log[|length - d(\gamma)|]\}];$
$];$
$\alpha := Slope\_Coeff[Regress[err]];$ $/ *$ see Section1.6 $* /$

Fig. 4.3. Pseudocode for the main program loop computing an α estimate in $d(\hat{\gamma}_2) - d(\gamma) = O(\delta^\alpha)$ based on collection of reduced data $\{\mathcal{Q}_{2j}\}_{j=m_1}^{m_2}$

Rys. 4.3. Pseudokod głównej pętli programu obliczającej oszacowanie α w $d(\hat{\gamma}_2) - d(\gamma) = O(\delta^\alpha)$ na podstawie rodziny danych zredukowanych $\{\mathcal{Q}_{2j}\}_{j=m_1}^{m_2}$

- Cumulative chord piecewise-quadratics $\hat{\gamma}_2$ (see Section 4.1; $k = 2$) with $m = 30$, 100, 200 yield errors 1.738×10^{-4}, 4.332×10^{-6}, 5.302×10^{-7} respectively in length estimates. The numerical estimate of order of convergence for length estimates, based on samples of up to 201 points is 3.03.
- Cumulative chord piecewise-cubics $\hat{\gamma}_3$ (see Section 4.1; $k = 3$) with $m = 30$, 99, 198 yield errors 9.514×10^{-6}, 8.741×10^{-8}, 5.670×10^{-9} respectively. The numerical estimate of order of convergence for length estimates, based on samples of up to 199 points is 3.96.

So the orders of approximation for length estimates given in Theorem 4.1 for cumulative chord piecewise-quadratics and piecewise-cubics appear to be sharp (at least for $n = 2$). Note that condition $m\delta = O(1)$ holds for sampling from Example 4.1 used also here. Although $\hat{\gamma}_{sp_k}$ (for $k = 2, 3$) are not C^1, differences in left and right derivatives (at t_3 in Figures 4.4(b), 4.4(d) and at t_2, t_4 in Figure 4.4(c)) are hardly discernible (even for sporadic data) for piecewise-4-point quadratics and cumulative chord piecewise-quadratics and piecewise-cubics. Such features are practically invisible when m is large. More examples can be found in [42]. □

Piecewise-4-point quadratics and cumulative chord piecewise-cubics have the same orders of convergence, but cumulative chord piecewise-cubics and piecewise-quadratics are more *generally applicable* (see Example 4.3): *curves need not be planar or convex and sampling need not be more-or-less uniform* for estimating orders of approximation.

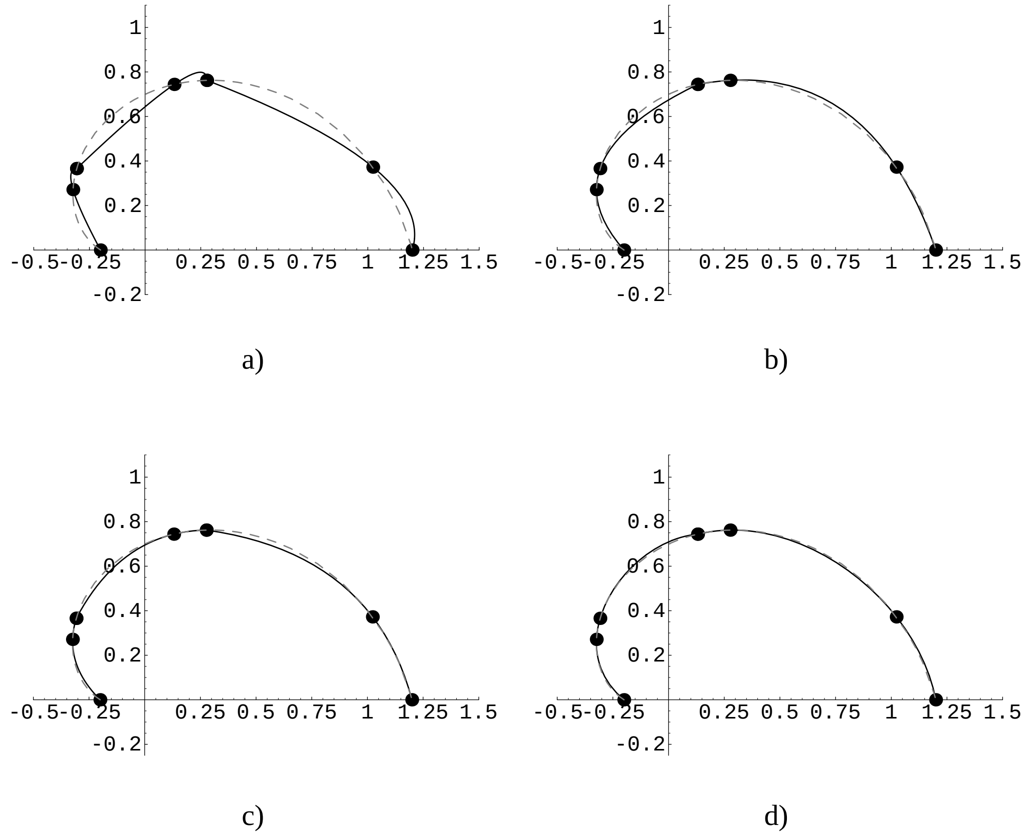


Fig. 4.4. *A spiral* γ_{sp} (dashed) for $\mathcal{Q}_6$ (Example 4.1) interpolated by *piecewise-* (solid): a) *uniform quadratic* $\tilde{\gamma}_{sp_2}$ $(\hat{t}_i = i)$: $d(\tilde{\gamma}_{sp_2}) = d(\gamma_{sp}) + 2.422 \times 10^{-3}$, b) *4-point quadratic* Q_{sp}: $d(Q_{sp}) = d(\gamma_{sp}) - 1.155 \times 10^{-2}$, c) *cumulative chord quadratic* $\hat{\gamma}_{sp_2}$: $d(\hat{\gamma}_{sp_2}) = d(\gamma_{sp}) - 2.459 \times 10^{-2}$, d) *cumulative chord cubic* $\hat{\gamma}_{sp_3}$: $d(\hat{\gamma}_{sp_3}) = d(\gamma_{sp}) - 1.048 \times 10^{-3}$

Rys. 4.4. *Spirala* γ_{sp} (linia przerywana) dla $\mathcal{Q}_6$ (Przykład 4.1) interpolowana *funkcją sklejaną przedziałowo-* (linia ciągła): a) *kwadratową* $\tilde{\gamma}_{sp_2}$: $d(\tilde{\gamma}_{sp_2}) = d(\gamma_{sp}) + 2.422 \times 10^{-3}$ $(\hat{t}_i = i)$, b) *4-punktową kwadratową* Q_{sp}: $d(Q_{sp}) = d(\gamma_{sp}) - 1.155 \times 10^{-2}$, c) *kwadratową na skumulowanej długości cięciwy* $\hat{\gamma}_{sp_2}$: $d(\hat{\gamma}_{sp_2}) = d(\gamma_{sp}) - 2.459 \times 10^{-2}$, d) *kubiczną na skumulowanej długości cięciwy* $\hat{\gamma}_{sp_3}$: $d(\hat{\gamma}_{sp_3}) = d(\gamma_{sp}) - 1.048 \times 10^{-3}$

Example 4.3. Let $\gamma_{cf} : [0,1] \to \mathbb{R}^2$ be *the cubic*, with inflection point $(0,0)$, given by

$$\gamma_{cf}(t) = (t - 0.5, 4(t - 0.5)^3) . \tag{4.20}$$

Given m, take $\{t_i\}_{i=0}^{m}$ to be

$$\frac{i}{m} \quad \text{or} \quad \frac{(i-1)}{m} + \frac{1}{m^2} \tag{4.21}$$

according as i is even or odd. Then sampling (4.21) is not more-or-less uniform. The plot for cumulative chord piecewise-quadratic interpolation of $-\log|d(\hat{\gamma}_{cf_2}) - d(\gamma_{cf})|$ against

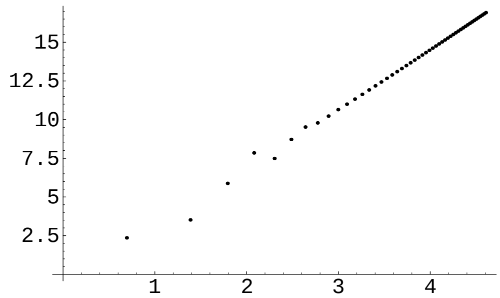


Fig. 4.5. Plot of $\mathcal{D} = (\log(m), -\log|d(\hat{\gamma}_{cf_2}) - d(\gamma_{cf})|)$ for a *cubic* γ_{cf} (4.20) with *inflection point*, sampled as in Example 4.3 ($4 \le m \le 100$) and interpolated by *cumulative chord piecewise-quadratics* $\hat{\gamma}_{cf_2}$. The slope of linear regression line applied to the set of $\mathcal{D}$: $\alpha = 3.86 > 3$ (see Theorem 4.1 for $k = 2$)

Rys. 4.5. Wykres zbioru punktów $\mathcal{D} = (\log(m), -\log|d(\hat{\gamma}_{cf_2}) - d(\gamma_{cf})|)$ dla interpolacji *krzywej kubicznej* γ_{cf} (4.20) z *punktem przegięcia* próbkowanej jak w Przykładzie 4.3 ($4 \le m \le 100$) i interpolowanej *przedziałowo-kwadratowymi funkcjami sklejanymi na bazie skumulowanej parametryzacji długością cięciwy* $\hat{\gamma}_{cf_2}$. Współczynnik nachylenia prostej otrzymanej z liniowej regresji dla $\mathcal{D}$: $\alpha = 3.86 > 3$ (patrz Twierdzenie 3.1 dla $k = 2$)

$\log(m)$ shown in Figure 4.5, for $m = 4, 6, \ldots, 100$, appears almost linear, with least squares estimate of slope 3.86. Theorem 4.1 says the slope should be at least 3. $\square$

Example 4.4. Figure 4.6(a) shows a cumulative chord piecewise-quadratic interpolant $\hat{\gamma}_{h_2}$ of 9 points on *the elliptical helix* $\gamma_h : [0, 2\pi] \to \mathbb{R}^3$, given by

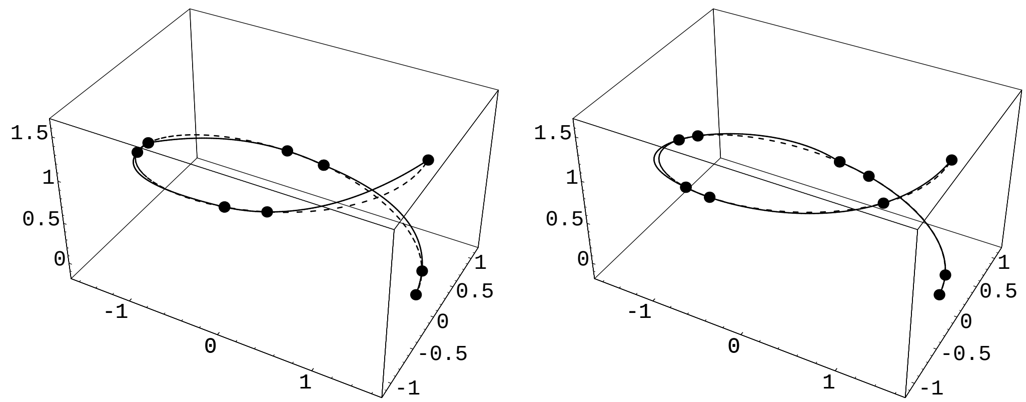
$$\gamma_h(t) = (1.5\cos t, \sin t, t/4) . \tag{4.22}$$

Although sampling is uneven, sparse, and not available for interpolation, $\hat{\gamma}_{h_2}$ seems very close to γ_h: $d(\gamma_h) = 8.090$ and $d(\hat{\gamma}_{h_2}) = 8.019$. For (not more-or-less uniform) samplings where $\{t_i\}_{i=0}^m$ is

$$\frac{2\pi i}{m} \quad \text{or} \quad \frac{2\pi(i-1)}{m} + \frac{2\pi}{m^{3/2}} \tag{4.23}$$

according as i is even or odd, and $m = 50, 52, \ldots, 200$, the order of convergence for length with cumulative piecewise-quadratics is estimated as 3.91. Theorem 4.1 asserts at least 3. Figure 4.6(b) shows a cumulative chord piecewise-cubic interpolant $\hat{\gamma}_{h_3}$ of 10 points on the helix γ_h. Approximation of γ_h by $\hat{\gamma}_{h_3}$ is visually very good, and $d(\hat{\gamma}_{h_3}) = 8.179$. Using $m = 75, 78, \ldots, 300$ the numerical estimate of order of convergence for length is 4.013 confirming sharpness of Theorem 4.1 (at least for $n = 3$) in respect of cumulative chord piecewise-cubics without conditions on sampling. $\square$

Next we verify sharpness of Theorem 4.1 for cumulative chord piecewise-quadratics $\hat{\gamma}_2$ with sampling conditions of the form (4.5).



a) b)

Fig. 4.6. *An elliptical helix* γ_h (4.22) (dashed) sampled as in (4.23) and interpolated by *cumulative chord* (solid): a) *piecewise-quadratic* $\hat{\gamma}_{h_2}$ for data $\mathcal{Q}_8$: $d(\hat{\gamma}_{h_2}) = d(\gamma_h) - 7.029 \times 10^{-2}$, b) *piecewise-cubic* $\hat{\gamma}_{h_3}$ for data $\mathcal{Q}_9$: $d(\hat{\gamma}_{h_3}) = d(\gamma_h) + 8.952 \times 10^{-2}$

Rys. 4.6. *Helikoida eliptyczna* γ_h (4.22) (linia przerywana) próbkowana w zgodzie z formułą (4.23) i interpolowana *funkcją sklejaną na bazie skumulowanej długości cięciwy* (linia ciągła): a) *przedziałowo-kwadratową* $\hat{\gamma}_{h_2}$ dla punktów $\mathcal{Q}_8$: $d(\hat{\gamma}_{h_2}) = d(\gamma_h) - 7.029 \times 10^{-2}$, b) *przedziałowo-kubiczną* $\hat{\gamma}_{h_3}$ dla punktów $\mathcal{Q}_9$: $d(\hat{\gamma}_{h_3}) = d(\gamma_h) + 8.952 \times 10^{-2}$

Example 4.5. Let $\gamma_c : [0,1] \to \mathbb{R}^2$ be the cubic given by (1.25). For m, we consider now ε-uniform sampling $(1.24)(i)$. Then cumulative chord piecewise-quadratic interpolation for $m = 40, 42, \ldots, 200$, with $\varepsilon = 0.0, 0.1, 0.25, 0.5, 0.75, 1, 3$, yields convergence orders rates for length estimations as $\alpha_0 = 2.978$, $\alpha_{0.1} = 3.087$, $\alpha_{0.25} = 3.246$, $\alpha_{0.5} = 3.504$, $\alpha_{0.75} = 3.756$, $\alpha_1 = 4.008$, and $\alpha_3 = 3.970$, respectively. We found no additional increase in convergence order for $\varepsilon > 1$. This confirms sharpness of (4.6). $\square$

Remark 4.2. Notice that constraint (1.51) is *a necessary condition* for Theorem 4.1 to hold. Indeed, for γ_c defined by (1.25) and sampled according to (1.17) (for which $m\delta \neq O(1)$) cumulative chord piecewise-quadratics γ_{c_2} give and estimate for $d(\gamma)$ approximation (with $2 \leq m \leq 100$) equal to $2.09 < 3$, claimed by Theorem 4.1 and $k = 2$. A similar effect appears for $k = 3$. Experiments show that regularity of γ may not be a necessary condition for Theorem 4.1 to hold. This remains an open problem. However, as previously, the proof of Theorem 4.1 relies on assumption that γ is regular and thus the latter is not abandoned.

4.5 Discussion and motivation for Chapter 5

Cubic and quartic orders of approximation for trajectory and length estimation are proved and confirmed to be sharp experimentally (at least for $d(\gamma)$ estimation of some planar and space curves) for cumulative chord piecewise-quadratic and cubics sampled in $\mathbb{R}^n$ and according to general admissible sampling condition $\mathcal{V}_G^m$ (see Definition 1.2). For cumulative chord piecewise-quadratic and $\mathcal{V}_\varepsilon^m$ (see Definition 1.4) an extra sharp acceleration eventuates for $d(\gamma)$ estimation with convergence order equal to $\min\{4, 3+\varepsilon\}$ (for $\varepsilon \geq 0$). Experiments also show *a good performance of cumulative chords on sporadic data*, *i.e.* excellent γ and $d(\gamma)$ approximation and *small jumps in derivatives* at the junction knots, where two consecutive chords are "glued" together. Of course, the asymptotic analysis does not cover the case when m is small.

Unlike Theorem 3.1, Theorem 4.1 holds for any sufficiently smooth regular curve γ (not necessarily convex) in arbitrary Euclidean space $\mathbb{R}^n$, and is applicable even without tight conditions on sampling (only $m\delta = O(1)$ being required, which is also implicitly fulfilled by $\mathcal{V}_\varepsilon^m$ or $\mathcal{V}_{mol}^m$ - see Definition 1.5). Cumulative chord piecewise-cubics approximate at least to order 4, as do the piecewise-4-point quadratics of Theorem 3.1. Of course, the piecewise-quadratic Lagrange interpolation used with $\hat{t}_i = i$ is also outperformed by cumulative chord piecewise-quadratics and cubics (see Theorems 2.1 and 4.1). This also eventuates on sporadic data (see Figure 4.4). Condition (1.51) is shown to be a necessary one for Theorem 4.1 to hold.

Remark 4.3. Notice also that cumulative chord piecewise-quadratics and piecewise-cubics approximate to the same order as the piecewise-quadratic and the piecewise-cubic Lagrange interpolants used with $\{t_i\}_{i=0}^m$ given (see Theorems 1.2 and 4.1). Cumulative chord piecewise-quadratics also match length estimates for ε-uniform sampling, where the $\{t_i\}_{i=0}^m$ are given and $r = 2$ (see Theorems 1.3 and 4.1). The last two properties yield *a positive answer to Question I* from Subsection 1.3.2, at least for $r = 2, 3$. In other words, by applying cumulative chords, we can now in a general case *compensate for the loss of information contained in reduced data* while passing from its non-reduced data $(\mathcal{Q}_m, \{t_i\}_{i=0}^m)$ counterpart to $\mathcal{Q}_m$.

Recall that for the non-reduced admissible samplings $\mathcal{V}_G^m$ (where $\{t_i\}_{i=0}^m$ are known) there is an increase by factor one in convergence rates for Lagrange interpolation with order r incremented (see Theorem 1.2). A similar effect is proved in this chapter for cumulative chords $\hat{\gamma}_k$ with $k = 2, 3$ (for reduced samplings $\mathcal{V}_G^m$ - see Theorem 4.1). Thus the natural question arises:

Does further acceleration in convergence orders (to quintic ones) for γ and $d(\gamma)$ estimation (for fitting reduced data $\mathcal{Q}_m$) occur for higher order cumulative chords and the general class of admissible samplings $\mathcal{V}_G^m$?

A key ingredient in the proof of Theorem 4.1 is to show that all coefficients of quadratic and cubic ψ^i are $O(1)$. This is not necessarily the case for the higher degree ψ^i needed to extend the proof for higher order approximations *e.g.* by *cumulative-chord piecewise--quartics* $\hat{\gamma}_4$. Indee, the next example shows that, for general admissible samplings $\mathcal{V}_G^m$ the polynomial $\psi^i : [t_i, t_{i+4}] \to [\hat{t}_i, \hat{t}_{i+4}]$ of degree at most 4 (see (1.8)), satisfying

$$\psi^i(t_{i+j}) = \hat{t}_{i+j} \tag{4.24}$$

for $j = 0, 1, 2, 3, 4$ can have the divided difference $\psi^i[t_i, t_{i+1}, t_{i+2}, t_{i+3}, t_{i+4}]$ *unbounded*. Again, as for $k = 2, 3$ (see (4.12)), we suppressed here subscript 4 for ψ^i introduced by (4.24).

Example 4.6. Given C^4 $\gamma : [0, T] \to \mathbb{R}^n$, for m divisible by 4 and $i = 0, 4, 8, \ldots, m-4$, consider (very nearly uniform) samplings of the form

$$t_i = \frac{i}{m}, \ t_{i+1} = \frac{i+1}{m}, \ t_{i+2} = \frac{i+2}{m} + \frac{1}{2m}, \ t_i = \frac{i+3}{m}, \ t_{i+4} = \frac{i+4}{m} .$$

The proof of Lemma 4.2 shows that the

$$\psi^i[t_{i+j}, t_{i+j+1}, t_{i+j+2}, t_{i+j+3}] = O(1) ,$$

for $j = 0, 1$, and consequently

$$\alpha \equiv \psi^i[t_i, t_{i+1}, t_{i+2}, t_{i+3}, t_{i+4}] = O(m) .$$

We show now that the latter cannot have slower asymptotic. Writing $d_j = \|q_{i+j} - q_i\|$ for $j = 1, 2, 3, 4$, a calculation gives

$$\alpha = \frac{m^4}{90}(-3d_1 + 7d_2 - 25d_3 + 5d_4) . \tag{4.25}$$

As in the proof of Lemma 4.2,

$$d_1 = \frac{1}{m}(1 - \frac{\kappa}{24m^2} + O(\frac{1}{m^3})) , \qquad d_2 = \frac{3}{2m}(1 - \frac{3\kappa}{32m^2} + O(\frac{1}{m^3})) ,$$
$$d_3 = \frac{1}{2m}(1 - \frac{\kappa}{96m^2} + O(\frac{1}{m^3})) , \qquad d_4 = \frac{1}{m}(1 - \frac{\kappa}{24m^2} + O(\frac{1}{m^3})) ,$$

where κ is evaluated at t_i. Substituting into (4.25),

$$\alpha = -\frac{\kappa}{96}m + O(1) .$$

So, except when γ is affine (when $\kappa \equiv 0$), α is unbounded. □

Despite negative result from Example 4.6, the next chapter shows that further acceleration in convergence orders for γ and $d(\gamma)$ estimation is still possible for *cumulative chord piecewise-quartics* $\hat{\gamma}_4$ if special subsamplings of $\mathcal{V}_G^m$ (see Definition 1.2) are considered.

Chapter 5

Cumulative chord piecewise-quartics

Abstract

We discuss the problem of estimating the trajectory of a smooth regular curve γ in $\mathbb{R}^n$ and its length $d(\gamma)$ from ordered samples of reduced data $\mathcal{Q}_m$ by using cumulative chord piecewise-quartics. The corresponding convergence orders are established for different types of reduced data including ε-uniform and more-or-less uniform samplings. (e.g. ranging from quartic to quintic (or from quartic to the 6th order) or quartic orders, respectively). The latter extends previous results on cumulative chord piecewise-quadratics and piecewise-cubics. As shown herein further acceleration on convergence rates with cumulative chord piecewise-quartics is obtainable only for special samplings (e.g. for ε-uniform samplings). On the other hand, convergence rates for more-or-less uniform samplings coincide with those already established for cumulative chord piecewise-cubics. The results for length estimation are experimentally confirmed to be sharp for some planar and space curves. A good performance of cumulative chord piecewise-quartics extends also on sporadic data not covered by the asymptotic analysis from this monograph. This work is published in [27], [28] and [29].

5.1 Main results

In the last chapter we established Theorem 4.1 which gives a positive feedback to Question I (see Subsection 1.3.2) at least for $r = 2, 3$. In an attempt to obtain faster convergence rates than cubic/quartic (rendered by cumulative chord piecewise-quadratics or piecewise-cubics) the natural question arises:

Can the claim of Theorem 4.1 be extended to piecewise-quartic cumulative chords, and if yes, does further acceleration in convergence (e.g. to quintic orders) occur?

In general, as indicated in Remark 4.6, the proof of Theorem 4.1 cannot be directly extended to the case $r = 4$, since the fourth divided difference $\psi^i[t_i, t_{i+1}, t_{i+2}, t_{i+3}, t_{i+4}]$

(see (4.24)) can be *unbounded*.

The answer to the above question, is given here for two subfamilies of $\mathcal{V}_G^m$ (see Definition 1.2), *i.e.* for ε*-uniform samplings* $\mathcal{V}_\varepsilon^m$ (see Definition 1.4) and *more-or-less uniform samplings* $\mathcal{V}_{mol}^m$ (see Definition 1.5) - see Theorems 5.1 and 5.2. The following two main results are here established for regular curves in $\mathbb{R}^n$, the first for $\{t_i\}_{i=0}^m \in \mathcal{V}_\varepsilon^m$ (with $\varepsilon > 0$) (see [28]):

Theorem 5.1. *Suppose γ is a regular C^{4+l} curve in $\mathbb{R}^n$, where $l = 1,2$ and sampled ε-uniformly with $\varepsilon > 0$. Let $\hat{\gamma}_4 : [0,\hat{T}] \to \mathbb{R}^n$ be the cumulative chord piecewise-quartic approximation*[1] *defined by reduced data $\mathcal{Q}_m$. Then there is a piecewise-C^∞ reparameterization*[2] *$\psi : [0,T] \to [0,\hat{T}]$, with*

$$\boxed{\boxed{\hat{\gamma}_4 \circ \psi \;=\; \gamma \;+\; O(\delta^{\min\{5,4+\varepsilon\}}) \quad and \quad d(\hat{\gamma}_4) \;=\; d(\gamma) \;+\; O(\delta^{\min\{4+l,4+l\varepsilon\}})\;.}} \tag{5.1}$$

and the second for $\{t_i\}_{i=0}^m \in \mathcal{V}_{mol}^m$ (see [28]):

Theorem 5.2. *Let $\gamma \in C^5$ be a regular curve in $\mathbb{R}^n$ sampled more-or-less uniformly as in* (1.28) *(or equivalently as in* (1.29)*) and let $\hat{\gamma}_4 : [0,\hat{T}] \to \mathbb{R}^n$ be the cumulative chord piecewise-quartic defined by reduced data $\mathcal{Q}_m$. Then there is a piecewise-C^∞ reparameterization*[3] *$\psi : [0,T] \to [0,\hat{T}]$, with*

$$\boxed{\boxed{\hat{\gamma}_4 \circ \psi \;=\; \gamma \;+\; O(\delta^4) \qquad and \qquad d(\hat{\gamma}_4) \;=\; d(\gamma) \;+\; O(\delta^4)\;.}} \tag{5.2}$$

First, in Section 5.2 we prove some auxiliary results (see Lemmas 5.1, 5.2 and 5.3) holding for cumulative chord piecewise-quartics. Subsequently, both Theorems 5.1 and 5.2 are proved in Section 5.3. Finally, in Section 5.4 we experiment with different samplings and curves to verify both claims of Theorems 5.1 and 5.2.

5.2 Cumulative chord piecewise-quartics

For any $i = 0,4,8,\ldots,m-4$, let $\psi^i : [t_i, t_{i+4}] \to [\hat{t}_i, \hat{t}_{i+4}]$ be the quartic polynomial, where $\hat{t}_{i+j}$ $(j = 0,1,2,3,4)$ are defined as in (4.1), and $\gamma_4^i : [\hat{t}_i, \hat{t}_{i+4}] \to \mathbb{R}^n$ be the cumulative chord quartic both satisfying (see also (1.8))

$$\boxed{\psi^i(t_{i+j}) \;=\; \hat{t}_{i+j} \quad \text{and} \quad \gamma_4^i(\hat{t}_{i+j}) \;=\; q_{i+j}, \quad \text{for} \quad 0 \le j \le 4\,,}$$

respectively.

[1]See (4.1), (4.2) and (4.3) with $k = 4$ for more details.
[2]See Section 5.2 for more details.
[3]See Section 5.2 for more details.

As already mentioned, the proof of Theorem 4.1 exploits boundedness of the 1st, 2nd and 3rd divided differences of ψ_k^i (with $k = 2,3$), where $\psi_k^i : [t_i, t_{i+k}] \to [\hat{t}_i, \hat{t}_{i+k}]$ is a polynomial of order k interpolating at $\psi_k^i(t_{i+j}) = \hat{t}_{i+j}$, for $0 \leq j \leq k$. In an attempt to extend Theorem 4.1 to cumulative chord piecewise-quartics one would hope that boundedness of all divided differences of ψ^i is preserved. On the contrary, as shown in the Example 4.6

$$\boxed{\psi^i[t_i, t_{i+1}, t_{i+2}, t_{i+3}, t_{i+4}] \;=\; O(\delta^{-1})}$$

is sharp and thus the 4th divided difference of ψ^i can be unbounded. In order to extend Theorem 4.1 to $k = 4$ we prove now three auxiliary lemmas holding for arbitrary regular $\gamma \in C^4$ in $\mathbb{R}^n$ and interpolated with ε-uniform sampling subsumed in $\mathcal{V}_G^m$.

Lemma 5.1. *Let $\{t_i\}_{i=0}^m$ satisfy* (1.15) *with $\varepsilon > 0$ and γ be regular (and thus $\|\dot{\gamma}\| = 1$). Then*

$$\begin{aligned}
\psi^i[t_{i+j}, t_{i+j+1}] &= 1 + O(\delta^2)\,, & j = 0,1,2,3\\
\psi^i[t_{i+j}, t_{i+j+1}, t_{i+j+2}] &= O(\delta^{\min\{2,1+\varepsilon\}})\,, & j = 0,1,2\\
\psi^i[t_{i+j}, t_{i+j+1}, t_{i+j+2}, t_{i+j+3}] &= -\frac{\kappa^2}{24}\frac{(t_{i+j+3} - 3t_{i+j+2} + 3t_{i+j+1} - t_{i+j})}{t_{i+j+3} - t_{i+j}}\\
&\quad + O(t_{i+j+3} - t_{i+j}) = O(1)\,, & j = 0,1\\
\psi^i[t_i, t_{i+1}, t_{i+2}, t_{i+3}, t_{i+4}] &= O(\delta^{\min\{0,\varepsilon-1\}})\,, & (5.3)
\end{aligned}$$

where κ is the curvature of γ evaluated at t_i (i.e. $\|\ddot{\gamma}(t_i)\|$).

Proof. The first formula in (5.3) was proved in Theorem 4.1. For the second one, and for $0 \leq j \leq 2$ we have (see (5.3)):

$$\psi^i[t_{i+j}, t_{i+j+1}, t_{i+j+2}] \;=\; -\frac{\kappa^2}{24}(t_{i+j+2} - 2t_{i+j+1} + t_{i+j}) \;+\; O((t_{i+4} - t_i)^3)\,.$$

Hence Taylor's Theorem (applied at t_{i+j}) combined with (1.3) and (1.15) yields:

$$\psi^i[t_{i+j}, t_{i+j+1}, t_{i+j+2}] \;=\; O(\frac{1}{m^2}) \;+\; O(\frac{1}{m^{1+\varepsilon}}) \;+\; O(\delta^3)\,.$$

The latter combined with $T = \sum_{i=1}^m t_i - t_{i-1} \leq m\delta$ renders that any term $\rho = O(1/m^\alpha)$ satisfies $\rho = O(\delta^\alpha)$ (where $\alpha > 0$), and thus

$$\psi^i[t_{i+j}, t_{i+j+1}, t_{i+j+2}] \;=\; O(\delta^2) \;+\; O(\delta^{1+\varepsilon}) \;+\; O(\delta^3) \;=\; O(\delta^{\min\{2,1+\varepsilon\}})\,.$$

The third formula in (5.3) is proved in Theorem 4.1. For the last one, coupling (4.8) (for $k=4$) with the third formula in (5.3) leads to for $h_{i,4}=\psi^i[t_i,t_{i+1},t_{i+2},t_{i+3},t_{i+4}]$

$$h_{i,4} = \frac{-\kappa^2}{24}\left(\frac{(t_{i+4}-3(t_{i+3}-t_{i+2})-t_{i+1})(t_{i+3}-t_i)}{(t_{i+4}-t_i)(t_{i+3}-t_i)(t_{i+4}-t_{i+1})} - \frac{(t_{i+3}-3(t_{i+2}-t_{i+1})-t_i)(t_{i+4}-t_{i+1})}{(t_{i+4}-t_i)(t_{i+3}-t_i)(t_{i+4}-t_{i+1})}\right), \tag{5.4}$$

up to a $O(1)$ term. Notice, that to derive (5.4) we also invoke Taylor's Theorem yielding $\kappa^2(t_{i+1})=\kappa^2(t_i)+O(1)(t_{i+1}-t_i)$. As sampling is ε-uniform, Taylor's Theorem applied to ϕ from (1.15) at t_{i+j} (for $j=0,1,2,3,4$) in the neighborhood of t_i yields

$$t_{i+j} = \phi(\frac{iT}{m}) + \frac{jd}{m} + O(\frac{1}{m^2}) + O(\frac{1}{m^{1+\varepsilon}}) = \phi(\frac{iT}{m}) + \frac{jd}{m} + \bar{b}_j, \tag{5.5}$$

where $\bar{b}_j=O(\frac{1}{m^{\min\{2,1+\varepsilon\}}})$ and $d=T\dot{\phi}(\frac{i}{m})$. Substituting the latter into (5.4) gives

$$h_{i,4} = -\frac{\kappa^2}{24}\left(\frac{m^2(3d+(\bar{b}_3-\bar{b}_0)m)(\bar{b}_4-3\bar{b}_3+3\bar{b}_2-\bar{b}_1)}{(3d+(\bar{b}_4-\bar{b}_1)m)(4d+(\bar{b}_4-\bar{b}_0)m)(3d+(\bar{b}_3-\bar{b}_0)m)} - \frac{m^2(3d+(\bar{b}_4-\bar{b}_1)m)(\bar{b}_3-3\bar{b}_2+3\bar{b}_1-\bar{b}_0)}{(3d+(\bar{b}_4-\bar{b}_1)m)(4d+(\bar{b}_4-\bar{b}_0)m)(3d+(\bar{b}_3-\bar{b}_0)m)}\right) \tag{5.6}$$

up to $O(1)$ term. Note now that $d>0$ is separated from zero as ϕ is a diffeomorphism over the compact interval $[0,T]$ and thus $d^{-1}=O(1)$. The latter combined with $\varepsilon>0$ yields $\theta=O(\frac{1}{m^{\min\{1,\varepsilon\}}})$, where $\theta=\frac{(\bar{b}_4-\bar{b}_1)m}{3d}$. Hence $|\theta|<1$, asymptotically. Thus by geometric expansion

$$\frac{1}{(3d+(\bar{b}_4-\bar{b}_1)m)} = \frac{1}{3d}\left(1+\sum_{k=1}^{\infty}O(\frac{1}{m^{\min\{k,k\varepsilon\}}})\right) = \frac{1}{3d}+O(\frac{1}{m^{\min\{1,\varepsilon\}}}) = O(1).$$

Similarly, $(3d+(\bar{b}_3-\bar{b}_0)m)^{-1}=O(1)$ and $(4d+(\bar{b}_4-\bar{b}_0)m)^{-1}=O(1)$. The latter coupled with $\kappa=O(1)$ and (5.6) render

$$h_{i,4}=O(m^2((3d+(\bar{b}_3-\bar{b}_0)m)(\bar{b}_4-3\bar{b}_3+3\bar{b}_2-\bar{b}_1)-(3d+(\bar{b}_4-\bar{b}_1)m)(\bar{b}_3-3\bar{b}_2+3\bar{b}_1-\bar{b}_0)),$$

up to a $O(1)$ term. A simple inspection reveals

$$\begin{aligned} h_{i,4} &= O(3dm^2(\bar{b}_4-4\bar{b}_3+6\bar{b}_2-4\bar{b}_1+\bar{b}_0)) \\ &\quad + O(3m^3(\bar{b}_1^2+\bar{b}_2\bar{b}_3-\bar{b}_3^2+\bar{b}_2\bar{b}_4-\bar{b}_1\bar{b}_4-\bar{b}_1\bar{b}_2+\bar{b}_0\bar{b}_3-\bar{b}_0\bar{b}_2)) + O(1) \\ &= O(m^{\max\{0,1-\varepsilon\}}) + O(m^{\max\{-1,1-2\varepsilon\}}) \\ &= O(m^{\max\{0,1-\varepsilon\}}). \end{aligned} \tag{5.7}$$

By (5.5) we have for $m \geq m_0$ and some positive constants k_1 and k_2 (independent from m) the following holds

$$t_{i+1} - t_i \leq \frac{d}{m} + \frac{k_1}{m^2} + \frac{k_2}{m^{1+\varepsilon}} ,$$

and therefore by (1.3) we have that

$$\delta = O(\frac{1}{m}) . \tag{5.8}$$

Hence if $\rho = O(m^\alpha)$ then $\rho = O(\frac{1}{\delta^\alpha})$, for $\alpha > 0$. Combining the latter with (5.7) yields the last equation in (5.3). The proof is complete. □

Note that the above proof fails for $\varepsilon = 0$ as then $\theta = O(1)$, and therefore $|\theta| < 1$ may not hold.

Differentiating ψ^i expressed as in (4.9) for $k = 4$ and using Lemma 5.1 leads to

Lemma 5.2. $\dot{\psi}^i = 1 + O(\delta^2)$, $\ddot{\psi}^i = O(\delta)$, $\frac{d^3\psi^i}{dt^3} = O(1)$, *and* $\frac{d^4\psi^i}{dt^4} = O(\delta^{\min\{0,\varepsilon-1\}})$.

In particular, asymptotically, ψ^i is a C^∞ diffeomorphism. Now we can justify the following:

Lemma 5.3. $\frac{d^j\hat{\gamma}_4^i}{ds^j} = O(1)$, *for* $j = 1, 2, 3$ *and* $\frac{d^4\hat{\gamma}_4^i}{ds^4} = O(\delta^{\min\{0,\varepsilon-1\}})$.

Proof. As $\hat{\gamma}_4^i(\hat{t}_{i+j}) = \gamma \circ (\psi^i)^{-1}(\hat{t}_{i+j})$ (where $j = 0, 1, 2, 3, 4$) the corresponding divided differences of $\hat{\gamma}_4^i$ and $\chi_4^i = \gamma \circ (\psi^i)^{-1}$ at $\hat{t}_{i+j}$ are equal and thus by (4.9), for each $s \in [\hat{t}_i, \hat{t}_{i+4}]$, we have (by Newton Interpolation Formula)

$$\begin{aligned} \hat{\gamma}_4^i(s) &= \chi_4^i[\hat{t}_i] + (s - \hat{t}_i)\chi_4^i[\hat{t}_i, \hat{t}_{i+1}] + (s - \hat{t}_i)(s - \hat{t}_{i+1})\chi_4^i[\hat{t}_i, \hat{t}_{i+1}, \hat{t}_{i+2}] \\ &+ (s - \hat{t}_i)(s - \hat{t}_{i+1})(s - \hat{t}_{i+2})\chi_4^i[\hat{t}_i, \hat{t}_{i+1}, \hat{t}_{i+2}, \hat{t}_{i+3}] \\ &+ (s - \hat{t}_i)(s - \hat{t}_{i+1})(s - \hat{t}_{i+2})(s - \hat{t}_{i+3})\chi_4^i[\hat{t}_i, \hat{t}_{i+1}, \hat{t}_{i+2}, \hat{t}_{i+3}, \hat{t}_{i+4}] . \end{aligned} \tag{5.9}$$

The Chain Rule combined with (4.10) and Lemma 5.2 yield

$$\begin{aligned}
\chi_4^i[\hat{t}_i, \hat{t}_{i+1}] &= O(\frac{\dot{\gamma}}{\dot{\psi}^i}) = O(1) , \\
\chi_4^i[\hat{t}_i, \hat{t}_{i+1}, \hat{t}_{i+2}] &= O(\frac{\ddot{\gamma}}{(\dot{\psi}^i)^2}) + O(\frac{\dot{\gamma}\ddot{\psi}^i}{(\dot{\psi}^i)^3}) = O(1) , \\
\chi_4^i[\hat{t}_i, \hat{t}_{i+1}, \hat{t}_{i+2}, \hat{t}_{i+3}] &= O(\frac{\frac{d^3\gamma}{dt^3}}{(\dot{\psi}^i)^3}) + O(\frac{\ddot{\gamma}\ddot{\psi}^i + \dot{\gamma}\frac{d^3\psi^i}{dt^3}}{(\dot{\psi}^i)^4}) + O(\frac{\dot{\gamma}(\ddot{\psi}^i)^2}{(\dot{\psi}^i)^5}) \\
&= O(1) , \\
\chi_4^i[\hat{t}_i, \hat{t}_{i+1}, \hat{t}_{i+2}, \hat{t}_{i+3}, \hat{t}_{i+4}] &= O(\frac{\frac{d^4\gamma}{dt^4}}{(\dot{\psi}^i)^4}) + O(\frac{\frac{d^3\gamma}{dt^3}\ddot{\psi}^i + \ddot{\gamma}\frac{d^3\psi^i}{dt^3} + \dot{\gamma}\frac{d^4\psi^i}{dt^4}}{(\dot{\psi}^i)^5}) \\
&\quad + O(\frac{\dot{\gamma}(\ddot{\psi}^i)^3}{(\dot{\psi}^i)^7}) + O(\frac{\ddot{\gamma}(\ddot{\psi}^i)^2 + \dot{\gamma}\ddot{\psi}^i\frac{d^3\psi^i}{dt^3}}{(\dot{\psi}^i)^6}) \\
&= O(\delta^{\min\{0,\varepsilon-1\}}) . \qquad (5.10)
\end{aligned}$$

Cumulative chord parameterization (4.1) combined with the Mean Value Theorem, for $s \in [\hat{t}_i, \hat{t}_{i+4}]$ and $0 \le k \le 3$, yield

$$|s - \hat{t}_{i+k}| \le \hat{t}_{i+4} - \hat{t}_i = \sum_{j=0}^{3} \|\gamma(t_{i+j+1}) - \gamma(t_{i+j})\| \le 4 \sup_{0 \le t \le T} \|\dot{\gamma}(t)\|\delta = O(\delta) . \qquad (5.11)$$

Combining respective derivatives in (5.9) with (5.10) and (5.11) proves Lemma 5.3. ☐

Let $\psi : [0, T] \to [0, \hat{T}]$ be the track-sum of $\{\psi^{4i}\}_{i=0}^{\frac{m}{4}-1}$ (see Remark 1.1). This function is used to reparameterize $\hat{\gamma}_4$ (a track-sum of $\{\hat{\gamma}_4^{4i}\}_{i=0}^{\frac{m}{4}-1}$). We pass now to proving the main results *i.e.* Theorems 5.1 and 5.2.

5.3 Proof of main results

Proof. We adapt the previous proof (see Theorem 4.1). Assume first that $l = 1$. Define now a C^5 function (a reparameterized $\hat{\gamma}_4^i$)

$$\boxed{f^i \equiv \hat{\gamma}_4^i \circ \psi^i - \gamma : [t_i, t_{i+4}] \to \mathbb{R}^n .}$$

As $f^i(t_{i+j}) = \vec{0}$ for $0 \le j \le 4$, by Remark 1.5 we obtain

$$f^i(t) = (t - t_i)(t - t_{i+1})(t - t_{i+2})(t - t_{i+3})g^i(t) \quad \text{and} \quad g^i(t) = (t - t_{i+4})h^i(t) , \qquad (5.12)$$

where $g^i, h^i : [t_i, t_{i+k}] \to \mathbb{R}^n$ are C^1 and C^0, respectively. Moreover, Remark 1.5 shows that $h^i = O(\frac{d^5 f^i}{dt^5})$ and thus as $\deg(\hat{\gamma}_4^i) = \deg(\psi^i) = 4$, the Chain Rule combined with Lemmas 5.2 and 5.3 yield

$$h^i = O(\delta^{\min\{0,\varepsilon-1\}}) \ . \tag{5.13}$$

Hence $g^i = O(\delta^{\min\{1,\varepsilon\}})$. Again the proof of Lemma 1.1 (applied to multiple zeros) yields $\dot{g}^i = O(\frac{d^5 f^i}{dt^5})$ and thus $\dot{g}^i = O(\delta^{\min\{0,\varepsilon-1\}})$. The latter coupled with (5.12) render

$$\boxed{f^i \ = \ O(\delta^{\min\{5,4+\varepsilon\}}) \qquad \text{and} \qquad \dot{f}^i \ = \ O(\delta^{\min\{4,3+\varepsilon\}}) \ .} \tag{5.14}$$

In particular $\hat{\gamma}_4 \circ \psi$ approximates γ uniformly with $O(\delta^{\min\{5,4+\varepsilon\}})$ errors, which completes the proof of first formula in (5.1).

To compare lengths, write $\hat{\gamma}_4^i \circ \psi^i \equiv \tilde{\gamma}^i$. Then as $\|\dot{\gamma}\| = 1$ we have

$$\dot{f}^i(t) \ = \ \langle \dot{f}^i(t), \dot{\gamma}(t) \rangle \dot{\gamma}(t) \ + \ v(t) \ ,$$

where $v(t)$ is the projection of $\dot{f}^i(t)$ onto the line orthogonal to $\dot{\gamma}(t)$. Thus as the following holds $\dot{f}^i(t) = \dot{\tilde{\gamma}}^i(t) - \dot{\gamma}(t)$ we have

$$\dot{\tilde{\gamma}}^i(t) \ = \ (1 + \langle \dot{f}^i(t), \dot{\gamma}(t) \rangle) \dot{\gamma}(t) \ + \ v(t) \ . \tag{5.15}$$

By (5.14) and (5.15) we have $v = O(\delta^{\min\{4,3+\varepsilon\}})$. Thus as $\|\dot{\gamma}\| \equiv 1$, $\langle \dot{\gamma}(t), v(t) \rangle = 0$ and $\langle v(t), v(t) \rangle (1 + \langle \dot{f}^i(t), \dot{\gamma}(t) \rangle)^{-2} < 1$ asymptotically, by Taylor's Theorem we obtain

$$\|\dot{\tilde{\gamma}}^i(t)\| = (1 + \langle \dot{f}^i(t), \dot{\gamma}(t) \rangle) \|\dot{\gamma}(t)\| \ + \ O(\delta^{\min\{8,6+2\varepsilon\}}) \ .$$

Hence

$$I_i = \int_{t_i}^{t_{i+4}} (\|\dot{\tilde{\gamma}}^i(t)\| - \|\dot{\gamma}(t)\|) \, dt = \int_{t_i}^{t_{i+4}} \langle \dot{f}^i(t), \dot{\gamma}(t) \rangle \, dt \ + \ O(\delta^{\min\{9,7+2\varepsilon\}})$$

which, on integration by parts as $f_i(t_i) = f_i(t_{i+4}) = \vec{0}$, becomes

$$I_i = -\int_{t_i}^{t_{i+4}} \langle f^i(t), \ddot{\gamma}(t) \rangle \, dt \ + \ O(\delta^{\min\{9,7+2\varepsilon\}}) \ . \tag{5.16}$$

Thus by (5.14) and (5.16) $I_i = O(\delta^{\min\{6,5+\varepsilon\}})$, and therefore

$$\int_{t_i}^{t_{i+4}} \|\dot{\gamma}(t)\| \, dt \ - \ d(\tilde{\gamma}^i) = O(\delta^{\min\{6,5+\varepsilon\}}) \ .$$

The latter combined with (5.8) yield

$$\boxed{d(\hat{\gamma}_4) - d(\gamma) \ = \ \Sigma_{j=0}^{\frac{m}{4}-1} d(\tilde{\gamma}^{4j}) \ - \ d(\gamma) \ = \ \frac{m}{4} O(\delta^{\min\{6,5+\varepsilon\}}) \ = \ O(\delta^{\min\{5,4+\varepsilon\}}) \ .}$$

This completes the proof of (5.1) for $l = 1$.

When $l = 2$ we can increase the order of $d(\gamma)$ estimation. Indeed, by Remark 1.5, function h^i is C^1 with $\dot{h}^i = O(\frac{d^6 f^i}{dt^6})$. Thus as $\deg(\hat{\gamma}_4^i) = deg(\psi^i) = 4$ the Chain Rule combined with Lemmas 5.2 and 5.3 render

$$\dot{h}^i = O(\delta^{\min\{0,\varepsilon-1\}}) \ . \tag{5.17}$$

Therefore by (5.13), (5.17) and Taylor's Theorem applied at t_i to

$$r(t) \ = \ \langle h^i(t), \ddot{\gamma}(t)\rangle$$

we have

$$\begin{aligned} r(t) \ &= \ r(t_i) \ + \ O(\delta)O(\langle \dot{h}^i, \ddot{\gamma}\rangle) \ + \ O(\delta)O(\langle h^i, \frac{d^3\gamma}{dt^3}\rangle) \\ &= \ a_i \ + \ O(\delta^{\min\{1,\varepsilon\}}) \ , \end{aligned} \tag{5.18}$$

where by (5.13) the constant

$$a_i \ \equiv \ \langle h^i(t_i), \ddot{\gamma}(t_i)\rangle \ = \ O(\delta^{\min\{0,\varepsilon-1\}}) \ . \tag{5.19}$$

Hence by (5.12) and (5.18)

$$\langle f^i(t), \ddot{\gamma}(t)\rangle \ = \ a_i(t-t_i)(t-t_{i+1})(t-t_{i+2})(t-t_{i+3})(t-t_{i+4}) \ + \ O(\delta^{\min\{6,5+\varepsilon\}}) \ . \tag{5.20}$$

As shown in [32] for sampling (1.15) the following holds

$$\begin{aligned} \int_{t_i}^{t_{i+4}} (t-t_i)(t-t_{i+1})(t-t_{i+2})(t-t_{i+3})(t-t_{i+4})\, dt \ &= \ O(\frac{1}{m^{6+\min\{1,\varepsilon\}}}) \\ &= \ O(\delta^{6+\min\{1,\varepsilon\}}) \ , \end{aligned}$$

upon resorting to (5.8). The latter together with (5.16) (5.19), and (5.20) yield

$$I_i \ = \ O(\delta^{\min\{7,5+2\varepsilon\}}) \ + \ O(\delta^{\min\{7,6+\varepsilon\}}) \ + \ O(\delta^{\min\{9,7+2\varepsilon\}}) \ = \ O(\delta^{\min\{7,5+2\varepsilon\}}) \ .$$

Thus again by (5.8) we have

$$\boxed{d(\hat{\gamma}_4) - d(\gamma) \ = \ \Sigma_{j=0}^{\frac{m}{4}-1} d(\tilde{\gamma}^{4j}) - d(\gamma) \ = \ \tfrac{m}{4} O(\delta^{\min\{7,5+2\varepsilon\}}) \ = \ O(\delta^{\min\{6,4+2\varepsilon\}}) \ .}$$

The proof of (5.1) for $l = 2$ is complete. □

Remark 5.1. Though the proof of Lemma 5.1 fails for $\varepsilon = 0$ still Theorem 5.1 can still be extended to 0-uniform samplings which are also more-or-less uniform. Indeed (1.28) (or equivalently (1.29)) combined with the definition of fourth divided differences and

$$\psi^i[t_{i+k}, t_{i+k+1}, t_{i+k+2}, t_{i+k+3}] \;=\; O(1)\;, \quad \text{for} \quad k = 0, 1$$

yield

$$\psi^i[t_i, t_{i+1}, t_{i+2}, t_{i+3}, t_{i+4}] \;=\; O(\delta^{-1})\;.$$

Repeating the proof of Lemmas 5.2, 5.3 and Theorem 5.1 proves Theorem 5.2.

Note that for *uniform sampling* $t_i = \frac{iT}{m}$ (here $\varepsilon = \infty$ and $\phi \equiv id$) formula (5.3) yields the fourth divided differences of $\psi^i = O(1)$. This combined with the proof of Theorem 5.1 yields

$$\boxed{\hat\gamma_4 \circ \psi \;=\; \gamma \;+\; O(\delta^5) \quad \text{and} \quad d(\hat\gamma_4) \;=\; d(\gamma) \;+\; O(\delta^6)\;,}$$

which coincides with the uniform piecewise-quartic interpolation with guessed $\hat t_i = i/m$.

Of course, the condition $m\delta = O(1)$ is imposed implicitly on $\{t_i\}_{i=0}^m$ in Theorems 5.1 and 5.2 as both samplings (1.15) and (1.28) (or (1.29)) satisfy it.

We comment now on *superiority* of using *cumulative chords* over *piecewise-linear interpolation* ($r = 1$), for which contrary to $r > 1$ the choice of $\{\hat t_i\}_{i=0}^m$ is not important.

Remark 5.2. Recall that difference in convergence rates between Theorem 4.1 and those from Theorem 2.1 has its geometrical reason. For a given triple of points (q_i, q_{i+1}, q_{i+2}) various choices of tabular points $(\hat t_i, \hat t_{i+1}, \hat t_{i+2})$ yield geometrically different Lagrange quadratic curves $\tilde\gamma_{i,2} : [\hat t_i, \hat t_{i+2}] \to \mathbb{R}^n$ satisfying $\tilde\gamma_{i,2}(\hat t_{i+j}) = q_{i+j}$, where $j = 0, 1, 2$ (see (1.8) and Example 1.1). In particular, this holds for t_i guessed according to either (4.1) or uniform distribution. Such geometrical difference occurs also for a higher-order interpolation and justifies the need for a care in choosing suitable tabular points. Exceptionally, a piecewise-linear interpolation does not depend on the choice of tabular points $(\hat t_i, \hat t_{i+1})$. Indeed, for a given pair of points (q_i, q_{i+1}) there exists exactly one geometrical straight line passing through those points. Thus, if *e.g. piecewise-linear cumulative chord* is used $\hat\gamma_1^i : [0, \hat t_{i+1}] \to \mathbb{R}^n$ (see (4.1), (4.2) and (4.3) with $k = 1$: $\hat t_{i+1} = \|q_{i+1} - q_i\|$) defined as

$$\hat\gamma_1^i(s) \;=\; \frac{q_{i+1} - q_i}{\|q_{i+1} - q_i\|} s \;+\; q_i$$

we obtain $\|\hat\gamma_1^{i'}\| = 1 = O(1)$. Furthermore, for $\psi^i : [t_i, t_{i+1}] \to [0, \hat t_{i+1}]$ set to

$$\psi^i(t) \;=\; \frac{\|q_{i+1} - q_i\|}{t_{i+1} - t_i} t \;-\; \frac{t_i \|q_{i+1} - q_i\|}{t_{i+1} - t_i}$$

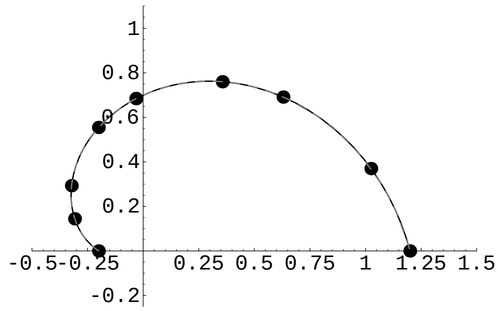


Fig. 5.1. Interpolating *a spiral* γ_{sp} (1.18) (dashed) sampled as in (1.24)(i) ($\varepsilon = 0.5$) by *a cumulative chord piecewise-quartic* $\hat{\gamma}_{sp_4}$ (solid), for 2 successive quintuples of $\mathcal{Q}_8$ (dotted) with length estimate: $d(\hat{\gamma}_{sp_4}) = d(\gamma_{sp}) + 2.370 \times 10^{-4}$

Rys. 5.1. Interpolacja *spirali* γ_{sp} (1.18) (linia przerywana) próbkowanej w takt (1.24)(i) ($\varepsilon = 0.5$) *przedziałowo-wielomianową funkcją sklejaną rzędu czwartego, na bazie skumulowanej parametryzacji długością cięciwy* $\hat{\gamma}_{sp_4}$ (linia ciągła), dla 2 kolejnych piątek punktów ze zbioru $\mathcal{Q}_8$ (wytłuszczone) z oszacowaniem długości: $d(\hat{\gamma}_{sp_4}) = d(\gamma_{sp}) + 2.370 \times 10^{-4}$

the Mean Value Theorem yields $\dot{\psi}^i = O(1)$. A similar argument as in Theorem 5.1 renders

$$\boxed{\hat{\gamma}_1 = \gamma + O(\delta^2) \quad \text{and} \quad d(\hat{\gamma}_1) = d(\gamma) + O(\delta^2)\ .}$$

Despite quadratic approximation orders for piecewise-linear cumulative chord $\hat{\gamma}_1$ (where $k = 1$), there are some advantages in using piecewise-k-cumulative chord interpolations ($k = 2, 3, 4$). Firstly, the corresponding convergence rates are faster. Secondly, the performance of $\hat{\gamma}_1$ is very poor on sporadic data with bad approximation of γ and $d(\gamma)$ and with big and visible discontinuities in smoothness at tabular points. Lastly, for $\hat{\gamma}_1$ there is no convergence in curvature estimation (the curvature of $\hat{\gamma}_1$ vanishes).

5.4 Experiments

We verify now experimentally the sharpness of Theorem 5.1 and 5.2 (see also [27] and [29]) tested only for the length estimation, $n = 2, 3$ and for the generic case of regularity of curve γ *i.e.* $l = 2$ (in fact here $l = \infty$). We omit the derivation of the *pseudocode* for $\hat{\gamma}_4$ as being analogous to the previous chapter.

First, *two planar curves* $\gamma_{sc}, \gamma_{sp} : [0, 1] \to \mathbb{R}^2$ representing, respectively *a semicircle* (see (1.9)) and *a spiral curve* (see (1.18)) (see Figure 5.1; dashed) are tested, with true lengths amounting to $d(\gamma_{sc}) = \pi$ and $d(\gamma_{sp}) = 2.452$, respectively. The unknown ε-uniform knot parameters $\{t_i\}_{i=0}^m$ satisfying (1.24)(i) are chosen merely to synthetically generate ordered sequences of interpolation points $\mathcal{Q}_m$. Finally, one *space curve* $\gamma_h : [0, 2\pi] \to \mathbb{R}^3$ representing *an elliptical helix* (see (4.22)) (see Figure 5.2; dashed) is tested with true length

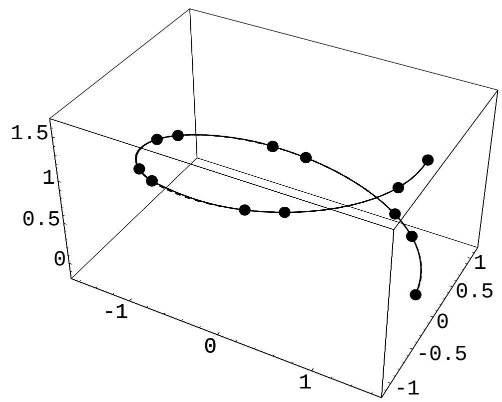


Fig. 5.2. Interpolating *an elliptical helix* γ_h (4.22) (dashed) sampled as in (1.24)(i) ($\varepsilon = 0.1$; rescaled by factor 2π) by *a cumulative chord piecewise-quartic* $\hat{\gamma}_{h_4}$ (solid), for 3 successive quintuples of $\mathcal{Q}_{12}$ (dotted) with length estimate: $d(\hat{\gamma}_{h_4}) = d(\gamma_h) - 1.646 \times 10^{-2}$

Rys. 5.2. Interpolacja *helikoidy eliptycznej* γ_h (4.22) (linia przerywana) próbkowanej w takt wzoru (1.24)(i) ($\varepsilon = 0.1$; przeskalowane czynnikiem 2π) *przedziałowo-wielomianową funkcją sklejaną rzędu czwartego na bazie skumulowanej parametryzacji długością cięciwy* $\hat{\gamma}_{h_4}$ (linia ciągła), dla 3 kolejnych piątek punktów z $\mathcal{Q}_{12}$ (wytłuszczone) z oszacowaniem długości: $d(\hat{\gamma}_{h_4}) = d(\gamma_h) - 1.646 \times 10^{-2}$

$d(\gamma_h) = 8.090$. For the helix γ_h the experiments are carried out with the following two families of ε-uniform samplings *i.e.* with (1.24)(i) (rescaled by factor 2π) and with

$$t_i = \frac{2\pi i}{m} + (Random[\ ] - 0.5)\frac{2\pi}{m^{1+\varepsilon}} \quad for \quad 1 \le i \le m-1; \quad t_0 = 0; \ t_m = 2\pi \ , \tag{5.21}$$

where $Random[\ ]$ takes the pseudo-random values from the interval $[0,1]$. Different sampling points are generated with $\varepsilon_0 = 0$, $\varepsilon_{1/10} = 0.1$, $\varepsilon_{1/5} = 0.2$, $\varepsilon_{1/4} = 0.25$, $\varepsilon_{1/3} = 0.33$, $\varepsilon_{1/2} = 0.5$, $\varepsilon_{2/3} = 0.66$, $\varepsilon_{3/4} = 0.75$, $\varepsilon_{9/10} = 0.9$, $\varepsilon_1 = 1$, and $\varepsilon_\infty = \infty$ (here $O(\frac{1}{m^\infty})$ vanishes), respectively. Note also that as $T = \sum_{i=1}^m (t_i - t_{i-1}) \le m\delta$ to verify sharpness of (5.1) and (5.2) in terms of $O(\delta^\alpha)$ it is sufficient to confirm both of them in terms of $O(\frac{1}{m^\alpha})$. The experiments are carried out for $m = 4k$ with bounds on $k_{min} = 3 \le k \le k_{max} = 70$. From the set of absolute errors $E_m(\gamma) = |d(\gamma) - d(\hat{\gamma}_4)|$, for $4 * k_{min} \le m \le 4 * k_{max}$, the estimate of convergence rate $O(\frac{1}{m^\alpha})$ is computed by applying *a linear regression* to pairs of points $(\log(m), -\log(E_m(\gamma)))$, with inequalities $4 * k_{min} \le m \le 4 * k_{max}$ - see Table 5.4.

The results from Table 5.4 confirm sharpness of (5.1). Note that the case when $\varepsilon = 0$ is covered by (5.2) (at least for sampling $(1.24)(i)$ satisfying (1.28) or (1.29)). Visibly some α's are slightly smaller as compared with those predicted by Theorems 5.1 and 5.2. Reaching a high accuracy in performed computation with $m << \infty$, hinders the exact verification of herein presented results proved merely for $m \approx \infty$. As it happens with cumulative chord piecewise-quadratics and piecewise-cubics (see Example 4.2), Figures

	ε_0	$\varepsilon_{1/10}$	$\varepsilon_{1/5}$	$\varepsilon_{1/4}$	$\varepsilon_{1/3}$	$\varepsilon_{1/2}$	$\varepsilon_{2/3}$	$\varepsilon_{3/4}$	$\varepsilon_{9/10}$	ε_1	ε_∞
$\alpha_{\gamma_{sc}}$ for (1.24)(i)	3.98	4.19	4.39	4.48	4.65	4.97	5.30	5.42	5.71	6.05	5.99
$\alpha_{\gamma_{sp}}$ for (1.24)(i)	3.95	4.14	4.31	4.38	4.49	4.68	5.27	5.67	6.08	6.02	5.96
α_{γ_h} for (1.24)(i)[a]	4.02	4.19	4.39	4.48	4.66	5.02	5.35	7.09	5.84	5.97	6.01
α_{γ_h} for (5.21)	4.11	4.30	4.46	4.54	4.85	5.33	5.72	5.75	5.78	5.95	6.01
$\alpha_{Th.5.1,5.2}$	4.00	4.20	4.40	4.50	4.66	5.00	5.33	5.50	5.80	6.00	6.00

[a] For γ_h sampling (1.24)(i) is rescaled by factor 2π

Tab. 5.1. *Cumulative chord piecewise-quartic* estimates of α in $d(\gamma)$ approximation for γ_{sc}, γ_{sp}, and γ_h defined by (1.9), (1.18) and (4.22), respectively (here $12 \leq m \leq 280$)

5.1 and 5.2 demonstrate highly accurate curve and length estimation by cumulative chord piecewise-quartics yielded on sporadic data (*i.e.* when m is small). Here for (1.24)(i) and curves γ_{sp} and γ_h we set $\varepsilon = 0.5$ and $\varepsilon = 0.1$, respectively. Figures 5.1 and 5.2 show also that at junction points q_{4j} (where $1 \leq j \leq \frac{m}{4} - 1$), between two consecutive quartic chords $\hat{\gamma}_4^j$ and $\hat{\gamma}_4^{j+4}$, the discontinuities in *geometrical smoothness* of $\hat{\gamma}_{sp_4}$ and of $\hat{\gamma}_{h_4}$ are indiscernible (i.e at point q_{4j+4} we have $\hat{\gamma}_4^{j'}(\hat{t}_4) \approx \hat{\gamma}_4^{j+4'}(0)$).

5.5 Discussion and motivation for Chapter 6

We proved (see Theorem 5.1) and verified experimentally that for ε*-uniform samplings cumulative chord piecewise-quartics yield an increase in convergence rates* for estimation of γ in $\mathbb{R}^n$ and $d(\gamma)$ over cumulative chord piecewise-cubics and piecewise-quadratics. The corresponding estimates are confirmed to be sharp (at least for length estimation and $n = 2, 3$). For *more-or-less uniform samplings* such *accelerations may not occur* and the convergence orders are equal to 4 which coincide with the respective orders established for cumulative chord piecewise-cubics (see Theorems 4.1 and 5.2). As also shown experimentally, at least for some planar and space curves, cumulative chord *piecewise-quartics perform well on sporadic data* (not covered by the asymptotic analysis). A good approximation of γ and $d(\gamma)$ by $\hat{\gamma}_4$ appears even for $m << \infty$. Similarly, the *negligible discontinuities in smoothness* between consecutive chords $\hat{\gamma}_4^i$ and $\hat{\gamma}_4^{i+4}$ occur for a small number of interpolation junction points. The latter is important in various applications, where either only sparse data is given or there is a need to process data quickly. Recall that neither *convexity* nor *specific dimensionality* of γ (*i.e.* n is arbitrary) is needed for cumulative chord interpolation.

However, comparing already fast convergence orders established by Theorem 4.1 for cumulative chord piecewise-quadratics and piecewise-cubics ($\alpha = 3, 4$, respectively), with respective orders from Theorems 5.1 and 5.2, it is evident that for the general class of ad-

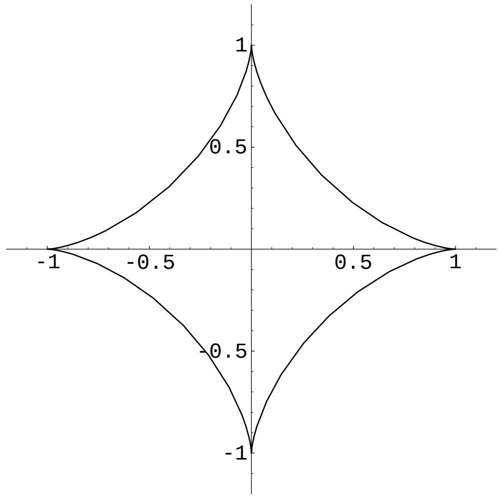


Fig. 5.3. *An asteroid* γ_a (5.22) with *cusps* at four planar points $(1,0)$, $(0,1)$, $(-1,0)$ and $(0,-1)$

Rys. 5.3. *Asteroida* γ_a (5.22) z *„rogami”* w czterech punktach na płaszczyźnie $(1,0)$, $(0,1)$, $(-1,0)$ i $(0,-1)$

missible samplings $\mathcal{V}_G^m$ (see Definition 1.2) *no further acceleration in convergence orders* eventuates for *cumulative chord piecewise-quartics* and reduced data $\mathcal{Q}_m$. Thus for handling reduced data, it suffices to resort to cumulative chords with $k = 2, 3$. The latter does not happen for *non-reduced data fitting* $(\mathcal{Q}_m, \{t_i\}_{i=0}^m)$ (for $\{t_i\}_{i=0}^m$ known) used with piecewise-r-degree Lagrange interpolation (see Theorem 1.2), where the acceleration occurs for $\mathcal{V}_G^m$.

As already mentioned the cumulative chord piecewise-polynomials are generically not C^1 (*analytically smooth*) at junction points, where two consecutive chords $\hat{\gamma}_k^i$ and $\hat{\gamma}_k^{i+k}$ are glued together (here $k = 2, 3, 4$ but can be arbitrary). They are most likely to provide discontinuities in the geometrical smoothness of the trajectory of γ (*e.g.* with cusps or corners). The latter is important for different applications *e.g.* in geometric modeling.

Remark 5.3. Recall that a C^1 curve $\gamma : [0, T] \to \mathbb{R}^n$ may not necessarily yield *a geometrically smooth* trajectory (*e.g.* with no cusps or corners). Indeed, take *e.g.* the C^1 *asteroid*

$$\gamma_a(t) \;=\; (\cos^3(t), \sin^3(t)) \,, \quad \text{for} \quad t \in [0, 2\pi] \,. \tag{5.22}$$

which has four *cusps* (see Figure 5.3) - the undesirable features to model the data. Even worse the *corners* for a C^1 curve $\gamma : [0, \bar{T}] \to \mathbb{R}^n$ may also occur. Take *e.g.* a C^1 singular *piecewise-linear curve* $\gamma_{pl} : [-1, 1] \to \mathbb{R}^2$ defined as:

$$\gamma_{pl}(t) \;=\; \begin{cases} (-t^2, t^2) & : \quad t \in [-1, 0) \\ (0, 0) & : \quad t = 0 \\ (t^2, t^2) & : \quad t \in (0, 1] \end{cases}$$

which clearly has a corner at $t = 0$. Evidently, for C^1 curve $\tilde{\gamma} : [0, \hat{T}] \to \mathbb{R}^n$ such singularities (cusps or corners) can only occur, where $\tilde{\gamma}'(\hat{t}) = 0$. Thus to eliminate such singularities it suffices to show that the resulting interpolant $\tilde{\gamma} \in C^1$ (is *analytically smooth*) and forms *a regular curve* (*i.e.* $\tilde{\gamma}'(\hat{t}) \neq \vec{0}$).

Though the jump in derivative for cumulative chord piecewise-quadratics, cubics and quartics at junction points between two consecutive chords $\hat{\gamma}_k^i$ and $\hat{\gamma}_k^{i+k}$ ($k = 2, 3, 4$) is minor, both for sparse and dense data (see (4.18) and (5.14)) still the question arises:

Can we define a smooth interpolant (C^1 and regular) based on cumulative chords to fit reduced data $\mathcal{Q}_m$ in $\mathbb{R}^n$ with fast convergence rates?

In the next chapter we propose the scheme which gives *a possible answer* to the above question. In the first step the scheme estimates (based on cumulative chord cubics $\hat{\gamma}_3^i$) the corresponding velocities $v(q_i)$ of $\dot{\gamma}$ at sampling points $\mathcal{Q}_m$. Subsequently, the pair of sequences $(\mathcal{Q}_m, \{v(q_i)\}_{i=0}^m)$, determines a unique Hermite piecewise-cubic $\hat{\gamma}_H$ (asymptotically regular: *i.e.* for sufficiently large m) interpolating smoothly $(\mathcal{Q}_m, \{v(q_i)\}_{i=0}^m)$. Respective quartic orders of convergence are established and confirmed experimentally to be sharp.

Chapter 6

Smooth cumulative chord cubics

Abstract

Regular cumulative chord C^1 piecewise-cubics, for reduced data $\mathcal{Q}_m$ from regular curves in $\mathbb{R}^n$, are constructed as follows. In the first step derivatives at given ordered interpolation points $\mathcal{Q}_m$ are estimated from the previously discussed overlapping family of cumulative chord cubics. Then Hermite interpolation is used to construct a C^1 piecewise-cubic interpolant γ_H yielding also a regular curve in $\mathbb{R}^n$ (with no cusps and corners). Quartic orders of approximation are established, and their sharpness is verified through numerical experiments (at least for $n = 2, 3$ and length estimation). Good performance of the interpolant is also confirmed experimentally on sparse data not covered by the asymptotic analysis presented herein. This work is published in [30] and [31].

6.1 Main result

Recall that Theorem 4.1 holds for any sufficiently smooth regular curve γ (not necessarily convex) in Euclidean space $\mathbb{R}^n$ of arbitrary dimension, and is applicable without extra conditions on sampling. Cumulative chord piecewise-cubics (or piecewise-quadratics) approximate at least to order 4 (or 3) which matches the same orders from Theorem 1.2, where the t_i are given (compare also Theorem 1.3 with Theorem 4.1). We remark here that cumulative chord piecewise-quartics yield further convergence speed-up only for special families of samplings (see Theorem 5.1).

Again, for an integer $k \geq 1$, set

$$\hat{t}_0 = 0 \quad \text{and} \quad \hat{t}_j = \hat{t}_{j-1} + \|q_j - q_{j-1}\| , \tag{6.1}$$

for $j = 1, 2, \ldots, m$. For k dividing m and $i = 0, k, 2k, \ldots, m-k$, let $\hat{\gamma}_k$ be the curve satisfying

$$\hat{\gamma}_k(\hat{t}_j) = q_j , \tag{6.2}$$

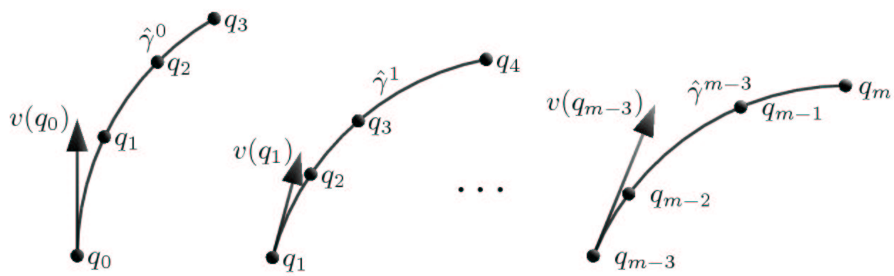


Fig. 6.1. Estimation of velocities $\dot{\gamma}$ of γ at points $\{q_i\}_{i=0}^{m-3}$ with *cumulative chord cubics* $\{\hat{\gamma}_3^i\}_{i=0}^{m-3}$ by setting $\dot{\gamma}(t_i) \approx v(q_i) = \hat{\gamma}_3^{i'}(\hat{t}_i)$ (for $0 \leq i \leq m-3$)

Rys. 6.1. Estymacja wektorów prędkości $\dot{\gamma}$ dla γ w punktach $\{q_i\}_{i=0}^{m-3}$, z zastosowaniem interpolacji *kubicznych* $\{\hat{\gamma}_3^i\}_{i=0}^{m-3}$ na *bazie skumulowanej parametryzacji długością cięciwy*; przyjęto $\dot{\gamma}(t_i) \approx v(q_i) = \hat{\gamma}_3^{i'}(\hat{t}_i)$ (gdzie $0 \leq i \leq m-3$)

for all $j = 0, 1, 2, \dots, m$, and whose restriction $\hat{\gamma}^i$ to each $[\hat{t}_i, \hat{t}_{i+k}]$ is a polynomial of degree at most k. Call $\hat{\gamma}_k$ the *cumulative chord piecewise-degree-k polynomial* approximation to γ defined by $\mathcal{Q}_m = (q_0, q_1, \dots, q_m)$. Unfortunately, cumulative chord piecewise-polynomials are usually not C^1 at knot points t_{kj} (*not smooth*), where $j \equiv 0 \ mod\, k$. The purpose of the present chapter is to *rectify this deficiency* for *cumulative chord piecewise-cubics* ($k = 3$), as follows:

1. First, for each $i = 0, 1, \dots, m-3$, let $\hat{\gamma}_3^i : [\hat{t}_i, \hat{t}_{i+3}] \to \mathbb{R}^n$ be the cumulative chord cubic interpolating $q_i, q_{i+1}, q_{i+2}, q_{i+3}$ at $\hat{t}_i, \hat{t}_{i+1}, \hat{t}_{i+2}, \hat{t}_{i+3}$ (see (6.1)), respectively. Those cumulative chords cubics permit to approximate the derivatives of γ at $\{q_i\}_{i=0}^m$ (or more precisely at $\{t_i\}_{i=0}^m$). More specifically, for each subinterval $[\hat{t}_i, \hat{t}_{i+3}]$ we estimate the velocity $v(q_i)$ of γ at q_i as $v(q_i) = \hat{\gamma}_3^{i'}(\hat{t}_i)$ - see Figure 6.1. Note that for the last three points $(q_{m-2}, q_{m-1}, q_m) \subset \mathcal{Q}_m$ the respective derivative estimation is obtained by passing, for each $i = m, m-1, m-2$, the *reverse cumulative chord cubic interpolants* $\hat{\gamma}_{3-}^i : [\hat{t}_i, \hat{t}_{i-3}] \to \mathbb{R}^n$ (here knots are $\hat{t}_i = 0$, $\hat{t}_{i-1} = \|q_i - q_{i-1}\|$, $\hat{t}_{i-2} = \hat{t}_{i-1} + \|q_{i-2} - q_{i-1}\|$, and $\hat{t}_{i-3} = \hat{t}_{i-2} + \|q_{i-3} - q_{i-2}\|$) satisfying reverse order interpolation conditions $\hat{\gamma}_{3-}^i(\hat{t}_{i-k}) = q_{i-k}$, for $k = 0, 1, 2, 3$, respectively.

2. Then let $\gamma_H^i : [\hat{t}_i, \hat{t}_{i+1}] \to \mathbb{R}^n$ be the Hermite cubic polynomial satisfying:

$$\gamma_H^i(\hat{t}_{i+k}) = q_{i+k} \quad \text{and} \quad {\gamma_H^i}'(\hat{t}_{i+k}) = \hat{\gamma}_3^{i+k'}(\hat{t}_{i+k}) , \quad \text{for} \quad k = 0, 1 \tag{6.3}$$

defined by *Newton's Interpolation Formula* (see Chapter 1 of [13]; see also (6.6)) as

$$\begin{aligned} \gamma_H^i(\hat{t}) \;=\;& \gamma_H^i[\hat{t}_i] + \gamma_H^i[\hat{t}_i, \hat{t}_i](\hat{t} - \hat{t}_i) + \gamma_H^i[\hat{t}_i, \hat{t}_i, \hat{t}_{i+1}](\hat{t} - \hat{t}_i)^2 \\ &+ \gamma_H^i[\hat{t}_i, \hat{t}_i, \hat{t}_{i+1}, \hat{t}_{i+1}](\hat{t} - \hat{t}_i)^2(\hat{t} - \hat{t}_{i+1}) \,. \end{aligned} \tag{6.4}$$

Note that by (6.6) $v(q_i)$ and $v(q_{i+1})$ are implicitly involved in (6.4), as divided differences $\gamma_H^i[\hat{t}_{i+k}, \hat{t}_{i+k}] = v(q_{i+k})$, for $k = 0, 1$. Let $\gamma_H : [0, \hat{T}] \to \mathbb{R}^n$ (called *cumulative chord* C^1 *piecewise-cubic*) be the track-sum of the $\{\gamma_H^i\}_{i=0}^{m-1}$ (see Remark 1.1).

Here is our *main result* holding for general class of admissible samplings $\{t_i\}_{i=0}^m \in \mathcal{V}_G^m$ (see [30]):

Theorem 6.1. *Suppose* γ *is a regular* C^4 *curve in* $\mathbb{R}^n$. *Let* $\gamma_H : [0, \hat{T}] \to \mathbb{R}^n$ *be the cumulative chord* C^1 *piecewise-cubic defined by* $\mathcal{Q}_m$ *as in* (6.3). *Then there is a piecewise-*C^∞ *reparameterization*[1] $\phi : [0, T] \to [0, \hat{T}]$, *such that*

$$\boxed{\gamma_H \circ \phi \;=\; \gamma \;+\; O(\delta^4) \quad and \quad d(\gamma_H) \;=\; d(\gamma) \;+\; O(\delta^4)\ .} \tag{6.5}$$

The second formula in (6.5) *needs an extra assumption* $m\delta = O(1)$.

Note that, there is a version of this construction, and of Theorem 6.1, for cumulative chord piecewise-quadratics, with orders of convergence decreased by 1, *i.e.* to cubic approximation orders. After some preliminaries and auxiliary lemmas in Section 6.2, Theorem 6.1, is proved in Section 6.3. Then, in Section 6.4, the sharpness of Theorem 6.1 is illustrated for some planar and space curves ($n = 2, 3$).

6.2 Divided differences and cumulative chords

We have already introduced the notion of divided difference for different tabular points $\{t_i\}_{i=0}^m$ (see Section 4.2). For the need of this chapter one has to extend the notion of divided differences to the situation, where some of the tabular points coincide (see *e.g.* already introduced (6.4)). Given recursive definition of (4.7) and (4.8) (and symmetry of divided differences - [52]) it suffices to cover the case when all t_i coincide. More specifically, when *tabular points* $t_i, t_{i+1}, \ldots, t_{i+k}$ coincide ($k+1$-times) and $\gamma \in C^k$ we have (see [52])

$$\gamma[t_i, t_i, \ldots, t_i] \;=\; \frac{\gamma^{(k)}(t_i)}{k!}\ . \tag{6.6}$$

Let γ be C^r and regular, where $r \geq k+1$ and k is 2 or 3. After a C^r reparameterization, as in Chapter 1; Proposition 1.1.5, of [26] we can assume for proving purposes that γ is parameterized by arc-length, namely $\|\dot{\gamma}\|$ is identically 1. Consider the cubic Lagrange interpolants (see (1.8)) $\psi^i : [t_i, t_{i+3}] \to [\hat{t}_i, \hat{t}_{i+3}]$ and $\psi^{i+1} : [t_{i+1}, t_{i+4}] \to [\hat{t}_{i+1}, \hat{t}_{i+4}]$ satisfying

$$\boxed{\psi^{i+k}(t_{i+k+j}) \;=\; \hat{t}_{i+k+j}\ ,} \tag{6.7}$$

[1] See Section 6.2 for more details.

for each $k = 0, 1$ with $j = 0, 1, 2, 3$. By Newton's Interpolation Formula

$$\begin{aligned}\psi^{i+k}(t) &= \psi^{i+k}(t_{i+k}) + (t - t_{i+k})\psi^{i+k}[t_{i+k}, t_{i+k+1}] \\ &+ (t - t_{i+k})(t - t_{i+k+1})\psi^{i+k}[t_{i+k}, t_{i+k+1}, t_{i+k+2}] \\ &+ (t - t_{i+k})(t - t_{i+k+1})(t - t_{i+k+2}) \\ &\quad \cdot \psi^{i+k}[t_{i+k}, t_{i+k+1}, t_{i+k+2}, t_{i+k+3}] .\end{aligned}$$

Recall that the following holds (see Chapter 4):

Lemma 6.1. *If γ is C^4 then, for $k = 0, 1$ and $t \in [t_{i+k}, t_{i+k+3}]$ we have the following: $\dot{\psi}^{i+k} = 1 + O(\delta^2)$, $\ddot{\psi}^{i+k} = O(\delta)$ and $\frac{d^3\psi^{i+k}}{dt^3} = O(1)$.*

In particular, ψ^{i+k} (for $k = 0, 1$) is a C^∞ diffeomorphism for δ small, which we assume from now on. Similarly for $\hat{\gamma}_3$ defined in (6.2) we have (see Chapter 4):

Lemma 6.2. *If γ is C^4 then, for $k = 0, 1$ and $s \in [\hat{t}_i, \hat{t}_{i+k+3}]$, $\hat{\gamma}_3^{i+k}, \frac{d\hat{\gamma}_3^{i+k}}{ds}, \frac{d^2\hat{\gamma}_3^{i+k}}{ds^2}, \frac{d^3\hat{\gamma}_3^{i+k}}{ds^3}$ are $O(1)$.*

Let $\psi : [0, T] \to [0, \hat{T}]$ be the track-sum of the $\{\psi^{3i}\}_{i=0}^{\frac{m}{3}-1}$ (see Remark 1.1). The following lemma is used in Chapter 4 to prove Theorem 4.1:

Lemma 6.3. *If γ is C^4 then, for $k = 0, 1$ and $t \in [t_i, t_{i+k+3}]$,*

$$\gamma(t) - (\hat{\gamma}_3^{i+k} \circ \psi^{i+k})(t) = O(\delta^4) \text{ and } \dot{\gamma}(t) - \frac{d(\hat{\gamma}_3^{i+k} \circ \psi^{i+k})}{dt}(t) = O(\delta^3) . \tag{6.8}$$

Let now $\phi^i : [t_i, t_{i+1}] \to [\hat{t}_i, \hat{t}_{i+1}]$ be *a Hermite cubic* satisfying

$$\boxed{\phi^i(t_i) = \hat{t}_i , \quad \phi^i(t_{i+1}) = \hat{t}_{i+1} , \quad \dot{\phi}^i(t_i) = \dot{\psi}^i(t_i) , \quad \dot{\phi}^i(t_{i+1}) = \dot{\psi}^{i+1}(t_{i+1}) ,} \tag{6.9}$$

given, by Newton's Interpolation Formula, as

$$\begin{aligned}\phi^i(t) &= \phi^i[t_i] + \phi^i[t_i, t_i](t - t_i) + \phi^i[t_i, t_i, t_{i+1}](t - t_i)^2 \\ &+ \phi^i[t_i, t_i, t_{i+1}, t_{i+1}](t - t_i)^2(t - t_{i+1}) .\end{aligned} \tag{6.10}$$

Then for the track-sum $\phi : [0, \hat{T}] \to \mathbb{R}^n$ of the $\{\phi^i\}_{i=0}^{m-1}$ (see Remark 1.1), which is used later to reparameterize γ_H we have:

Lemma 6.4. $\phi^i[t_i, t_i] = 1 + O(\delta^2)$, $\phi^i[t_i, t_i, t_{i+1}] = O(\delta)$, *and* $\phi^i[t_i, t_i, t_{i+1}, t_{i+1}] = O(1) + \frac{O(\delta^2)}{(t_{i+1} - t_i)^2}$.

Proof. Formulas (4.10), (6.9) combined with Lemma 6.1 render

$$\phi^i[t_i,t_i] \;=\; \dot\phi(t_i) \;=\; \dot\psi^i(t_i) \;=\; 1 \;+\; O(\delta^2)\;,$$

and analogously

$$\begin{aligned}\phi^i[t_i,t_i,t_{i+1}] &= \frac{\psi^i(t_{i+1})-\psi^i(t_i)}{(t_{i+1}-t_i)^2} - \frac{\dot\psi^i(t_i)}{t_{i+1}-t_i}\\ &= \psi^i[t_i,t_i,t_{i+1}]\\ &= O(\ddot\psi^i)\\ &= O(\delta)\;.\end{aligned}$$

Similarly

$$\phi^i[t_i,t_i,t_{i+1},t_{i+1}] \;=\; \frac{\dot\psi^{i+1}(t_{i+1})-\psi^i[t_i,t_{i+1}]}{(t_{i+1}-t_i)^2} - \frac{\psi^i[t_i,t_i,t_{i+1}]}{t_{i+1}-t_i}\;. \tag{6.11}$$

By Lemma 6.1, $\dot\psi^{i+k}(t_{i+1}) \;=\; 1+O(\delta^2)$, for $k \;=\; 0,1$ and thus the following holds $\dot\psi^{i+1}(t_{i+1}) = \dot\psi^i(t_{i+1}) + O(\delta^2)$. Hence, (4.10), (6.11) and Lemma 6.1 yield

$$\begin{aligned}\phi^i[t_i,t_i,t_{i+1},t_{i+1}] &= \frac{O(\delta^2)}{(t_{i+1}-t_i)^2} + \frac{\dot\psi^i(t_{i+1})-\psi^i[t_i,t_{i+1}]}{(t_{i+1}-t_i)^2} - \frac{\psi^i[t_i,t_i,t_{i+1}]}{t_{i+1}-t_i}\\ &= \frac{O(\delta^2)}{(t_{i+1}-t_i)^2} + \psi^i[t_i,t_i,t_{i+1},t_{i+1}]\\ &= \frac{O(\delta^2)}{(t_{i+1}-t_i)^2} + O(\frac{d^3\psi^i}{dt^3})\\ &= \frac{O(\delta^2)}{(t_{i+1}-t_i)^2} + O(1)\;.\end{aligned} \tag{6.12}$$

The proof is complete. □

Note that additional assumption about more-or-less uniformity for $\{t_i\}_{i=0}^m$ simplifies (6.12) to $O(1)$. This, however is not needed as the factor $1/(t_{i+1}-t_i)^2$ fed through into the further analysis is eventually attenuated in (6.28).

Lemma 6.5. *For* $t \in [t_i,t_{i+1}]$ *we have* $\dot\phi^i \;=\; 1+O(\delta^2)$, $\ddot\phi^i \;=\; O(\delta)+\frac{O(\delta^2)}{t_{i+1}-t_i}$, *and* $\frac{d^3\phi^i}{dt^3} = O(1)+\frac{O(\delta^2)}{(t_{i+1}-t_i)^2}$.

Proof. Differentiating (6.10) accordingly together with Lemma 6.4 and with the inequality $|(t-t_{i+k})(t_{i+1}-t_i)^{-1}| \leq 1$ (for $k=0,1$) completes the proof. □

Thus, asymptotically, each ϕ^i is a diffeomorphism. Similarly, the following holds for the interpolant γ_H^i:

Lemma 6.6. *$\gamma_H^i[\hat{t}_i, \hat{t}_i] = O(1)$, $\gamma_H^i[\hat{t}_i, \hat{t}_i, \hat{t}_{i+1}] = O(1)$, and $\gamma_H^i[\hat{t}_i, \hat{t}_i, \hat{t}_{i+1}, \hat{t}_{i+1}] = O(1) + \frac{O(\delta^2)}{(\hat{t}_{i+1}-\hat{t}_i)^2}$.*

Proof. By (6.3), (4.10), and Lemma 6.2

$$\gamma_H^i[\hat{t}_i, \hat{t}_i] = {\gamma_H^i}'(\hat{t}_i) = \hat{\gamma}_3^{i'}(\hat{t}_i) = O(1) ,$$

and similarly for each j-th component of γ_H^i and $\hat{\gamma}^i$ (where $1 \leq j \leq n$) and some $\tilde{t}_j^i \in [t_i, t_{i+1}]$ we have

$$\begin{aligned} \gamma_H^i[\hat{t}_i, \hat{t}_i, \hat{t}_{i+1}]_j &= \frac{\hat{\gamma}_{3j}^i(\hat{t}_{i+1}) - \hat{\gamma}_{3j}^i(\hat{t}_i)}{(\hat{t}_{i+1} - \hat{t}_i)^2} - \frac{\hat{\gamma}_{3j}^{i'}(\hat{t}_i)}{\hat{t}_{i+1} - \hat{t}_i} \\ &= \hat{\gamma}_3^i[\hat{t}_i, \hat{t}_i, \hat{t}_{i+1}]_j \\ &= O(\frac{\hat{\gamma}_{3j}^{i''}(\tilde{t}_j^i)}{2}) \\ &= O(1) . \end{aligned}$$

Evidently the latter extends to the vector form

$$\gamma_H^i[\hat{t}_i, \hat{t}_i, \hat{t}_{i+1}] = O(1) .$$

Analogously, (6.3) yields

$$\begin{aligned} \gamma_H^i[\hat{t}_i, \hat{t}_i, \hat{t}_{i+1}, \hat{t}_{i+1}] &= \frac{\hat{\gamma}_3^{i+1'}(\hat{t}_{i+1})}{(\hat{t}_{i+1} - \hat{t}_i)^2} - \frac{\hat{\gamma}_3^i(\hat{t}_{i+1}) - \hat{\gamma}_3^i(\hat{t}_i)}{(\hat{t}_{i+1} - \hat{t}_i)^3} \\ &\quad - \frac{\hat{\gamma}_3^i(\hat{t}_{i+1}) - \hat{\gamma}_3^i(\hat{t}_i)}{(\hat{t}_{i+1} - \hat{t}_i)^3} + \frac{\hat{\gamma}_3^{i'}(\hat{t}_i)}{(\hat{t}_{i+1} - \hat{t}_i)^2} . \end{aligned} \tag{6.13}$$

Taylor's Theorem combined with Lemma 6.2 yield

$$\begin{aligned} \hat{\gamma}_3^i(\hat{t}_i) &= \hat{\gamma}_3^i(\hat{t}_{i+1}) - \hat{\gamma}_3^{i'}(\hat{t}_{i+1})(\hat{t}_{i+1} - \hat{t}_i) + \frac{\hat{\gamma}_3^{i''}(\hat{t}_{i+1})}{2}(\hat{t}_{i+1} - \hat{t}_i)^2 \\ &\quad + O((\hat{t}_{i+1} - \hat{t}_i)^3) . \end{aligned}$$

Thus the first two terms in (6.13) read

$$\frac{\hat{\gamma}_3^{i+1'}(\hat{t}_{i+1}) - \hat{\gamma}_3^{i'}(\hat{t}_{i+1})}{(\hat{t}_{i+1} - \hat{t}_i)^2} + \frac{\hat{\gamma}_3^{i''}(\hat{t}_{i+1})}{2(\hat{t}_{i+1} - \hat{t}_i)} + O(1) . \tag{6.14}$$

Furthermore, by (6.8)

$$\begin{aligned} \hat{\gamma}_3^{i+1'}(\hat{t}_{i+1})\dot{\psi}^{i+1}(t_{i+1}) - \dot{\gamma}(t_{i+1}) &= O(\delta^3) , \\ \hat{\gamma}_3^{i'}(\hat{t}_{i+1})\dot{\psi}^{i}(t_{i+1}) - \dot{\gamma}(t_{i+1}) &= O(\delta^3) \end{aligned}$$

which combined with $\psi^{i+k}(t_{i+1}) = 1 + O(\delta^2)$, for $k = 0, 1$ (see Lemma 6.1) yields

$$\hat{\gamma}_3^{i+k'}(\hat{t}_{i+1}) = \dot{\gamma}(t_{i+1}) \;-\; \hat{\gamma}_3^{i+k'}(\hat{t}_{i+1})O(\delta^2) \;+\; O(\delta^3)\;.$$

The latter combined with Lemma 6.2 reduces (6.14) (and thus the first two terms of (6.13)) to

$$\frac{\hat{\gamma}_3^{i+1'}(\hat{t}_{i+1})}{(\hat{t}_{i+1}-\hat{t}_i)^2} - \frac{\hat{\gamma}_3^{i}(\hat{t}_{i+1})-\hat{\gamma}_3^{i}(\hat{t}_i)}{(\hat{t}_{i+1}-\hat{t}_i)^3} \;=\; \frac{\hat{\gamma}_3^{i'}(t_{i+1})O(\delta^2)-\hat{\gamma}_3^{i+1'}(t_{i+1})O(\delta^2)}{(\hat{t}_{i+1}-\hat{t}_i)^2} + \frac{O(\delta^3)}{(\hat{t}_{i+1}-\hat{t}_i)^2} \;+\; \frac{\hat{\gamma}_3^{i''}(\hat{t}_{i+1})}{2(\hat{t}_{i+1}-\hat{t}_i)}\,, \quad (6.15)$$

up to $O(1)$ term. On the other hand, Taylor's Theorem and Lemma 6.2 yield

$$\hat{\gamma}_3^{i}(\hat{t}_{i+1}) = \hat{\gamma}_3^{i}(\hat{t}_i) \;+\; \hat{\gamma}_3^{i'}(\hat{t}_i)(\hat{t}_{i+1}-\hat{t}_i) \;+\; \frac{\hat{\gamma}_3^{i''}(\hat{t}_i)}{2}(\hat{t}_{i+1}-\hat{t}_i)^2 \;+\; O((\hat{t}_{i+1}-\hat{t}_i)^3)$$

and hence the last two terms in (6.13) read

$$\frac{\hat{\gamma}_3^{i'}(\hat{t}_i)}{(\hat{t}_{i+1}-\hat{t}_i)^2} - \frac{\hat{\gamma}_3^{i}(\hat{t}_{i+1})-\hat{\gamma}_3^{i}(\hat{t}_i)}{(\hat{t}_{i+1}-\hat{t}_i)^3} \;=\; O(1) \;-\; \frac{\hat{\gamma}_3^{i''}(\hat{t}_i)}{2(\hat{t}_{i+1}-\hat{t}_i)}\;. \quad (6.16)$$

Adding (6.15) and (6.16) reduces (6.13) to

$$\begin{aligned}\gamma_H^i[\hat{t}_i,\hat{t}_i,\hat{t}_{i+1},\hat{t}_{i+1}] \;&=\; \frac{\hat{\gamma}_3^{i''}(\hat{t}_{i+1})-\hat{\gamma}_3^{i''}(\hat{t}_i)}{2(\hat{t}_{i+1}-\hat{t}_i)} + O(1)\\ &\quad + \frac{\hat{\gamma}_3^{i'}(\hat{t}_{i+1})O(\delta^2)-\hat{\gamma}_3^{i+1'}(\hat{t}_{i+1})O(\delta^2)+O(\delta^3)}{(\hat{t}_{i+1}-\hat{t}_i)^2}\,,\end{aligned}$$

which by Taylor's Theorem combined with Lemma 6.2 renders

$$\begin{aligned}\gamma_H^i[\hat{t}_i,\hat{t}_i,\hat{t}_{i+1},\hat{t}_{i+1}] \;&=\; O(\frac{d^3\hat{\gamma}_3^i}{d\hat{t}^3}) \;+\; O(1) \;+\; \frac{O(1)O(\delta^2)+O(1)O(\delta^2)+O(\delta^3)}{(\hat{t}_{i+1}-\hat{t}_i)^2}\\ &=\; O(1) \;+\; \frac{O(\delta^2)}{(\hat{t}_{i+1}-\hat{t}_i)^2}\;.\end{aligned}$$

The proof is complete. □

Lemma 6.7. *For* $\hat{t} \in [\hat{t}_i, \hat{t}_{i+1}]$ $\gamma_H^{i'} = O(1)$, $\gamma_H^{i''} = O(1) + \frac{O(\delta^2)}{t_{i+1}-t_i}$, *and* $\frac{d^3\gamma_H^i}{d\hat{t}^3} = O(1) + \frac{O(\delta^2)}{(t_{i+1}-t_i)^2}$.

Proof. The Mean Value Theorem coupled with the definition of cumulative chords $\hat{t}_i$, render for each $\hat{t} \in [\hat{t}_i, \hat{t}_{i+1}]$

$$\hat{t}_{i+1} - \hat{t}_i = O(t_{i+1} - t_i) = O(\delta) \quad \text{and thus} \quad \hat{t} - \hat{t}_{i+k} = O(t_{i+1} - t_i) = O(\delta) , \tag{6.17}$$

with $k = 0, 1$. Upon differentiating (6.4), Lemma 6.6 and (6.17) yield

$$\begin{aligned} \gamma_H^{i\prime}(\hat{t}) &= \gamma_H^i[\hat{t}_i, \hat{t}_i] + 2\gamma_H^i[\hat{t}_i, \hat{t}_i, \hat{t}_{i+1}](\hat{t} - \hat{t}_i) \\ &\quad + \gamma_H^i[\hat{t}_i, \hat{t}_i, \hat{t}_{i+1}, \hat{t}_{i+1}](2(\hat{t} - \hat{t}_i)(\hat{t} - \hat{t}_{i+1}) + (\hat{t} - \hat{t}_i)^2) \\ &= O(1) + O(1)O(\delta) + O(\delta^2) \\ &= O(1) , \end{aligned}$$

and similarly

$$\begin{aligned} \gamma_H^{i\prime\prime}(\hat{t}) &= 2\gamma_H^i[\hat{t}_i, \hat{t}_i, \hat{t}_{i+1}] + 2\gamma_H^i[\hat{t}_i, \hat{t}_i, \hat{t}_{i+1}, \hat{t}_{i+1}](2(\hat{t} - \hat{t}_i) + (\hat{t} - \hat{t}_{i+1})) \\ &= O(1) + \frac{O(\delta^2)}{\hat{t}_{i+1} - \hat{t}_i} . \end{aligned} \tag{6.18}$$

As by (6.1) $\hat{t}_{i+1} - \hat{t}_i = \|\gamma(t_{i+1}) - \gamma(t_i)\|$ Taylor's Theorem combined with $\|\dot{\gamma}\| = 1$, the Binomial Theorem and geometric expansion yield for $\vec{w} = \gamma(t_i)(t_{i+1} - t_i) + O((t_{i+1} - t_i)^2)$

$$\begin{aligned} (\hat{t}_{i+1} - \hat{t}_i)^{-1} &= (\langle \vec{w} | \vec{w} \rangle)^{-1/2} \\ &= \frac{1}{(t_{i+1} - t_i)\sqrt{1 + O(t_{i+1} - t_i)}} \\ &= \frac{1}{(t_{i+1} - t_i)(1 + O(t_{i+1} - t_i))} \\ &= \frac{1}{(t_{i+1} - t_i)}(1 + O(\delta) + O(\delta^2) + \ldots + O(\delta^l) + \ldots) \\ &= \frac{O(1)}{t_{i+1} - t_i} , \end{aligned}$$

where $l \in \mathbb{N}$. Coupling the latter with (6.18) renders

$$\gamma_H^{i\prime\prime}(\hat{t}) = O(1) + \frac{O(\delta^2)}{t_{i+1} - t_i} .$$

Analogously, by (6.4) and Lemma 6.6

$$\frac{d^3\gamma_H^i}{d\hat{t}^3}(\hat{t}) = 6\gamma_H^i[\hat{t}_i, \hat{t}_i, \hat{t}_{i+1}, \hat{t}_{i+1}] = O(1) + \frac{O(\delta^2)}{(t_{i+1} - t_i)^2} .$$

The proof is complete. □

6.3 Proof of main result

Proof. To prove Theorem 6.1 we compare γ with $\gamma_H \circ \phi$ and $\dot{\gamma}$ with $(\gamma_H \circ \phi)^{(1)} = \gamma_H' \dot{\phi}$ over each subinterval $[t_i, t_{i+1}]$. We first show that over $[0, T]$ we have:

$$\boxed{\gamma(t) \; - \; (\gamma_H^i \circ \phi^i)(t) \; = \; O(\delta^4) \; .} \tag{6.19}$$

Indeed, for γ_H reparameterized with ϕ and for $\hat{\gamma}$ reparameterized with ψ, by Lemma 6.3 we have

$$\begin{aligned} \gamma(t) \; - \; (\gamma_H^i \circ \phi^i)(t) \; &= \; \gamma(t) \; - \; (\hat{\gamma}_3^i \circ \psi^i)(t) \; + \; (\hat{\gamma}_3^i \circ \psi^i)(t) \; - \; (\gamma_H^i \circ \phi^i)(t) \\ &= \; O(\delta^4) \; + \; (\hat{\gamma}_3^i \circ \psi^i)(t) \; - \; (\gamma_H^i \circ \phi^i)(t) \; , \end{aligned} \tag{6.20}$$

over $t \in [t_i, t_{i+1}]$. Newton's Interpolation Formula for the C^∞ function

$$\boxed{\rho^i \; = \; \hat{\gamma}_3^i \circ \psi^i \; - \; \gamma_H^i \circ \phi^i} \tag{6.21}$$

yields

$$\begin{aligned} \rho^i(t) \; = \; & \rho^i[t_i] \; + \; \rho^i[t_i, t_i](t - t_i) \; + \; \rho^i[t_i, t_i, t_{i+1}](t - t_i)^2 \\ & + \rho^i[t_i, t_i, t_{i+1}, t_{i+1}](t - t_i)^2(t - t_{i+1}) \\ & + (t - t_i)^2(t - t_{i+1})^2 \rho^i[t_i, t_i, t_{i+1}, t_{i+1}, t] \; , \end{aligned} \tag{6.22}$$

for $t \in [t_i, t_{i+1}]$. We prove now that

$$\boxed{\rho^i(t) \; = \; O(\delta^4) \; ,} \tag{6.23}$$

which combined with (6.20) is sufficient to prove the first claim of Theorem 6.1. Indeed, note that by (6.7), (6.9) combined with (6.2) and (6.3) the divided differences of ρ^i appearing in (6.22) satisfy

$$\begin{aligned} \rho^i[t_i] \; &= \; \hat{\gamma}_3^i(\psi^i(t_i)) \; - \; \gamma_H^i(\phi^i(t_i)) \\ &= \; \hat{\gamma}_3^i(\hat{t}_i) \; - \; \gamma_H^i(\hat{t}_i) \\ &= \; \vec{0} \, , \end{aligned} \tag{6.24}$$

$$\begin{aligned} \rho^i[t_i, t_i] \; &= \; \dot{\rho}^i(t_i) \\ &= \; \hat{\gamma}_3^{i'}(\hat{t}_i)\dot{\psi}^i(t_i) \; - \; {\gamma_H^i}'(\hat{t}_i)\dot{\phi}^i(t_i) \\ &= \; \hat{\gamma}_3^{i'}(\hat{t}_i)\dot{\psi}^i(t_i) \; - \; \hat{\gamma}_3^{i'}(\hat{t}_i)\dot{\psi}^i(t_i) \\ &= \; \vec{0} \, , \end{aligned} \tag{6.25}$$

and hence as for (6.24) we have $\rho^i(t_{i+1}) = 0$ and thus

$$\rho^i[t_i, t_i, t_{i+1}] \; = \; \frac{\rho^i(t_{i+1}) - \rho^i(t_i)}{(t_{i+1} - t_i)^2} \; - \; \frac{\rho^i[t_i, t_i]}{t_{i+1} - t_i} \; = \; \vec{0} \, , \tag{6.26}$$

and furthermore by (6.8)

$$\begin{aligned}
\rho^i[t_i,t_i,t_{i+1},t_{i+1}] &= \frac{\rho^i[t_i,t_{i+1},t_{i+1}]}{t_{i+1}-t_i} \\
&= \frac{\dot{\rho}^i(t_{i+1})}{(t_{i+1}-t_i)^2} \\
&= \frac{\hat{\gamma}_3^{i\prime}(\hat{t}_{i+1})\dot{\psi}^i(t_{i+1}) - \dot{\gamma}(t_{i+1}) + \dot{\gamma}(t_{i+1}) - {\gamma_H^i}'(\hat{t}_{i+1})\dot{\phi}^i(t_{i+1})}{(t_{i+1}-t_i)^2} \\
&= \frac{O(\delta^3)}{(t_{i+1}-t_i)^2} + \frac{\dot{\gamma}(t_{i+1}) - \hat{\gamma}_3^{i+1\prime}(\hat{t}_{i+1})\dot{\psi}^{i+1}(t_{i+1})}{(t_{i+1}-t_i)^2} \\
&= \frac{O(\delta^3)}{(t_{i+1}-t_i)^2} + \frac{O(\delta^3)}{(t_{i+1}-t_i)^2} \\
&= \frac{O(\delta^3)}{(t_{i+1}-t_i)^2}\,. \qquad (6.27)
\end{aligned}$$

Combining (6.22) with (6.24), (6.25), (6.26) and (6.27) yields, for $t \in [t_i, t_{i+1}]$,

$$\boxed{\rho^i(t) = O(\delta^4) + (t-t_i)^2(t-t_{i+1})^2\rho^i[t_i,t_i,t_{i+1},t_{i+1},t]\,.} \qquad (6.28)$$

To prove (6.23) it suffices now to show

$$\boxed{(t-t_i)^2(t-t_{i+1})^2\rho^i[t_i,t_i,t_{i+1},t_{i+1},t] = O(\delta^4)\,,}$$

for $t \in [t_i, t_{i+1}]$. In doing so recall that by (4.10) for each component ρ_j^i $(1 \le j \le n)$

$$\rho^i[t_i,t_i,t_{i+1},t_{i+1},t]_j = \frac{1}{4!}\frac{d^4\rho_j^i}{dt^4}(\tilde{t}_j^i)\,, \qquad (6.29)$$

for some $\tilde{t}_j^i \in [t_i, t_{i+1}]$. As the degrees of $\hat{\gamma}_3^i$, ψ^i, γ_H^i and ϕ^i do not exceed 3, the Chain Rule combined with (6.29) and Lemmas 6.1, 6.2, 6.5, and 6.7 yield for all derivatives of ψ^i, ϕ^i evaluated at $\tilde{t}_j^i$ and for all derivatives of $\hat{\gamma}_{3j}^i$ and ${\gamma_H}_j^i$ evaluated at $\psi^i(\tilde{t}_j^i)$ and at $\phi^i(\tilde{t}_j^i)$, respectively

$$\begin{aligned}
\rho^i[t_i,t_i,t_{i+1},t_{i+1},t]_j &= 6\hat{\gamma}_{3j}^{i\prime\prime\prime}{\dot{\psi}^i}{}^2\ddot{\psi}^i + 4\hat{\gamma}_{3j}^{i\prime\prime}\dot{\psi}^i\frac{d^3\psi^i}{dt^3} + 3\hat{\gamma}_{3j}^{i\prime\prime}{\ddot{\psi}^i}{}^2 \\
&\quad - 6{\gamma_H}_j^{i\prime\prime\prime}{\dot{\phi}^i}{}^2\ddot{\phi}^i - 4{\gamma_H}_j^{i\prime\prime}\dot{\phi}^i\frac{d^3\phi^i}{dt^3} - 3{\gamma_H}_j^{i\prime\prime}{\ddot{\phi}^i}{}^2 \\
&= O(1) + \frac{O(\delta^2)}{t_{i+1}-t_i} + \frac{O(\delta^3)}{(t_{i+1}-t_i)^2} + \frac{O(\delta^4)}{(t_{i+1}-t_i)^3}\,. \qquad (6.30)
\end{aligned}$$

The latter extends to the vector form of $\rho^i[t_i, t_i, t_{i+1}, t_{i+1}, t]$. Hence, for each $t \in [t_i, t_{i+1}]$, as the inequality $|(t - t_{i+k})(t_{i+1} - t_i)^{-1}| \leq 1$ holds with $k = 0, 1$ we have

$$
\begin{aligned}
(t-t_i)^2(t-t_{i+1})^2\rho^i[t_i,t_i,t_{i+1},t_{i+1},t] &= (t-t_i)^2(t-t_{i+1})^2 O(1) \\
&\quad + (t-t_i)^2(t-t_{i+1})O(\delta^2) \\
&\quad + (t-t_i)^2 O(\delta^3) + (t-t_i)O(\delta^4) \\
&= O(\delta^4)\ .
\end{aligned}
$$

This together with (6.28), (6.22) and (6.20) proves *the first claim of Theorem* 6.1 *i.e.* the formula (6.19).

To prove *the second claim of Theorem* 6.1 we first justify the following:

$$\boxed{\dot{\gamma}(t) \ - \ \frac{d(\gamma_H^i \circ \phi^i)}{dt}(t) \ = \ O(\delta^3)\ .} \tag{6.31}$$

Indeed by Lemma 6.3 and (6.21), for $t \in [t_i, t_{i+1}]$, we have

$$
\begin{aligned}
\dot{\gamma}(t) \ - \ \frac{d(\gamma_H^i \circ \phi^i)}{dt}(t) &= \dot{\gamma}(t) \ - \ \frac{d(\hat{\gamma}_3^i \circ \psi^i)}{dt}(t) \\
&\quad + \frac{d(\hat{\gamma}_3^i \circ \psi^i)}{dt}(t) \ - \ \frac{d(\gamma_H^i \circ \phi^i)}{dt}(t) \\
&= O(\delta^3) + \dot{\rho}^i(t)\ .
\end{aligned} \tag{6.32}
$$

By (6.22), (6.24), (6.25), and (6.26) we obtain

$$
\begin{aligned}
\dot{\rho}^i(t) &= \rho^i[t_i,t_i,t_{i+1},t_{i+1}](2(t-t_i)(t-t_{i+1}) + (t-t_i)^2) \\
&\quad + 2\rho^i[t_i,t_i,t_{i+1},t_{i+1},t]((t-t_i)(t-t_{i+1})^2 \ + \ (t-t_i)^2(t-t_{i+1})) \\
&\quad + \frac{d\rho^i[t_i,t_i,t_{i+1},t_{i+1},t]}{dt}(t-t_i)^2(t-t_{i+1})^2\ ,
\end{aligned}
$$

and furthermore by (6.27), (6.30), the symmetry and the continuity of divided differences for $t \in [t_i, t_{i+1}]$ (see Chapter 1 of [12]):

$$
\begin{aligned}
\dot{\rho}^i(t) &= O(\delta^3) \\
&\quad + \lim_{h\to 0} \frac{\rho^i[t_i,t_i,t_{i+1},t_{i+1},t+h] - \rho^i[t,t_i,t_i,t_{i+1},t_{i+1}]}{h} \\
&\quad \cdot (t-t_i)^2(t-t_{i+1})^2 \\
&= O(\delta^3) \ + \ \rho^i[t_i,t_i,t_{i+1},t_{i+1},t,t](t-t_i)^2(t-t_{i+1})^2\ .
\end{aligned} \tag{6.33}
$$

As previously, for each $1 \leq j \leq n$, we have

$$\rho^i[t_i,t_i,t_{i+1},t_{i+1},t,t]_j \ = \ \frac{1}{5!}\frac{d^5\rho_j^i}{dt^5}(\tilde{t}_j^i)\ , \tag{6.34}$$

for some $\tilde{t}^i_j \in [t_i, t_{i+1}]$. The Chain Rule combined with (6.34) and Lemmas 6.1, 6.2, 6.5, and 6.7 yield for all derivatives of ψ^i, ϕ^i evaluated at $\tilde{t}^i_j$ and for all derivatives of $\hat{\gamma}^i_{3j}$ and $\gamma_H{}^i_j$ evaluated at $\psi^i(\tilde{t}^i_j)$ and at $\phi^i(\tilde{t}^i_j)$, respectively

$$\begin{aligned}
\rho^i[t_i, t_i, t_{i+1}, t_{i+1}, t, t]_j &= 15\hat{\gamma}^{i'''}_{3j}\dot{\psi}^i\ddot{\psi}^{i\,2} + 10\hat{\gamma}^{i'''}_{3j}\dot{\psi}^{i\,2}\frac{d^3\psi^i}{dt^3} \\
&\quad + 10\hat{\gamma}^{i''}_{3j}\ddot{\psi}^i\frac{d^3\psi^i}{dt^3} - 15\gamma_H{}^{i'''}_j\dot{\phi}^i\ddot{\phi}^{i\,2} \\
&\quad - 10\gamma_H{}^{i'''}_j\dot{\phi}^{i\,2}\frac{d^3\phi^i}{dt^3} - 10\gamma_H{}^{i''}_j\ddot{\phi}^i\frac{d^3\phi^i}{dt^3} \\
&= O(1) + \frac{O(\delta^3)}{t_{i+1}-t_i} + \frac{O(\delta^2)}{(t_{i+1}-t_i)^2} + \frac{O(\delta^4)}{(t_{i+1}-t_i)^3} \\
&\quad + \frac{O(\delta^4)}{(t_{i+1}-t_i)^4}\,.
\end{aligned}$$

The latter extends to the vector form of $\rho^i[t_i, t_i, t_{i+1}, t_{i+1}, t]$. Consequently for $t \in [t_i, t_{i+1}]$

$$\boxed{(t-t_i)^2(t-t_{i+1})^2\rho^i[t_i, t_i, t_{i+1}, t_{i+1}, t, t] = O(\delta^4)\,.}$$

which combined with (6.33) yields $\dot{\rho}^i = O(\delta^3)$ and thus by (6.32) proving (6.31).

Define now $f = \gamma_H \circ \phi - \gamma$ and let $v(t)$ be the projection of $\dot{f}$ onto the orthogonal space $\dot{\gamma}(t)^\perp$ to $\dot{\gamma}(t)$. As $\|\dot{\gamma}(t)\| = 1$ for each

$$\boxed{f^i = \gamma^i_H \circ \phi^i - \gamma : [t_i, t_{i+1}] \to \mathbb{R}^n}$$

we obtain (as previously)

$$\dot{f}^i(t) = \langle \dot{f}^i(t), \dot{\gamma}(t)\rangle\dot{\gamma}(t) + v(t)\,,$$

where by (6.31) and the latter $v(t) = O(\delta^3)$. Thus for reparameterized γ^i_H as

$$\boxed{\tilde{\gamma}^i_H = \gamma^i_H \circ \phi^i}$$

we have

$$\dot{\tilde{\gamma}}^i_H(t) = (1 + \langle \dot{f}^i(t), \dot{\gamma}(t)\rangle)\dot{\gamma}(t) + v(t)\,.$$

Hence as $\langle \dot{\gamma}(t), v(t)\rangle = 0$, $\|\dot{\gamma}\| = 1$ and $(1+\langle \dot{f}^i(t), \dot{\gamma}(t)\rangle)^{-2}\|v(t)\|^2 > -1+\varepsilon$ (for some fixed $\varepsilon > 0$):

$$\boxed{\|\dot{\tilde{\gamma}}^i_H(t)\| - \|\dot{\gamma}(t)\| = \langle \dot{f}^i(t), \dot{\gamma}(t)\rangle + O(\delta^6)\,.} \tag{6.35}$$

As $f^i(t_i) = f^i(t_{i+1}) = \vec{0}$, the integration by parts, (6.19) and (6.35) yield

$$\begin{aligned}\int_{t_i}^{t_{i+1}} (\|\dot{\tilde{\gamma}}_H^i(t)\| \; - \; \|\dot{\gamma}(t)\|) \, dt \;&=\; \int_{t_i}^{t_{i+1}} \langle \dot{f}^i(t), \dot{\gamma}(t)\rangle \, dt \; + \; O(\delta^7) \\ &=\; -\int_{t_i}^{t_{i+1}} \langle f(t), \ddot{\gamma}(t)\rangle \, dt \; + \; O(\delta^7) \\ &=\; O(\delta^5) \; .\end{aligned}$$

Consequently, over each subinterval $T_i = [t_i, t_{i+1}]$ as lengths $d(\gamma_H^i) = d(\tilde{\gamma}_H^i)$ (recall that by Lemma 6.5 $\phi^i : [t_i, t_{i+1}] \to [\hat{t}_i, \hat{t}_{i+1}]$ is a reparameterization as $\dot{\phi}^i = 1 + O(\delta^2) > 0$, asymptotically) and therefore

$$\boxed{d(\gamma_H^i) \; - \; d(\gamma_i) \; = \; O(\delta^5) \; ,}$$

where $d(\gamma_i) = \int_{t_i}^{t_{i+1}} \|\dot{\gamma}(t)\| \, dt$. Hence as $m\delta = O(1)$ we obtain

$$\boxed{d(\gamma_H) \; - \; d(\gamma) \; = \; \sum_{i=0}^{m-1} (d(\gamma_H^i) - d(\gamma_i)) \; = \; O(\delta^4) \; .}$$

The proof is complete. □

Remark 6.1. Recall that for the last three points $(q_{m-2}, q_{m-1}, q_m) \subset \mathcal{Q}_m$ the respective derivative estimation is performed by passing *reverse cumulative chord cubic interpolants* $\{\hat{\gamma}_{3-}^i\}_{i=m}^{i=m-2}$ introduced in Section 6.1. For completeness, we also need to use *reverse Lagrange cubics* (see (6.7)) $\psi_-^i : [t_i, t_{i-3}] \to [\hat{t}_i, \hat{t}_{i-3}]$ satisfying $\psi_-^i(t_{i-k}) = \hat{t}_{i-k}$, for $k = 0, 1, 2, 3$, respectively. These cubics reparameterize $\hat{\gamma}_{3-}^i$ exactly as ψ^i reparameterize $\hat{\gamma}_3^i$. The proof of Theorem 6.1 applies also to this non-generic case.

Finally, we comment on the *regularity* of the γ_H which is an important feature to exclude potential cusps or corners (see Remark 5.3) from the trajectory of γ.

Remark 6.2. Note that asymptotically (for sufficiently small δ_m, or sufficiently large m) the resulting C^1 interpolant γ_{H,δ_m} is regular. Assume otherwise. Then there is a subsequence $\delta_{m_l} < \delta_0$ tending to zero (for δ_0 fixed) with $\hat{t}_{m_l} \in [0, \hat{T}]$ such that $\dot{\gamma}_{H,\delta_{m_l}}(\hat{t}_{m_l}) = \vec{0}$. Thus by (6.31) and the Definition 1.3, for $t_{m_l} = \phi^{-1}(\hat{t}_{m_l})$ we have $\|\dot{\gamma}(t_{m_l})\| \leq K\delta_{m_l}^3$ (for all $\delta_{m_l} < \delta_0$). On the other hand since $\gamma \in C^1$ is regular and defined over compact set $[0, T]$ the continuous function $\|\dot{\gamma}(t)\|$ is separated from zero *i.e.* for each $t \in [0, T]$ $\|\dot{\gamma}(t)\| > \varepsilon_0 > 0$. Clearly, for sufficiently small δ_{m_l} (or equivalently sufficiently large m) $K\delta_{m_l}^3 < \varepsilon_0$, a contradiction. Thus γ_H is regular (asymptotically) and as it is also C^1, the trajectory of γ_H *has not cusps or corners* and has a smooth geometrical appearance (for sufficiently large m). Note also that this proof applies to the interpolants from previous chapters yielding no bad singularities (cusps or corners), with the exception to those junction points where the gluing process of two local interpolants is done. At such junction points the resulting interpolant is not C^1 and has visible edges (not analytically smooth).

```
TanCubChord[list_]
{
    Knots := CC_C_Formula[list];
                        /* see (4.1) with k = 3 and {t̂_{i+k} := t̂_{i+k} − t̂_i}^3_{k=0} */
    CC_C[s_] := Lag_Formula[list, Knots, 3, s];        /* see (1.8) with r = 3 */
    CC_C_Der := D[CC_C[s], s−> 0];
    return{CC_C_Der}
}
```

Fig. 6.2. Pseudocode for procedure **TanCubChord**, which for one input list $\{q_i\}_i^{i+3}$, returns the list of single tangent vector $v(q_i) = \hat{\gamma}_3^{i'}(0) \approx \dot{\gamma}(t_i)$

Rys. 6.2. Pseudokod dla procedury **TanCubChord**, która dla danych na wejściu (lista $\{q_i\}_i^{i+3}$) zwraca listę złożoną z jednego wektora stycznego $v(q_i) = \hat{\gamma}_3^{i'}(0) \approx \dot{\gamma}(t_i)$

6.4 Experiments

We experiment here (see also [30] and [31]) with sampling points obtained from smooth regular curves not necessarily parameterized by arc-length. The arc-length reparameterization is only needed as a technical tool substantially simplifying the proof of the Theorem 6.1. The tests are carried out in *Mathematica*.

6.4.1 Pseudocode

The interpolation scheme introduced in Section 6.1 can be implemented with the aid of two procedures.

Let $list$ consisting of quadruple of points (taken from reduced data $\mathcal{Q}_m^{i,3}$) be initialized to $list := \{q_i, q_{i+1}, q_{i+2}, q_{i+3}\}$ (*i.e.* $list[s] = q_{i+s}$, for $0 \leq s \leq 3$). The pseudocode for the similar procedure as in Section 1.6.1 (with $r = 3$ and, for $\mathcal{Q}_m^{i,3}$ with knots: $\hat{t}_i = 0$, $\hat{t}_{i+1} = \|q_{i+1} - q_i\|$, $\hat{t}_{i+2} = \hat{t}_{i+1} + \|q_{i+2} - q_{i+1}\|$, and $\hat{t}_{i+3} = \hat{t}_{i+2} + \|q_{i+3} - q_{i+2}\|$ - see (4.1)) reads as in Figure 6.2.

Clearly, the procedure **TanCub** returns the estimation of tangent vector $v(q_i)$ to $\dot{\gamma}(t_i)$, with the aid of cumulative chord cubic $\hat{\gamma}_3^i$ (or reverse cumulative chord cubic $\hat{\gamma}_{3-}^i$), where $\hat{\gamma}_3^{i'}(\hat{t}_i) = v(q_i)$, for $0 \leq i \leq m-3$ (or $\hat{\gamma}_{3-}^{i'}(\hat{t}_i) = v(q_i)$, for $m-2 \leq i \leq m$).

For $\{v(q_i)\}_{i=0}^m$ representing the estimates of $\{\dot{\gamma}(t_i)\}_{i=0}^m$ (see Section 6.1) assume now that $list1 := \{q_i, q_{i+1}\}$, $list2 := \{v_i, v_{i+1}\}$. The pseudocode for the procedure to derive Hermite cubic γ_H^i (and its length $d(\gamma_H^i)$) based on cumulative chord parameterization (one can assume[2] that $\hat{t}_i = 0$ and $\hat{t}_{i+1} = \|q_{i+1} - q_i\|$) reads as in Figure 6.3.

[2]Recall that $\bar{\gamma}(\hat{t}) = \hat{\gamma}(\hat{t} - \bar{T})$ has the same trajectory and length over $[-\bar{T}, \hat{T} - \bar{T}]$ as $\hat{\gamma}$ over $[0, \hat{T}]$

$\mathbf{HermiteCC}[list1\_, list2\_]$
$\{$
$Knots := CC\_L\_Formula[list1];$
$/*$ see (4.1) for $k=1$ and $\{\hat{t}_i = \hat{t}_{i+k} - \hat{t}_i\}_{k=0}^{1}$ $*/$
$Hermite[s\_] := H\_Formula[list1, list2, Knots, s];$ $/*$ see (6.4) $*/$
$Der\_CC\_H := D[Expand[Hermite[s], s]];$
$CC\_H\_Der\_List := CoefficientList[Der\_CC\_H, s];$
$Norm[s\_] := Derivatives\_Norm[CC\_H\_Der\_List, s];$
$Length := NIntegrate[Norm[s], \{s, 0, Knots[1]\}];$
$/*$ see (1.1) over $[0, Knots[1]]$ $*/$
$CC\_H\_Plot := ParametricPlot[Hermite[s], \{s, 0, Knots[1]\}];$
$\mathbf{return}\{Length, CC\_H\_Plot\}$
$\}$

Fig. 6.3. Pseudocode for procedure **HermiteCC**, which for two input lists $\{q_i\}_i^{i+1}$ and $\{v(q_i)\}_i^{i+1}$, returns the list of $\{d(\gamma_H^i), plot\}$, where $plot$ represents a discrete set of $\gamma_H^i(s)$, for $s \in [0, \|q_{i+1} - q_i\|]$ (according to the ParametricPlot format)

Rys. 6.3. Pseudokod dla procedury **HermiteCC**, która dla danych na wejściu (dwie listy $\{q_i\}_i^{i+1}$ oraz $\{v(q_i)\}_i^{i+1}$) zwraca listę $\{d(\gamma_H^i), plot\}$, gdzie $plot$ jest listą dyskretnego zbioru wartości $\gamma_H^i(t)$ dla $s \in [0, \|q_{i+1} - q_i\|]$ (zgodnie z formatem ParametricPlot)

The estimation of α (for $d(\gamma_H) - d(\gamma) = O(\delta^\alpha)$) which is computed by linear regression (see Section 1.6) from the collection of reduced data $\{\mathcal{Q}_j\}_{j=m_1}^{m_2}$, for $m_1 \leq m \leq m_2$ (with $m_1 \geq 4$), yields a modified pseudocode (see Section 1.6.1) shown in Figure 6.4.

Note that the first (the second) internal loop calculates lengths $d(\gamma_H^i)$ for $0 \leq i \leq j-4$ (for $j-2 \leq i \leq j-1$). The case $d(\gamma_H^{j-3})$ is treated separately. Recall that we index here any list from the label set to zero. As before, a slight modification of the main program loop yields the pseudocode to plot γ_H for a given $\mathcal{Q}_{j_{fix}}$. Again, the pseudocode for estimating α in $\gamma_H \circ \psi - \gamma = O(\delta^\alpha)$ is similar but as previously involves computationally expensive optimization procedure (see Subsection 1.3.1) to find (for $m_1 \leq m \leq m_2$) the following quantity $\sup_{t \in [0,T]} |\gamma_H \circ \psi(t) - \gamma(t)|$, where $\psi : [0,T] \to [0,\hat{T}]$ is a piecewise C^∞ reparameterization - see Section 6.2).

6.4.2 Testing

We first verify the sharpness of claim of Theorem 6.1 for some *planar curves*.

Example 6.1. Consider a spiral $\gamma_{sp} : [0,1] \to \mathbb{R}^2$ from (1.18) with length $d(\gamma_{sp}) = 2.452$. Cumulative chord C^1 piecewise-cubic γ_{sp_H} (see (6.3) and (6.4)) based on the 7-tuple $\mathcal{Q}_m$ with sampling (1.24)(i) (and with $\varepsilon = 0$) is shown in Figure 6.5 for which a good

```
For  [j = m_1; err = { }, j ≤ m_2, j = j + 1,
     D[j] := {q_0, q_1, q_2, ..., q_j};
     For  [i = 0; length = 0, i ≤ j − 4, i = i + 1,
          ListI := {D[j][i], D[j][i + 1], D[j][i + 2], D[j][i + 3]};
          ListII := {D[j][i + 1], D[j][i + 2], D[j][i + 3], D[j][i + 4]};
          List1 := {D[j][i], D[j][i + 1]};
          List2 := {TanCub[ListI][0], TanCub[ListII][0]};
          length+ = HermiteCC[List1, List2][0];
          ];
     For  [i = j, i > j − 2, i = i − 1,
          listI := {D[j][i], D[j][i − 1], D[j][i − 2], D[j][i − 3]};
          listII := {D[j][i − 1], D[j][i − 2], D[j][i − 3], D[j][i − 4]};
          list1 := {D[j][i], D[j][i − 1]};
          list2 := {TanCub[listI][0], TanCub[listII][0]};
          length+ = HermiteCC[list1, list2][1];
          ];
     length+ = HermiteCC[{D[j][j − 3], D[j][j − 2]}, {List2[1], list2[1]}][0];
     err := AppendTo[err, {Log[j], −Log[|length − d(γ)|]}];
     ];
α := Slope_Coeff[Regress[err]];                    / * see Section1.6 * /
```

Fig. 6.4. Pseudocode for the main program computing an α estimate in $d(\gamma_H) - d(\gamma) = O(\delta^\alpha)$ based on collection of reduced data $\{\mathcal{Q}_j\}_{j=m_1}^{m_2}$

Rys. 6.4. Pseudokod głównej pętli programu obliczającej oszacowanie α w $d(\gamma_H) - d(\gamma) = O(\delta^\alpha)$ na podstawie rodziny danych zredukowanych $\{\mathcal{Q}_j\}_{j=m_1}^{m_2}$

performance in trajectory and length estimation is confirmed on such sporadic data. The respective absolute errors in length estimates for $m = 48, 90, 150$, and 198 are 2.466×10^{-6}, 1.973×10^{-7}, 2.537×10^{-8} and 8.339×10^{-9}. The plot for cumulative chord C^1 piecewise-cubic interpolation of $-\log|d(\gamma_{sp_H}) - d(\gamma_{sp})|$ against $\log(m)$ in Figure 6.6, for $m = 18, 24, 30, \ldots, 198$, appears almost linear, with least squares estimate of slope 4.01. The claim of Theorem 6.1 for $d(\gamma)$ estimation is here confirmed (at least for $n = 2$). Note that since $\sum_{i=0}^{m-1}(t_{i+1} - t_i) = T$, by (1.3) we obtain $m\delta \geq T$. Thus to examine or test the orders of convergence in $O(\delta^\alpha)$ rates (where $\alpha > 0$), it suffices to show or verify the corresponding rates in $O(1/m^\alpha)$ asymptotics. □

The next example tests the scheme in question for *a non-convex* cubic curve $\gamma_{cnc} \subset \mathbb{R}^2$ having one *inflection point* at $(0, 0)$.

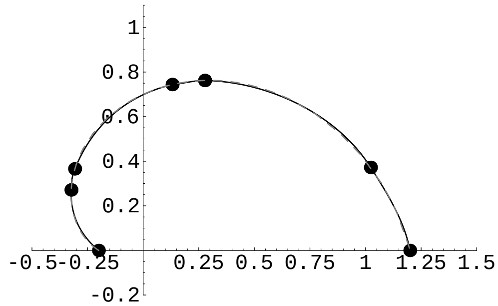


Fig. 6.5. Interpolating *a spiral* γ_{sp} (1.18) (dashed) sampled as in (1.24)(i) ($\varepsilon = 0$, $m = 6$) by *a cumulative chord* C^1 *piecewise-cubic* γ_{sp_H} (solid) with the following length estimate: $d(\gamma_{sp_H}) = d(\gamma_{sp}) + 6.122 \times 10^{-3}$

Rys. 6.5. Interpolacja *spirali* γ_{sp} (1.18) (linia przerywana) próbkowanej w takt (1.24)(i) ($\varepsilon = 0$, $m = 6$) *gładką przedziałowo-kubiczną funkcją sklejaną na bazie skumulowanej parametryzacji długością cięciwy* $\hat{\gamma}_{sp_4}$ (linia ciagła) z następującym oszacowaniem długości: $d(\gamma_{sp_H}) = d(\gamma_{sp}) + 6.122 \times 10^{-3}$

Example 6.2. Consider the following regular *cubic curve* $\gamma_{cnc} : [0,1] \to \mathbb{R}^2$ defined as

$$\gamma_{cnc}(t) = (2t-1, (2t-1)^3) . \tag{6.36}$$

Cumulative chord C^1 piecewise-cubic γ_{cnc_H} (see (6.3) and (6.4)) based on sampling (1.24)(i) (and $\varepsilon = 0$) yields the respective absolute errors in length approximation with $m = 48, 90, 150,$ 198 are 1.919×10^{-7}, 2.628×10^{-8}, 3.778×10^{-9}, 1.276×10^{-9}. As previously, linear regression applied to the set of points $-\log|d(\gamma_{cnc_H}) - d(\gamma_{cnc})|$ against $\log(m)$, for $m = 180, 186, \ldots, 300$, yields the estimate for convergence order of length approximation equal to 3.96. As in Example 6.1, the quartic order of convergence from Theorem 6.1 for $d(\gamma)$ estimation appears here to be experimentally confirmed (at least for $n = 2$). Again, cumulative chord C^1 piecewise-cubic γ_H based on the 7-tuple $\mathcal{Q}_m$ gives an excellent estimate on sporadic data for length $d(\gamma_{cnc_H}) = d(\gamma_{cnc}) + 3.505 \times 10^{-2}$, where $d(\gamma_{cnc}) = 3.096$. □

In the next step, we verify that Theorem 6.1 holds for not necessarily *more-or-less uniform* samplings (see (1.28) or (1.29)) which are required for proving some of the existing convergence results mentioned in the Theorems 3.1, 5.2.

Example 6.3. Apply cumulative chord C^1 piecewise-cubic $\gamma_{q_{b\,H}}$ (see (6.3) and (6.4)) for the following regular *quartic curve* $\gamma_q : [0,1] \to \mathbb{R}^2$ defined as

$$\gamma_{q_b}(t) = (t, (t+1)^4/8) \tag{6.37}$$

with $d(\gamma_{q_b}) = 1.4186$ and sampled according to:

$$t_i = \frac{i}{m}, \quad \text{for } i \text{ even}; \quad t_i = \frac{i}{m} + \frac{1}{m} - \frac{1}{m^2}, \quad \text{for } i \text{ odd}; \quad t_m = 1 . \tag{6.38}$$

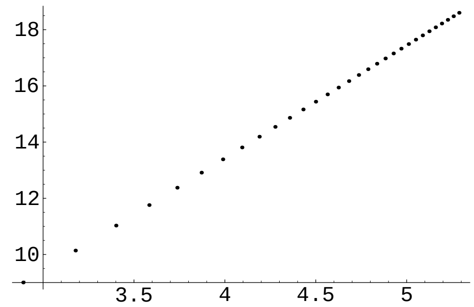


Fig. 6.6. Plot of $\mathcal{D} = (\log(m), -\log|d(\gamma_{sp_H}) - d(\gamma_{sp})|)$, for *a spiral* γ_{sp} sampled as in Example 6.1 ($18 \leq m \leq 198$) and interpolated by *cumulative chord* C^1 *piecewise-cubics* γ_{sp_H}. The slope of linear regression line applied to $\mathcal{D}$: $4.01 \approx 4$ (see Theorem 6.1)

Rys. 6.6. Wykres zbioru punktów $\mathcal{D} = (\log(m), -\log|d(\gamma_{sp_H}) - d(\gamma_{sp})|)$ dla interpolacji *spirali* γ_{sp} próbkowanej jak w Przykładzie 6.1 ($18 \leq m \leq 198$) i interpolowanej *gładkimi przedziałowo-kubicznymi funkcjami sklejanymi na bazie skumulowanej parametryzacji długością cięciwy* γ_{sp_H}. Współczynnik nachylenia prostej otrzymanej z liniowej regresji dla $\mathcal{D}$: $\alpha = 4.01 \approx 4$ (patrz Twierdzenie 6.1)

Clearly condition (1.28) (or equivalently (1.29)) does not hold for sampling (6.38). The respective absolute errors in length estimates for $m = 6, 48, 90, 150$, and 198 are as follows 1.912×10^{-5}, 2.557×10^{-9}, 3.483×10^{-10}, 5.394×10^{-11} and 1.884×10^{-11}. As previously, linear regression applied to the set of points $-\log|d(\gamma_{q_{b\,H}}) - d(\gamma_q)|$ against $\log(m)$, for $m = 480, 486, \ldots, 600$, yields the estimate for convergence order of length approximation equal to 4.04. Again, the rates established by Theorem 6.1 are confirmed here (at least for $n = 2$). ☐

Finally, we experiment with a regular elliptical helix *space curve*.

Example 6.4. Figure 6.7 shows a cumulative chord C^1 piecewise-cubic interpolant γ_{h_H} (see (6.3) and (6.4)) of 7 points on the *elliptical helix* $\gamma_h : [0, 2\pi] \to \mathbb{R}^3$, defined by (4.22) and sampled with t_i equal to either

$$\frac{2\pi i}{m} \quad \text{or} \quad \frac{\pi(2i-1)}{m} \tag{6.39}$$

according as i is even or odd. Although sampling is uneven, sparse, and not available for interpolation, γ_{h_H} seems very close to γ_h: $d(\gamma_h) = 8.090$ and $d(\gamma_{h_H}) = 8.045$. As previously, linear regression applied to the set of points $-\log|d(\gamma_{h_H}) - d(\gamma_h)|$ against $\log m$, for $m = 90, 96, \ldots, 300$, yields the estimate for convergence order of length approximation equal to 4.00. The respective absolute errors in length approximation for $m = 6, 48, 90, 150$, and 198 are 4.478×10^{-2}, 2.902×10^{-5}, 2.044×10^{-6}, 2.614×10^{-7} and 8.620×10^{-8}. Similarly, for sampling (6.38) linear regression applied to the set of points $-\log|d(\gamma_{h_H}) - d(\gamma_h)|$ against $\log(m)$, for $m = 240, 246, \ldots, 300$, yields the estimate for convergence order of length approximation equal to 4.08. ☐

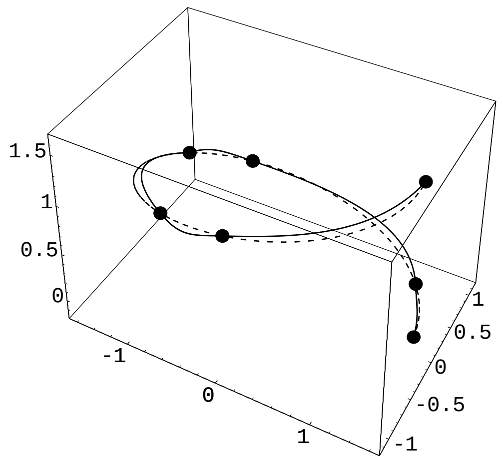


Fig. 6.7. Interpolating *an elliptical helix* γ_h (4.22) (dashed) sampled as in Example 6.4 (see (6.39) with $\mathcal{Q}_6$ (dotted) by *a cumulative chord* C^1 *piecewise-cubic* γ_{h_H} (solid) with length estimate: $d(\gamma_{h_H}) = d(\gamma_h) - 4.478 \times 10^{-2}$

Rys. 6.7. Interpolacja *helikoidy eliptycznej* γ_h (4.22) (linia przerywana) próbkowanej podobnie jak w Przykładzie 6.4 (patrz (6.39) z $\mathcal{Q}_6$ (wytłuszczone) *gładką przedziałowo-kubiczną funkcją sklejaną na bazie skumulowanej parametryzacji długością cięciwy* γ_{h_H} (linia ciągła) z oszacowaniem długości: $d(\gamma_{h_H}) = d(\gamma_h) - 4.478 \times 10^{-2}$

Remark 6.3. Note again that constraint (1.51) is *a necessary condition* for Theorem 6.1 to hold. Indeed, for γ_{sp} defined by (1.18) and sampled according to (1.17) (for which $m\delta \neq O(1)$) cumulative chord C^1 piecewise-cubic γ_{sp_H} (see (6.3) and (6.4)) gives and estimate of $d(\gamma_{sp})$ (with $6 \leq m \leq 95$) equal to $2.71 < 4$ (established for γ_{sp_H} in Theorem 6.1). Experiments show that regularity of γ may not be a necessary condition for Theorem 6.1 to hold. This remains an open problem. However, as previously, the proof of Theorem 6.1 relies on assumption that γ is regular and thus as previously the latter is not abandoned.

6.5 Discussion

The *quartic order* of convergence for length approximation given in Theorem 6.1 for regular cumulative chord C^1 piecewise-cubics is confirmed experimentally to be *sharp* (at least for $n = 2, 3$). Curves need *not be planar nor convex* and samplings apart from the natural condition (1.3) need *not be more-or-less uniform*. Condition $m\delta = O(1)$ (needed only for length estimation) is a necessary condition for Theorem 6.1 to hold (see Remark 6.3). The scheme in question also performs *well on sporadic data*, not covered by the asymptotic analysis presented herein. The trajectory of γ_H is *a geometrically smooth curve* (see Theorem 6.1 and Remark 6.2). The latter yields the main advantage of cumulative chord C^1 piecewise-cubics γ_H over cumulative chord piecewise-cubics $\hat{\gamma}_3$ (see 4.2 and 4.3). The other properties of γ_H are similar to $\hat{\gamma}_3$. In particular γ_H provides also a positive answer to the Question I from Subsection 1.3.2. It has also similar advantages as $\hat{\gamma}_3$ (see Section 4.5) over other interpolation schemes discussed in this monograph.

Chapter 7

Conclusion

We close this monograph with conclusion and short discussion.

7.1 Conclusion and main results

This monograph (with results *published* in [27], [28], [29], [30], [31], [32], [42], [43], [44], [45], [46] and [47]) achieves the following:

- the monograph analyzes the problem of *interpolating* the sequence of the general class of *reduced data* $\mathcal{Q}_m = (q_0, q_1, \ldots, q_m)$, obtained from admissible samplings of a regular, sufficiently smooth parametric curve $\gamma : [0, T] \to \mathbb{R}^n$, with interpolation knots $\{t_i\}_{i=0}^m$, for $q_i = \gamma(t_i)$, assumed to be unknown. Such an interpolation is coined as *non-parametric interpolation*.

- a focus is given to the *asymptotic analysis of* γ *and length* $d(\gamma)$ *estimation*, with different non-parametric interpolation schemes (*i.e. piecewise-polynomials*). Though the experiments show good performance of the interpolants for both *dense and sparse data*. *Literature review* and some *applications* are given. The *novelty of this work* (into the topic in question) is highlighted in each *chapter's abstract* together with the relevant *list of publications composing the backbone of this monograph*.

- first a short discussion covers the parametric case, for which the parameters from the pair $(\mathcal{Q}_m, \{t_i\}_{i=0}^m)$ are temporarily assumed to be known. As re-established, a piecewise-r-degree Lagrange interpolation (a classical parametric interpolation) yields an increment in approximation orders (from r to $r+1$) in γ and $d(\gamma)$ approximation. An extra acceleration is achievable for special subfamilies of admissible samplings (*e.g.* equidistant or ε-uniform) $-$ a new result.

- construction of a particular non-parametric interpolant, requires input of an explicit sequence of guessed knots $\{\hat{t}_i\}_{i=0}^m \approx \{t_i\}_{i=0}^m$. As proved in this work (a new result), *a blind choice of such knots* (*e.g.* $\hat{t}_i = i$) *i.e.* with no respect to the geometry of input data $\mathcal{Q}_m$, may have *serious consequences* on the quality of γ and $d(\gamma)$ estimation.

- a partial solution to the above problem yields *a piecewise-4-point quadratic interpolation* scheme Q (together with appropriately guessed $\{\hat{t}_i\}_{i=0}^m$ estimating the unknown interpolation knots $\{t_i\}_{i=0}^m$). Fast *quartic orders* of convergence for both γ and $d(\gamma)$ estimation are proved and verified to be *sharp* at least for length estimation. This also forms a new result. Good performance of Q is confirmed experimentally on both *dense and sporadic* reduced data $\mathcal{Q}_m$. Though the scheme is limited only to *planar and strictly convex curves sampled more-or-less uniformly*, it demonstrates that *the compensation for the loss of information* (while passing from non-reduced to reduced input data) is possible $-$ compare orders from Theorem 1.2 with $r = 3$ against the orders from Theorem 3.1.

- the latter results extend to *an arbitrary curve in* $\mathbb{R}^n$ by setting $\{\hat{t}_i\}_{i=0}^m$ as equal to the *cumulative chord length parameterization* (4.1). Then, as proved herein, *Lagrange piecewise-quadratics and piecewise-cubics* $\hat{\gamma}_k$ (for $k = 2, 3$) based on (4.1), approximate fast both γ and $d(\gamma)$ (*i.e.* at the $k+1$ order). The latter holds *sharply* within *the general class of admissible samplings* with no tight restriction on γ and n. Both cumulative chord piecewise-quadratics and piecewise-cubics $\hat{\gamma}_k$ (for $k = 2, 3$) perform well on *dense and sporadic data*. Extra acceleration in estimation orders for $d(\gamma)$ results also for $\hat{\gamma}_2$ and ε-uniform samplings. This above forms also a new result.

- cumulative chords $\hat{\gamma}_k$ (with $k = 2, 3$), with $\{\hat{t}_i\}_{i=0}^m$ as in (4.1), and piecewise-quadratic and piecewise-cubic Lagrange polynomials based on $\{t_i\}_{i=0}^m$ (if known), yield the same convergence orders in estimation of γ and $d(\gamma)$ $-$ see Theorem 1.2 with $r = 2, 3$ and Theorem 4.1 with $k = 2, 3$ (new results). The latter shows that, for *general class of curves in* $\mathbb{R}^n$ *the compensation for of loss of information*, when passing from non-reduced to reduced input data and applying the same interpolation scheme, is possible $-$ *i.e* piecewise-quadratic or piecewise-cubic Lagrange polynomials based on either $\{\hat{t}_i\}_{i=0}^m$ from (4.1) or $\{t_i\}_{i=0}^m$ (if known).

- *cumulative chord piecewise-quartics* $\hat{\gamma}_4$, with $\{\hat{t}_i\}_{i=0}^m$ defined as in (4.1), *do not yield*, in general, *quintic orders of convergence* in estimating γ and $d(\gamma)$. Recall that such an increment occurs for piecewise-r-degree Lagrange interpolation used with $\{t_i\}_{i=0}^m$ known (see Theorem 1.2). As proved in Theorems 5.1 and 5.2, *quartic or faster convergence orders* for $\hat{\gamma}_4$ hold only for *some subfamilies of admissible samplings* $\mathcal{Q}_m$ (*e.g.* for more-or-less or ε-uniform samplings). The latter forms also a new result. Thus to interpolate fast reduced data $\mathcal{Q}_m$ it suffices merely to apply either cumulative chord piecewise-quadratics or piecewise-cubics $\hat{\gamma}_k$ (for $k = 2, 3$).

- functions Q and γ_k (for $k = 2, 3, 4$) designed to fit reduced data $\mathcal{Q}_m$ *are not smooth at junction points* (where local interpolants are glued together). This blemish is removed by introducing *a cumulative chord smooth piecewise-cubic* interpolant γ_H. The herein proved *quartic convergence orders* in estimating both γ and $d(\gamma)$ (a new result) are verified to be *sharp* (at least for length). Function γ_H performs well on *both dense and sporadic data*. This again shows, for smooth interpolation, the possibility of *compensating for the loss of information* while passing from non--reduced to reduced data $-$ compare Theorem 1.2 (for $r = 3$) with Theorem 6.1.

This monograph shows that the *geometry of the distribution of the reduced data* $\mathcal{Q}_m$ *plays a vital role* in pinpointing the interpolation knots $\{\hat{t}_i\}_{i=0}^m \approx \{t_i\}_{i=0}^m$. As demonstrated herein, parameterization based on *cumulative chords* combined with $\hat{\gamma}_k$ (for $k = 2, 3, 4$) or γ_H, yield excellent interpolation schemes suitable to fit *reduced data (dense or sporadic)* which are derived from *general admissible samplings* of regular curves in $\mathbb{R}^n$.

7.2 Future research

The future *research extending* the topic in question and herein established results may include the following:

- an extension of this work to *other parameterizations* commonly used in curve modeling: *e.g. an exponential one* (see Chapter 1).
- a discussion for other *subfamilies of class of admissible samplings* $\mathcal{V}^m \subset \mathcal{V}_G^m$ for both reduced $\mathcal{Q}_m$ and non-reduced data $(\mathcal{Q}_m, \{t_i\}_{i=0}^m)$.
- an analysis from this monograph extended to *sporadic data* and/or to *non-regular* curves (revisiting the proofs of this monograph).
- an extension of the topic in question to other non-parametric interpolation schemes (*e.g. cubic splines*).
- an extension of the analysis to *curvature or torsion estimation* (revisiting the proofs of this monograph).
- an extension of the analysis from Chapter 2 to $r > 2$ - negative impact on approximation of γ and $d(\gamma)$ with of $\hat{t}_i = i$ and piecewise-r-degree Lagrange interpolation.
- an analysis to the *approximation techniques* applied *e.g.* for *noisy* or *digitized data*. Usually when noisy data are considered, the interpolation (adopted in this work for exact data) is *not a correct model* for data fitting. If *e.g.* additional information about γ is given (*e.g.* the curve belongs to the family of some curves like ellipses) then one can derive different costs functions (based on non-linear regression). The case of fitting noisy data (*e.g.* for curves or surfaces) requires more advanced mathematical and statistical models and techniques (exceeding the scope of this monograph) which usually yields difficult and computationally expensive non-linear optimization problems (see *e.g.* [8]). Still for approximation with piecewise-modeling cumulative chord parameterization can be invoked.
- implementation of fast methods for $m \approx \infty$. Parallel processing (see [1]) for piecewise-polynomial might be an option. Another approach (taking into account good performance of cumulative chords on sparse data) is to somehow coarsen the input data $\mathcal{Q}_m$ to $\mathcal{Q}_l$, where l is small.

There are many other avenues for this work to evolve (like *implicitly defined curves* or *surface and area estimation* from reduced data). Note however, that in contrast with regular curves (for which *arc-length* parameterization exists giving the chance for cumulative chord parameterization to work - as shown herein), there is no natural parameterization for surfaces or higher dimensional manifolds. Thus it should be pointed out that any extension of this work to higher dimensions or to noisy reduced data opens another Pandora's box worth thorough study, where different mathematical and statistical apparatus is to be invoked. This work shows however, the importance and large potential of the topic in question which this monograph merely covers and proves to be a "tip of the research iceberg".

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Symbols

$\| \cdot \|$ - standard norm in Euclidean space $\mathbb{R}^n$: Subsection 1.1

$\langle \cdot, \cdot \rangle$ - standard dot product in Euclidean space $\mathbb{R}^n$: (1.41)

$\perp$ - orthogonal: (1.41)

$\square$ - end of proof of theorem or lemma (or end of example): Chapter 1

$[0,T]^{m+1}$ - Cartesian $m+1$ product of interval $[0,T]$: Section 1.1

α_+ - parameter for Q^i: (3.4), (3.9)

β_+ - parameter for Q^i: (3.4), (3.9)

$\alpha^m_{\gamma,r}$ - computed convergence order: Subsection 1.6.3

γ - parametric curve $[0,T] \to \mathbb{R}^n$: Subsection 1.13555.

γ_a - asteroid: (5.22)

γ_c - cubic curve: (1.25)

γ_{cf} - cubic curve with inflection point: Example 4.3

γ_{cnc} - cubic curve: (6.36)

γ_{c_0} - cubic curve: Example 3.4

γ_{c_1} - cubic curve: Remark 3.3

γ_{c_2} - cubic curve: Remark 3.4

γ_h - elliptical helix (4.22)

γ_{pl} - piecewise-linear curve: Remark 5.3

γ_q - quartic curve: Remark 3.3

γ_{q_b} - quartic curve (6.37)

γ_{q_5} - quintic curve: (1.66)

γ_{sc} - semicircle: (1.1)

γ_{sp} - spiral: (1.18)

γ_{spl} - spiral (3.66)

$\gamma \circ \psi$ - composition of functions: Subsection 1.1

$\tilde{\gamma}$ - any interpolant for reduced or non-reduced data: Definition 1.3

γ^i_H - Hermite cubic: (6.3), (6.4)

γ_H - cumulative chord C^1 piecewise-cubic: Section 6.1

$\tilde{\gamma}_r$ - piecewise-r-degree Lagrange polynomial: Definition 1.1

$\tilde{\gamma}_{i,r}$ - Lagrange r-degree polynomial: (1.8)

$\tilde{\gamma}_{r,m}$ - piecewise-r-degree Lagrange polynomials: (1.65)

$\tilde{\gamma}_j$, $\tilde{\gamma}_i$ - reparameterized $\tilde{\gamma}_{i,r}$: (1.34), (2.18)

$\hat{\gamma}_k$ - cumulative chord piecewise-degree-k interpolant: (4.3)

$\hat{\gamma}^i_k$ - cumulative chord interpolant of degree-k for $\mathcal{Q}^{i,k}_m$: Section 4.1

$\hat{\gamma}^i_{3-}$ - reverse cumulative chord cubic: Section 6.1

$\gamma[t_i, t_{i+1}, \ldots, t_{i+k}]$ - k-th divided difference: (4.8)

$\gamma[t_i, t_i, \ldots, t_i]$ - k-th divided difference for $k+1$ coinciding knots: (6.6)

δ, δ_m : - maximum jump in $|t_{i+1} - t_i|$: Definition 1.2

ε - parameter in ε-uniform sampling: Definition 1.4

κ - curvature of curve in $\mathbb{R}^n$: (4.11)

λ - more-or-less uniform constant: (1.29)

ϕ - reparameterization: Definition 1.4

ϕ_1, ϕ_2, ϕ_3 - reparameterizations: Example 1.4

ψ - reparameterization: Subsection 1.1

C^k - class of k-times continuously differentiable functions $[0,T] \to \mathbb{R}^n$: Subsection 1.1

C^∞ - class of infinitely many times continuously differentiable functions $[0,T] \to \mathbb{R}^n$: Theorem 1.1

$d(\gamma)$ - length of curve γ: (1.1)

E_m - absolute errors: (1.65)

$\{f_i\}_{i=0}^{k_0-1}$ - track-sum of functions: Remark 1.1

$\mathcal{G}_{\tilde{\gamma}}$ - trajectory of $\tilde{\gamma}$: (1.5)

$k(t)$ - curvature of planar curves: (3.1)

K_l, K_u - constants for more-or-less uniform samplings: (1.28)

m - number of sampling points: (1.2)

n - dimension of sampling data: Subsection 1.1

$O(\delta^p)$ - approximation of order p: Definition 1.3 and (1.6)

$O(g(m))$ - approximation of order $g(m)$: Definition 1.4

P_r^j - r-degree Lagrange polynomial: (1.32)

q_i - interpolation point: (1.2)

Q - piecewise-4-point quadratic:
Q^i - 4-point quadratic: (3.3), (3.4)

$\mathcal{Q}_m$ - m sampling points $(q_i, q_{i+1}, \ldots, q_m)$: (1.2)

$\mathcal{Q}_m^{i,r}$ - r-tuple of sampling points $(q_i, q_{i+1}, \ldots, q_{i+r})$: Definition 1.1

r - degree of the Lagrange polynomial: (1.8)

$\mathbb{R}^n$ - n dimensional Euclidean space: Subsection 1.1

$\{t_i\}_{i=0}^m$ - parametric interpolation knots: Subsection 1.1

T - upper bound of $\gamma' s$ domain: Subsection 1.1

$\hat{T}$ - upper domain bound of non-parametric interpolant: Subsection 1.1)

$\{\hat{t}_i\}_{i=0}^m$ - non-parametric interpolation knots: Subsection 1.1

$v(q_i)$ - tangent vector at q_i obtained from cumulative chord piecewise-cubics - Section 6.1

$V_{\dot{\gamma}}^{\perp}(t)$ - orthogonal complement to $\dot{\gamma}(t)$: (1.41)

$\mathcal{V}_G^m$ - admissible samplings: Definition 1.2

$\mathcal{V}^m$ - any subfamily of admissible samplings: Subsection 1.3.3

$\mathcal{V}_0^m$ - 0-uniform samplings: Subsection 1.3.3

$\mathcal{V}_{\varepsilon>0}^m$ - ε-uniform samplings for $\varepsilon > 0$: Definition 1.4

$\mathcal{V}_{\varepsilon\geq 1}^m$ - ε-uniform samplings for $\varepsilon \geq 1$: Definition 1.4

$\mathcal{V}_U^m$ - uniform samplings: Subsection 1.3.1

$\mathcal{V}_\varepsilon^m$ - ε-uniform samplings: Definition 1.4

$\mathcal{V}_{mol}^m$ - more-or-less uniform samplings: Definition 1.5

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All manuscripts should be written in Polish or in English. Manuscript should be typed on one side paper only, and submitted in duplicate. The name and affiliation of each author should be followed by the title of the paper (as brief as possible). An abstract of not more than 50 words is required. The text should be logically divided under numbered headings and subheadings (up to four levels). Each table must have a title and should be cited in the text. Each figure should have a caption and have to be cited in the text. References should be cited with a number in square brackets that corresponds to a proper number in the reference list. The accuracy of the references is the author's responsibility. Abbreviations should be used sparingly and given in full at first mention (e.g. "Central Processing Unit (CPU)"). In case when the manuscript is provided in Polish (English) language, the summary and additional abstract (up to 300 words with reference to the equations, tables and figures) in English (Polish) should be added.

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